Computing the Optimal Number of Vessels for Circular Short Sea Schedules

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1. Introduction

Maritime shipping is a crucial to international trade as it is used by more than 80 percent of the world merchandise trade (Foresight, 2017) using different types of sea vessels including container or LoLo vessels (Lift on Lift off), RoRo vessels (Roll on Roll Off), reefer vessels and tankers, bulk and break vessels and other multi-purpose vessels. Furthermore, maritime shipping can also be one of two types of namely, short sea shipping and deep-sea shipping[1]. Short sea shipping is transport of goods along relatively short distances for example along the coast of a continent, crossing channels, straits or smaller seas, whereas deep sea shipping is the transport of freight across oceans. England, being an island, it is very dependent on short sea services for trade with Europe as illustrated by the various short sea routes in Figure 1. For example, there are more than 50 daily crossings between Dover and the Continent. In 2017, it is estimated that 85% of UK's international trade, or 380.2 million tonnes of freight tonnage[2], was moved by sea. Of that total, 71.9%, (273.5 million tonnes), was moved by short sea shipping.

Consider the case of short sea shipping where a vessel continuously shuttles between two ports on a fixed schedule as is the case between the ports of Calais and Dover. In this paper we consider the problem of determining the minimum number of ships required to maintain regular schedules for short sea shipping given a maximum waiting time for loads at the ports and look at special cases where the schedule is also circular. This is a crucial problem for short sea shipping for the UK. Social distancing due to Covid-19 and Brexit have greatly changed the capacity of some ports and shipping lines need to find alternative ports to introduce new services to maintain the same service levels and freight capacity.

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[1] https://www.opensea.pro/blog/short-sea-vs-deep-sea

[2] https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/762200/port-freight-statistics-2017.pdf

Figure 1: Major Ferry routes serving the UK(ref https://www.discoverferries.com/destinations/)

The paper gives a new concise mathematical model for finding the optimal number of ships for the general case where loads arriving at a port should wait no more than W periods before it leaves on a vessel. We also show that the general problem is NP-Complete, but a special case with no waiting time, i.e. $W=0$ can be easily solved in polynomial time.

2. Problem description, notation and hardness

2.1. The 2-Port, Circular-Schedule, Fleet Design (2-PCSFD) problem

Assume the case of a vessel that shuttles between two ports $Port\ 1$ and $Port\ 2$ on a continuous basis. The travel time, TR, between the two ports is known and constant. Let LU be the loading, unloading times, refuelling and other maintenance time incurred at each port. Assume that LU is also constant and is the same at each port. This means that if t is the departure time of the ship at $Port\ 1$ then the ship will next depart from $Port\ 1$ at time $t + 2(TR + LU)$ and again at $t + 4(TR + LU)$ and so on. We call this a *regular* schedule.

Let $p = 1, 2$ be the two ports and let l be a load arriving at a port at a time period t. Each load consists of a number of standardised containers or trailers. Let W be the maximum number of period that a load can wait at the ports including the period in which it arrives. Hence loads arriving within time period t must depart by the end of period $t + W - 1$. Furthermore, we assume that the load pattern arriving at a port repeat in time after every T period. The latter is referred to as the load cycle.

For short sea shipping, as is the case when crossing the English Channel (see Figure 1), T is usually a day and $TR + LU \leq 24$. In this paper, we restrict the analysis to the case where $TR + LU \leq T$. Let's define a *circular* ship schedule as one where the ship departure times from the ports are repeated over after a certain time period. In general, circular schedules can be achieved after T periods if $\frac{T}{TR+TU}$ is an even number or 2T if $\frac{T}{TR+TU}$ is an odd number (see Appendix for proof). For example, if T = 24 and $TR + LU = 12$, and the ship departs $Port\ 1$ at 6:00 am, then it will depart again from $Port\ 1$ the next day at 6:00 am. Here, circularity is achieved over a period of 24 hrs. On the other hand, if $TR + LU = 24$ hrs, a ship departing $Port\ 1$ at 6:00 am will depart from $Port\ 2$ at 6:00 am the next day and again from $Port\ 1$ at 6:00 am on Day 2. Here circularity is achieved over a period of 48 hrs.

The problem we consider is to find the minimum number of vessels required to maintain a circular schedule between the ports to transport all the loads to their destination, given vessel capacities and a maximum waiting time for the loads. We refer to this problem as the *2-Port, Circular-Schedule, Fleet Design* (2-PCSFD) problem.

2.2. Notation and Assumptions

Let

TR: Travel time between two ports;

LU: Time to load, unload and service the ship at port;

W: Number of maximum waiting periods for a load arriving at port before it leaves on a vessel;

C_i: Capacity of ship i;

T: Planning horizon to achieve circular regular schedule;

TE: The horizon period on which the problem will be solved;

Q_l': is the quantity of load $l = 1, \ldots, L$;

q_{lipt}: binary variable which equals to 1 if load l is departing on ship i at time t from $p = port(l)$, otherwise 0;

x_{ipt}: binary variable which equals to1 if ship i departs port p at time t, 0 otherwise;

y_{ipt}: binary variable which equals to1 if ship i arrives at port p at time t, 0 otherwise;

n_i: binary variable which equals to1 if ship i is used.

2.3. Time-Space network:

Fleet design and transport optimization problem with schedules usually embeds a time-space network for investigation of vessels trajectory and this technique has been widely used (Steinzen et al., 2010, Zhang et al., 2017). Let $G = (V, E)$ be our directed time-space network in which V is the set of vertices and E is the set of directed edges. There is a vertex $v \in V$ for each port p in each time period t, $v_{p,t}$ and let functions $port(v) = p$ and $time(v) = t$. We have three types of edges in the network. The first type of edge represents a vessel crossing between two ports and connects $v_{p,t}$ and $v_{d,t+TR}$ where $p \neq d$. The length of this edge is equal to the crossing time, *TR*, between two ports. The second edge represents the loading/unloading time of a vessel at a port. It connects an arrival vertex $v_{p,t}$ to a departure vertex $v_{p,t+LU}$ and has a length of LU.

The third type of edge represents waiting times for loads arriving at a port. A load arriving at time t will wait for a maximum of W periods and hence there is a series of edges connecting vertex $v_{p,t}$ to each of the vertices $v_{p,t'}$ where

$$t' = \{t + 1 \ldots t + W - 1\}, \quad \text{if } t + W - 1 \leq T$$

and if $t + W - 1 > T$, then

$$t' = \{t + 1 \ldots, t + W - 1 | \text{ if } t + W - 1 \leq T\} \cup \left\{1, \ldots mod\left(\frac{t+W-1}{T}\right) \middle| \, if \, t + W - 1 \leq T\right\},$$

Figure 2 gives an example of a time-space network where *TR= 3, LR = 2* and *W=4*. The crossing for a vessel leaving the first port at time 1 is represented by the edge from $v_{1,1}$ to $v_{2,4}$. Its loading/unloading time by the edge from $v_{2,4}$ to $v_{2,6}$ when it is ready to depart. A load arriving at port 1 in time 1 could wait until period 4 and hence the dotted arcs from $v_{1,1}$. Moreover, the figure shows a load arriving at the end of the time horizon, i.e. $t = T$, can wait up to periods 1, 2 and 3 on the next day.

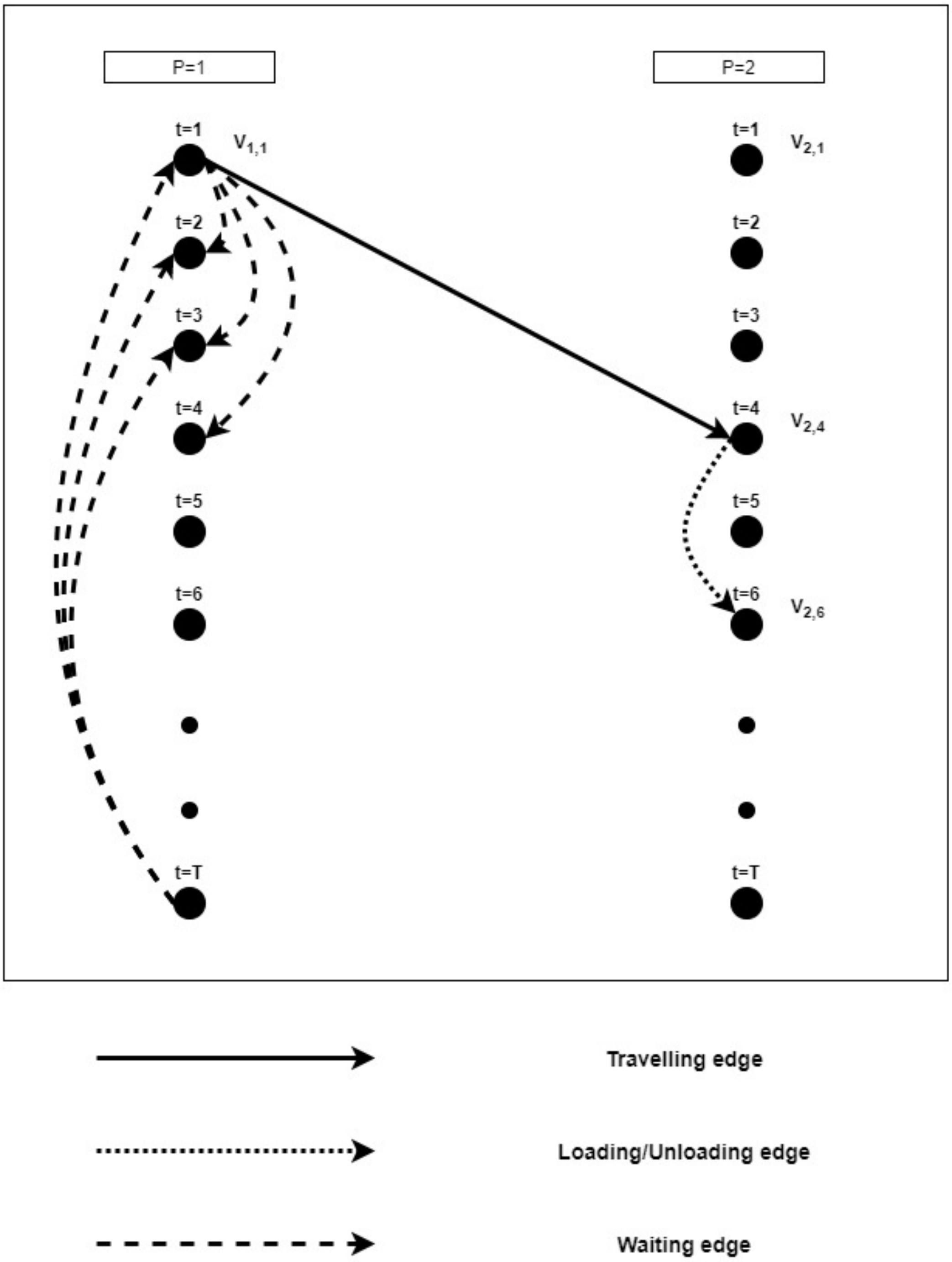

Figure 2 Example of a time-space network

2.4. Hardness of the problem

The decision version of the minimum number of the vessels for a regular service between two ports is in NP. We demonstrate the NP-completeness of the problem by reducing the NP-complete Bin Packing Problem (BPP) to our problem.

The decision version of the BPP is defined as follows: Given integer sets $N = \{1,2,\cdots,n\}$, a number k and constant c where $w_l \leq c, \ \forall\, l \in N$, can the set N be partitioned into k distinct subsets $S_{1,}S_{2,}$, ..., $S_{k,}$, such that, $\sum_{l \in S_i} w_l \leq c$ for i=1,..k.

Theorem: The decision version of the 2-PCSFD problem is NP-complete.

Proof: Given BPP, construct an instance $\mathfrak{I}$ of the *2-PCSFD* problem with a homogenous fleet of N vessels of capacity $C = c$. Set $TR = n$ and $LU = 0$. The time-space network is set as follows: we have two ports, P_1 and P_2 with a load cycle T = 2n with loads potentially arriving in each of time period t, $t = \{1,..,T\}$. Since $\frac{T}{TR+TU} = 2$, circularity is achieved after T periods. Loads $l = 1,\cdots,n$ arrive at P_1 in time period $t = 1$, and the quantity of each load $Q_l' = w_l$. Set the waiting time $W = n$, meaning that loads can wait a maximum of n periods, inclusive of the period in which it arrived, before being picked up by a vessel.
It can be observed that by construction of $\mathfrak{I}$:

1. All loads must be picked by time n at latest;
2. Because of the traveling time, vessels leaving P_1 at any time $t \leq n$ will only return to P_1 after time $t > n$;

The vessels in *2-PCSFD* are equivalent to the bins BPP. Let z be the minimum number of vessels in a solution to $\mathfrak{I}$ and let function $arg_l(q_{lipt})$ be defined as follows:

$$arg_l(q_{lipt}) = \begin{cases} l, & \text{if } q_{lipt} = 1 \text{ in the solution to } \mathfrak{I} \\ null, & otherwise \end{cases} \qquad (15)$$

The solution to the BPP is constructed by setting $k = z$ and sets $S_{1,}S_{2,}$, ..., $S_{k,}$ are defined as follows: $S_{i,} = \{arg_l(q_{li1t})\}$ for $i = 1,\cdots,k$, $l = 1,\cdots,n$ and $t = 1,\cdots,T$. Note that $\sum_{l \in S_i} w_l \leq c$ for $i = 1,\cdots,k$, is guaranteed since any vessel i in *2-PCSFD* departs P_1 only once in time horizon t=1, $\cdots,n$. Thus, BPP can be solved by solving the *2-PCSFD.*

For the rest of the study we will assume the following:

Assumption 1: $TR + LU \leq T$ and $\frac{T}{TR+TU}$ is integer.

Furthermore, if $\frac{T}{TR+TU}$ is even, circularity is reached after T periods and after 2T periods if it is odd. Having $\frac{T}{TR+TU}$ as an integer yields circular schedules which is a nice property to have as passengers can easily remember the schedules. This is always possible in practice by adjusting $TR + LU$ either by increasing or decreasing the speed of the ship or by adjusting LU, the time spent at the ports between arrivals and departures.

Assumption 2: $C_i \geq Max_l(Q_l')$

The ship capacity is greater than the maximum quantity of a load. Note that this is not as restrictive as it sounds since loads can be broken to accommodate ship sizes but there must be an adequate number of ships of the right size so that all the loads are carried within the maximum waiting time. This means that there could be two or more loads of different quantities to be picked up at a port in the same time period t.

3. Solution Approaches

We present a binary mathematical model for solving the 2-PCSFD problem and look at the special case for finding regular circular schedules under maximum load waiting times. Let i be a distinct ship and assume that we have N ships.

3.1. General Mathematical Model

$$BMP: Min \sum_{i}^{N} n_i \qquad (2)$$

s.t.

$$n_i \geq x_{ipt} \qquad for\ each\ i, p, t \qquad (3)$$

$$x_{ipt} + x_{ip't} \leq 1 \qquad for\ each\ i, p \neq p', t \qquad (4)$$

$$y_{ipt} + y_{ip't} \leq 1 \qquad for\ each\ i, p \neq p', t \qquad (5)$$

Enforcing regular schedules:

$$x_{ipt} = y_{ip'(t+TR)} \qquad for\ each\ i, p \neq p'\ and\ t \qquad (6)$$

$$y_{ipt} \leq x_{ip(t+LU)} \qquad for\ each\ i, p, t\ \in TE \qquad (7)$$

$$x_{ipt} + \sum_{t}^{t+2TR+2LU-1} x_{ipt} \leq 1 \qquad for\ each\ i, p \neq p'\ and\ t \qquad (8)$$

Enforcing max waiting time of loads and capacity of ship:

$$\begin{cases} \sum_{i=1}^{N} \sum_{t=t'}^{(t'+w-1)} q_{lipt} = 1 & if\ t \leq T, \\ \sum_{i=1}^{N} \sum_{t=1}^{(t'-T+w-1)} q_{lipmod(\frac{t}{T})} = 1 & if\ t > T, \end{cases} \qquad for\ each\ l, t' = avail(l), p = port(l) \quad (10)$$

$$\sum_{l=1}^{L} q_{lipt} \times Q_l' \leq C_i \qquad for\ each\ i, t\ and\ p \qquad (11)$$

$$x_{ipt} \geq q_{lipt} \qquad for\ each\ l, i, t\ and\ p = port(l) \qquad (12)$$

$x_{ipt}, y_{ipt}, n_i, q_{lipt} \in (0,1)$ (13)

The objective function minimizes the number of ships. Constraint (3) ensures that ship N_i is in service if it is to depart port p at time t. Constraints (4) and (5) guarantee that a ship cannot depart from or arrive at the two ports at the same time. Constraints (6) and (7) are used to enforce regular schedules. If a ship i departs port p at time t, i.e. x_{ipt}, then it should arrive at p' at t+TR. Similarly, if ship i arrives at a port p at time t (y_{ipt}), then it should depart at t+LU. Constraint (8) guarantees that if ship *i* leaves port *p* at time *t*, then it cannot depart again from port *p* until *t+2(TR+LU)*. Equation (9) ensures that load l available the port *p=port(l)* at time *t'=avail(l),* is carried by ship i at port p by time *t'+W-1*, where *W* is the max waiting time. Equation (10) guaranties that the load available to be shipped at time *t'=avail(l)*, is being carried out by one and only one of the ships departing port *p=port(l)* on or before time *t'+W-1*. Equation (11) makes sure that the total load on ship i departing port *p* at time t is less than the capacity of the ship. Equation (12) makes sure, that if the load *l* is being put on the ship *i* departing the port *p* at time *t*, then there is a departure for the ship *i* from port p at time t.

By assumption 1, circularity is achieved after either T or 2T periods. To solve BMP, a time interval of $TR + 2LU$ must be added to T to make the solutions feasible for loads arriving at a port between the $T - (TR + LU)$ and T according to equation 5 and 6. Hence, the effective planning horizon, TE, over which BMP will be solved is

$$TE = \begin{cases} 1,2,\cdots,(24 + TR + 2LU\ )\ if\ \frac{24}{LU+TR}\ is\ an\ even\ number \\ 1,2,\cdots,(48 + TR + 2LU)\ \ if\ \frac{24}{LU+TR}\ is\ an\ odd\ number \end{cases} \quad (14)$$

Note that less or equal sign is required in equation (7) because the effective planning horizon ends at some point and the ship will not depart if it cannot reach the other port before the end of the planning horizon.

3.2. Numerical Examples

Consider the example shown in Figure 3 where there is a total of 18 loads $l = (1, \ldots, 18)$ to be loaded from both ports $Port\ 1$ and $Port\ 2$ in a period of *T=24* hours. Here, loads $l = (1, \ldots, 10)$ are to be shipped from $Port\ 1$ in different time periods and loads $l = (11, \ldots, 18)$ are to be shipped from $Port\ 2$ in different time periods (see Figure 3). Notice that the numbering of the load, l, uniquely defines the time at which the load can be shipped and the port from which it is shipped. For l = 3, port(l) =1, avail(l) =6, Q'_3 = 300.

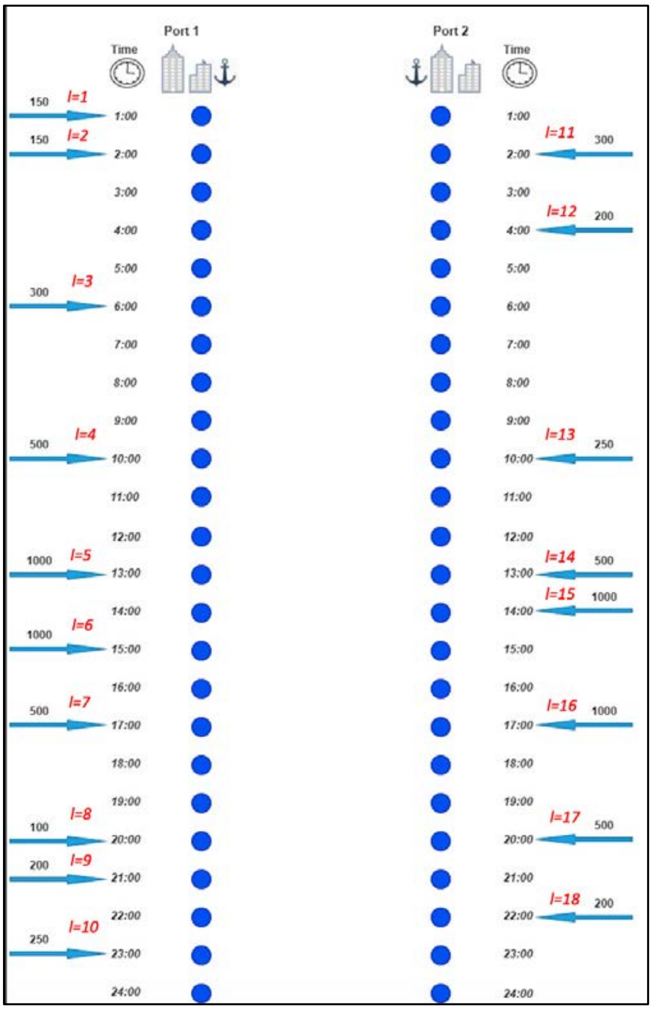

Figure 3: Example with 18 loads distributed over a period of 24 hours

For this example, assume that C_i=1000 for all ships and that $W = 4$ meaning load l must leave by time $avail(l) + W - 1$. Also, let *TR = 4*, and *LU = 2*, the following results are obtained. Since $\frac{24}{TR+LU}$ is an even number circularity is achieved in 24 hrs and TE = 1,…,32. The number of the ships required for this example is 4 and the solution is obtained by solving the BMP as shown in Figure 4.

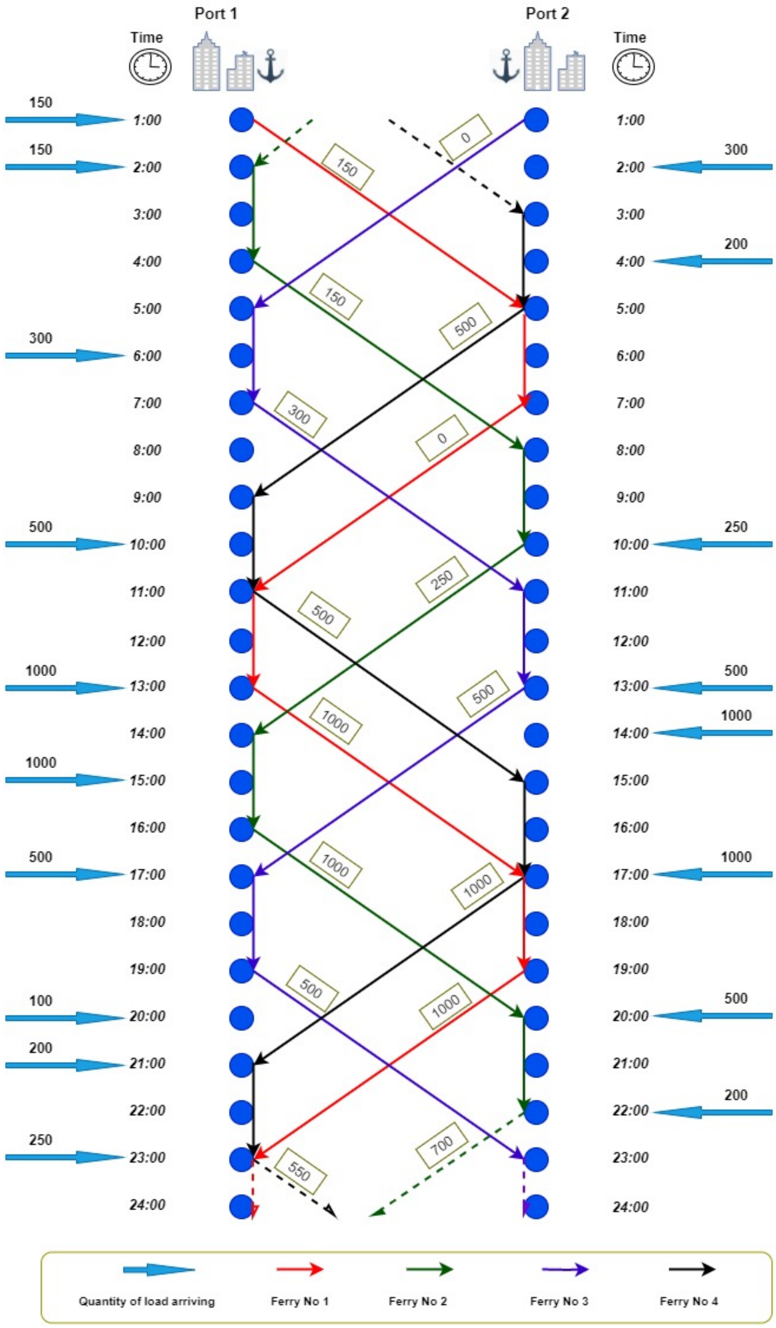

Figure 4: Solution with TR=4, LU=2 and W=4

Note that if *TR* is increased from 4 to 6 hrs and LU remains at 2 hrs, then since, $\frac{24}{Tr+Lt}$ is an odd number, $TE = 1, \ldots, 52$. Solving the problem, we find that the number of ships required is now 6 and the resulting schedule shown in Figure 5.

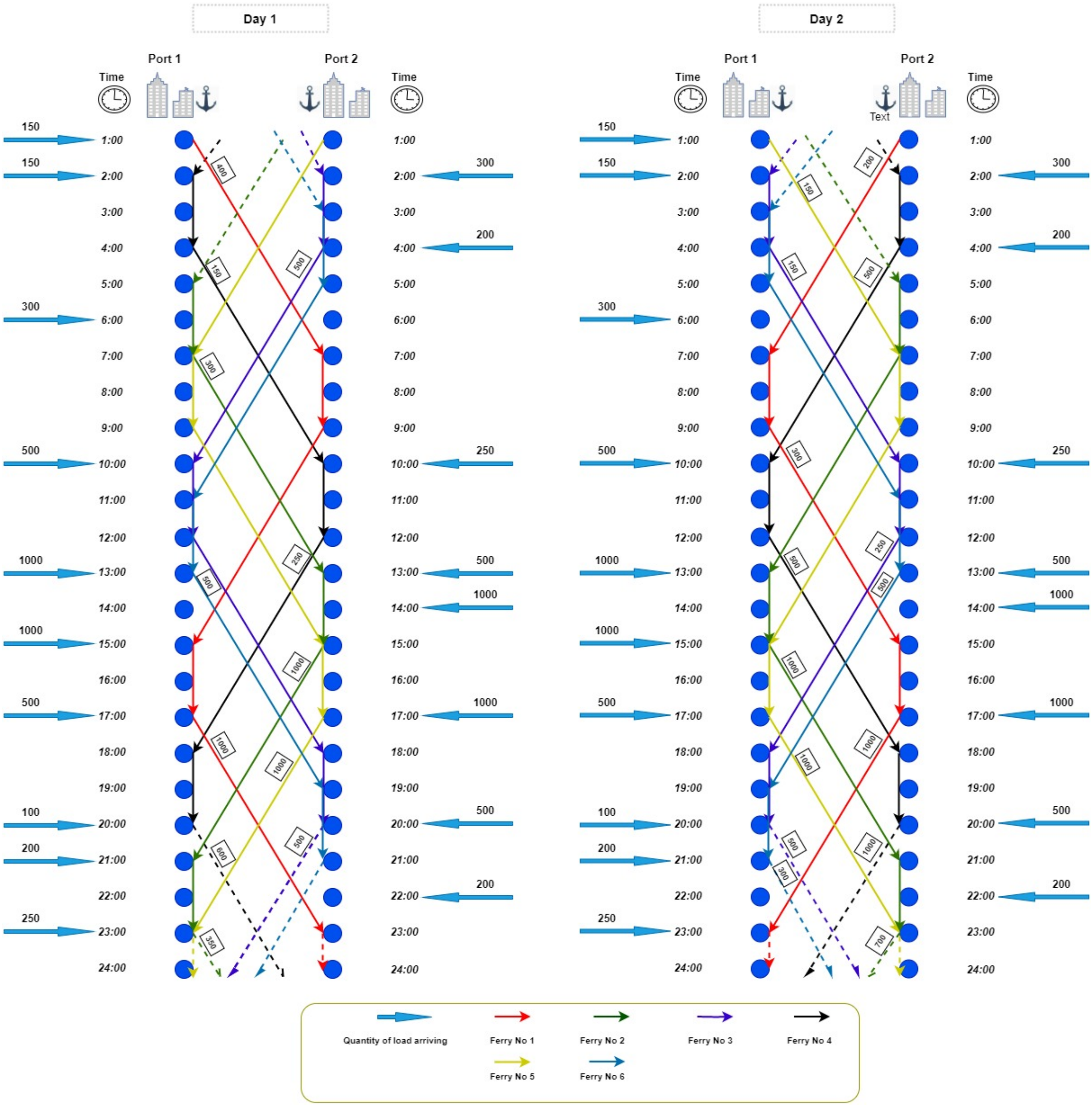

Figure 5: Solution with TR=6, LU=2 and W=4

Furthermore, assume a new load distribution based on Figure 6. Let $TR = 4$ and $LU = 2$. Waiting time remains 4 hrs and again since $\frac{24}{TR+LU}$ is an even number circularity is achieved in 24 hrs and $TE = 1,2,\cdots,32$. The optimal number of vessels is 4 and the circular schedule is shown in Figure 6. Note that the last load arriving at P_1 in time period 23, waits for the vessel departing at time period 2 the next day.

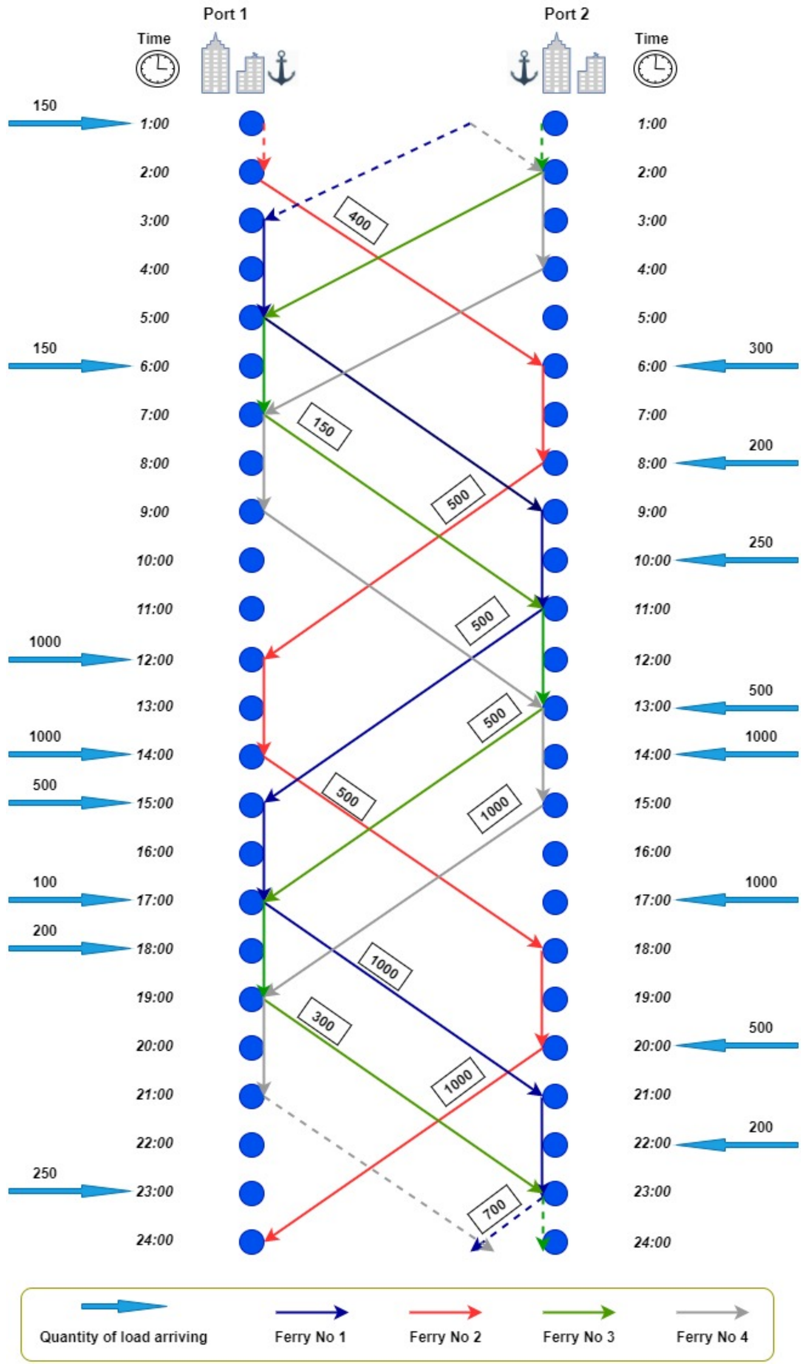

Figure 6 Solution with TR=4, LU=2, and W=4 and loads waiting to be moved the next day

3.3. Special Case with identical ships and No Waiting time for loads

Consider the special case where ports have a maximum of one load l with quantity Q_l' to be shipped within the same period as when the load becomes available. In this special case, W=0 and the problem can be solved by the algorithm given below to compute the optimal number of the ships.

Procedure for the special case of no-wait time for the loads:

Step 0: Create an ordered list L ($L = \{l_{(1)}, l_{(2)}, \cdots, l_{(L)}\}$), of the loads from both ports in non-decreasing order of arrival times at the ports. Set $l = 1, N = 0, i = 0$;

Step 1: Pick the first load $l_{(1)}$ from set L, that needs to depart at time t=avail($l_{(1)}$) from port($l_{(1)}$).

Step 2: Schedule a ship $i = i + 1$ to leave from that port($l_{(1)}$) at time t and construct the regular schedule for ship *i* departing port($l_{(l)}$) at t=avail($l_{(1)}$) based on TR and TU. The ship will visit a number of ports and return to port *p* at time t (either in 24 hrs or 48 hrs depending on if (24/ (travel + load/unload time)) is even or odd). Set *N* = *N*+*1*; Choose the ship *i* so, it's capacity is as close to the maximum load on the ports visited by *i*.

Step 3: Remove all the loads carried by ship *i* on its circuit from the set L.

Step 4: If L is empty **stop.** Else re-order the list L.

Step 5: If not go to step 1. n = n+1;

The above procedure returns the optimal number of ships. Consider the example shown in Figure 7 where TR = 4, LU = 2, C_i=1000, W=0 and the load arrivals as shown. We see that 11 vessels are required if loads are required to shipped in the same time period where they arrive.

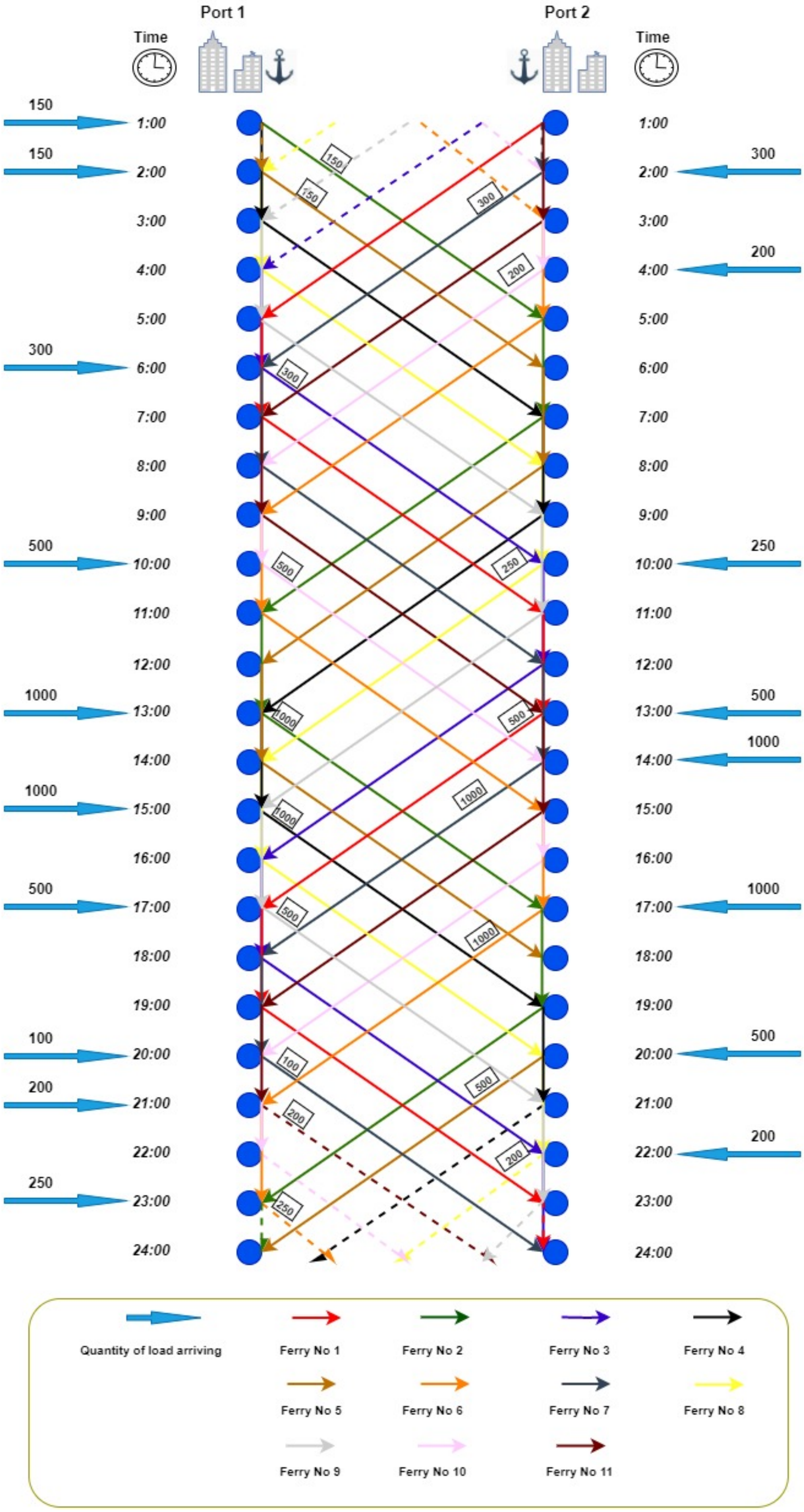

Figure 7 Solution with TR=4, LU=2 and W=0

4. Discussions and Conclusion

Finding new circular short sea services with countries on the West coast of Europe is a current and crucial issue for England. The supply chains between UK and Europe is tightly integrated with a constant flow of driver accompanied or unaccompanied trailers and containers moving between the English Channel and the North Sea on tight schedules. About 72%, of UK's sea trade in 2017 was moved by short sea shipping. As can be seen in Figure 8, when it comes to unitised freight (containers and trailers), 73.1% was moved through the ports in the South of England (e.g. Dover, Felixstowe, London, Southampton, etc), with 26.9% going through the Northern ports such as Hull, Immingham and Liverpool among others.

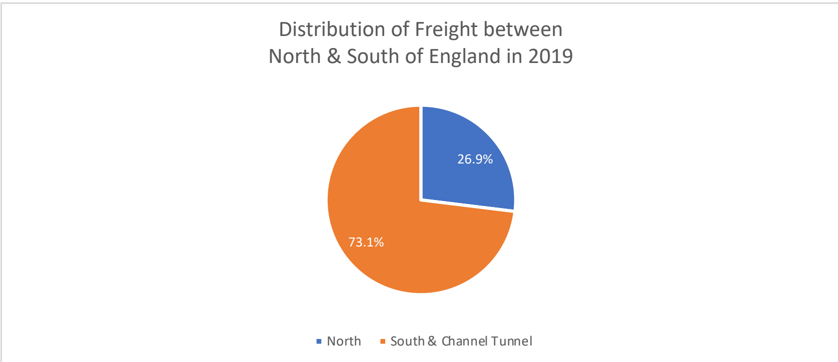

Figure 8: Percentage of unitised freight going through the Northern and Southern ports in UK in 2019

This percentage has remained relatively stable over the last five years as can be seen from Figure 9. But social distancing due to COVID -19 and the changes in custom regulations due to Brexit may have reduced the handling capacity of some ports. For example, the number of services on the Dover-Calais route has been reduced because of the ban of passenger travel due to COVID-19.

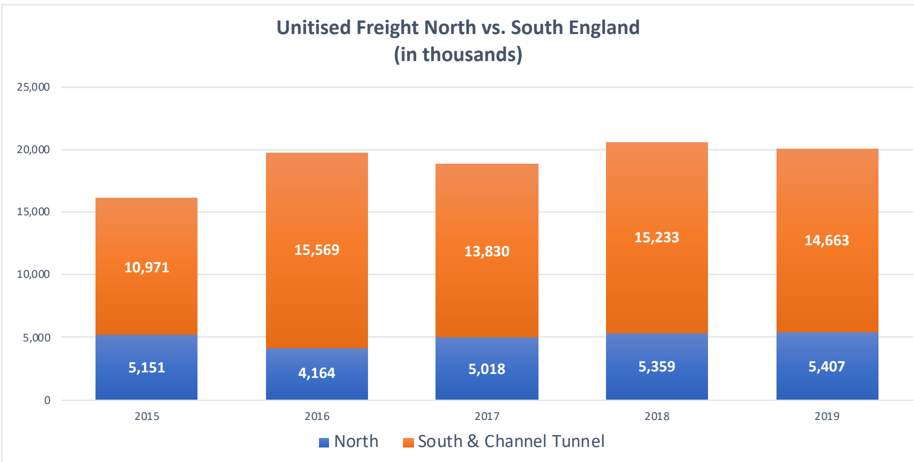

Figure 9: Evolution of unitised freight showing that the number of units going through the Northern and Southern ports remained nearly the same since 2015

Shipping lines are looking to add new services to other ports to boost capacity, service level and create more resilience in the freight network. Ports further north are readying themselves to receive these new vessels. For instance, the port of Liverpool has invested more than £400m to create a new deep-water container terminal[3], the ports of Hull and Immingham have invested more than £15m[4] and £33m[5], respectively, in gantry cranes, yard and other equipment for increasing the handling capacity of containers. Furthermore, in January 2021, a £47m[6] investment was announced for three new border control posts at the Humber ports. These shipping lines will have to develop new schedules based on the load pattern at the ports and the algorithms developed in this paper can help in optimising the number of vessels considering waiting times.

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Appendix

Claim 1: If $\frac{T}{TR+LU}$ is equal to an even number, then the number of periods required to achieve circularity is T.

Proof:

First observe that the ship will return to the first port it departs after an even number of trips. To achieve circularity, it needs to return to the same port in the same time period after T periods. Suppose there is a ferry that starts at time t and after n trips arrives at the same port.

In this case the number of trips n done by the ship is:

$$n = \frac{T}{TR + LU}$$

The proof for this claim is done through contradiction.

Let's assume that the ferry starts a port in period t and after T periods it has not completed its cycle, meaning it is not at the same time-space position as it started. So, if after T periods the time space position is t', then, $t' \neq t$.

From claim 1 we know that if n is an even number then t' is as follow:

$$t' = \left(t + n \times (TR + LU)\right) - T$$

By replacing n in above equation, we have:

$t' = \left(t + \frac{T}{Tr+Lt} \times (TR + LU)\right) - T$, and thus, $t' = (t + T) - T$

Which results in $t' = t$.

Claim 2: If $\frac{T}{TR+LU}$ is equal to an odd number, then the number of periods required to achieve circularity is 2T.

Proof:

Suppose there is a ferry that starts at time t and after m trips arrives at the same port.

In this case we can calculate m, the number of trips, as follow:

$$m = \frac{2T}{TR + LU}$$

Proceeding as in claim 1, let's assume that the ferry starts at t but returns to the same port at time t' where $t' \neq t$.

By construction we know that if m is equal to an odd number then t' is as follow:

$$t' = \big(t + m \times (TR + LU)\big) - 2T$$

By replacing m in above equation, we have:

$t' = \left(t + \frac{2T}{TR+LU} \times (TR + LU)\right) - 2T$, and thus, $t' = (t + 2T) - 2T$

Which results in $t' = t$.