Classification of book spreads in PG(5, 2)

T.P. McDonough[1], R. Shaw[2] and S. Topalova[3]

[1]Institute of Mathematics and Physics, Aberystwyth University,
Aberystwyth, SY23 3BZ, Wales tpd@aber.ac.uk
[2]Centre for Mathematics, University of Hull,
Hull, HU6 7RX, UK r.shaw@hull.ac.uk
[3]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
P.O. Box 323, 5000 Veliko Tarnovo, Bulgaria svetlana@math.bas.bg

Abstract

We classify all line spreads $\mathcal{S}_{21}$ in $\mathrm{PG}(5,2)$ of a special kind, namely those which are *book spreads*. We show that up to isomorphism there are precisely nine different kinds of book spreads and describe the automorphism groups which stabilize them. Most of the main results are obtained in two independent ways, namely theoretically by the first author, and by computer (without using anything of the theoretical proofs) by the other authors. We also discuss the importance of book spreads among all other spreads.

MSC2010: 51E20, 05B25, (05B05)
Key words: line spread, $\mathrm{PG}(5,2)$, *combinatorial design*
Running title: Book spreads

1 Introduction

A line spread in the projective space $\mathrm{PG}(5,q)$ consists of a set of q^4+q^2+1 lines which partition the points of the space. The task of classifying all line spreads in $\mathrm{PG}(5,q)$ is an extremely formidable one, and certainly requires computer help even for low values of q. Line spreads in $\mathrm{PG}(5,2)$ were considered in [13], where, with computer assistance, 131044 inequivalent spreads were found. Most of these spreads have very little symmetry and presumably their properties do not warrant further consideration. Indeed, see [13, Table I], as many as 128474 different kinds of line spreads in $\mathrm{PG}(5,2)$ have trivial automorphism group!

In some unpublished research in 2004 one of us, R. Shaw, considered line spreads in $\mathrm{PG}(5,2)$ of a special kind, which he termed *book spreads,* as defined in Section 1.1 below. In this 2004 work book spreads were classifyied into nine different kinds, and in the present paper we set out the main details of this classification, as summarized in the table (5.1) in Section 5.

We claim that book spreads are some of the most interesting kinds of spreads for the following reasons:

- They have rich automorphism groups because their lines partition not only the whole projective space, but also subspaces covering it. In $\mathrm{PG}(5,2)$ there are no other spreads which partition at least five 3-dimensional subspaces, book spreads account for 9 of the 26 orbits of line spreads having an automorphism group of order ≥ 36, and for 8 of the 16 orbits of line spreads having an automorphism group of order ≥ 72.

- The structure of book spreads makes them interesting for various constructions based on spreads. In particular in [13] spreads in $\mathrm{PG}(5,2)$ were used to obtain affine 2-(64,16,5) designs by Rahilly's construction [15]. The aim was to find 2-(64,16,5) designs with the smallest possible 2-rank and thus look for new counter examples to Hamada's conjecture [6] (see also [7], [14]). Only two minimal rank designs were found [13, Table III] and they were both constructed from book spreads, namely from those corresponding to $\mathcal{P}_0$ and $\mathcal{P}_3$ in Table 5.1 below. Further the orbits of book spreads might be used for the construction and study of parallelisms of book spreads.

- For higher parameters, for which computer classification is not possible, only spreads with certain additional properties can feasibly be considered, and thus bookspreads are a suitable type of spreads to study. In particular we expect that the $\mathrm{PG}(5,2)$ results in the present paper will be of use in some future work where we intend to investigate certain kinds of book spreads in $\mathrm{PG}(7,2)$, see Remark 1.1(i) below.

Spreads in projective spaces have been widely studied in the last several decades and quite many constructions of spreads have been found [12]. Classification results are known for spreads in $\mathrm{PG}(3,q)$ with certain automorphisms [9], [10], [11], for maximal partial spreads in $\mathrm{PG}(3,2)$ [17], $\mathrm{PG}(3,3)$[17], $\mathrm{PG}(3,4)$ [17], [18], and $\mathrm{PG}(4,2)$ [5], for spreads in $\mathrm{PG}(5,2)$ [13], and for maximal partial spreads of size 45 in $\mathrm{PG}(3,7)$ [2].

While working over $\mathrm{GF}(2)$ *we will identify the nonzero elements of a vector space* $V(n+1,2) = V_{n+1}$ *with the points of the associated projective space* $\mathbb{P}V_{n+1} = \mathrm{PG}(n,2)$. Consequently we identify $\mathrm{GL}(V_{n+1}) = \mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. We use $\langle u, v, \ldots \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$.

1.1 Books, quatrain books and book spreads in $\mathrm{PG}(5,2)$

Given a line μ in $\mathrm{PG}(5,2) = \mathbb{P}V_6$, let $\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be a set of five solids (3-flats) σ_i in $\mathrm{PG}(5,2)$ such that each 3-flat σ_i contains μ and such that each of the 60 points in the complement $\mu^c := \mathrm{PG}(5,2) \diagdown \mu$ of μ lies in one (and only one) of the five solids of $\mathcal{B}$. We will call $\mathcal{B}$ a *book of solids with spine* μ, and we will to the elements $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ of $\mathcal{B}$ as the *pages* of the book $\mathcal{B}$. In vector space terms, with $\mu = \mathbb{P}V_2$ and $\sigma_i = \mathbb{P}V_4^{(i)}$, the five 2-dimensional quotient spaces $\bar{V}_2^{(i)} := V_4^{(i)}/V_2$ thus form a spread for the 4-dimensional quotient space $\overline{V_4} := V_6/V_2$.

If $\mathcal{S}_4^{(i)}$ is a partial spread of four lines in σ_i such that $\Sigma^{(i)} := \{\mu\} \cup \mathcal{S}_4^{(i)}$ constitutes a line spread for σ_i then let us call such a $\mathcal{S}_4^{(i)}$ a *quatrain* $Q^{(i)}$ for the page σ_i. Suppose that in each of the five pages of the book $\mathcal{B}$ we 'write', i.e. choose, a quatrain $Q^{(i)}$. Thus equipped, we refer to $\mathcal{B}$ as a *quatrain book,* or a *Qbook,* and denote it by ${}^5\mathcal{B}$. Each quatrain book ${}^5\mathcal{B}$ thus determines a line spread $\mathcal{S}_{21} = \mathcal{S}_{21}({}^5\mathcal{B})$ for $\mathrm{PG}(5,2)$, whose elements consist of the lines of the five quatrains together with the spine of $\mathcal{B}$:

$$\mathcal{S}_{21} = \{\mu\} \cup Q^{(1)} \cup Q^{(2)} \cup Q^{(3)} \cup Q^{(4)} \cup Q^{(5)}. \tag{1.1}$$

Line spreads for $\mathrm{PG}(5,2)$ of this kind will be referred to as *book spreads.*

In $\mathrm{PG}(5,2)$ there are 651 choices for the spine μ of a book $\mathcal{B}$, and there are 56 choices for the set $\mathcal{B}$ of five solids σ_i through μ. (For, see Section 1.2.2, there are 56 spreads in the projective 3-space $\mathbb{P}(V_6/V_2)$.) Consequently in $\mathrm{PG}(5,2)$ there exist $651 \times 56 = 36,456 = 2^3.3.7^2.31$ books. Now the spine μ belongs to eight distinct spreads in each σ_i, so there are eight choices of quatrain for each of the five pages of the $36,456$ books. Hence in $\mathrm{PG}(5,2)$ there exist $36,456 \times 8^5 = 1,194,590,208 = 2^{18}.3.7^2.31$ quatrain books.

Remark 1.1 *(i) In the present paper we confine our attention solely to book spreads in* $\mathrm{PG}(5,2)$. *Of course there could well be some interest in investigating book spreads in* $\mathrm{PG}(5,q)$ *for* $q > 2$*, a book here containing* q^2+1 *pages, each page a 3-flat in* $\mathrm{PG}(5,q)$. *However such an investigation would surely be much more difficult than the present one, for the following two reasons. Firstly we would have to contend with the greater number of pages. Secondly in each* $\mathrm{PG}(3,q)$ *page we would need to consider, for* $q > 2$*, more than one kind of line spread, see [8, Section 17.1]; in contrast, as explained in detail in Section 1.2.2 below, in* $\mathrm{PG}(3,2)$ *there exists just one kind of spread. Higher dimensional generalizations may also be worth considering. For example in* $\mathrm{PG}(7,q)$ *one could consider line spreads contained in books consisting of* q^2+1 *pages of 5-flats sharing a common 3-flat as spine.*

(ii) Concerning higher dimensional generalizations, these could also involve plane spreads rather than line spreads. Thus in $\mathrm{PG}(8,q)$ *one could consider plane spreads contained in books consisting of* q^3+1 *pages of 5-flats sharing a common plane as spine. Concerning this last generalization,*

again the $q = 2$ case stands out: for in each of the nine $\mathrm{PG}(5,2)$ *pages of the* $\mathrm{PG}(8,2)$ *book we need to choose a spread of nine planes (one plane being the spine), and in* $\mathrm{PG}(5,2)$ there exists only one kind of spread of nine planes***:*** *see [16, Theorem 4.1].*

1.2 Useful background material

1.2.1 Normal spreads in *(i)* $\mathrm{PG}(3,2)$ and in *(ii)* $\mathrm{PG}(5,2)$

(i) In $\mathrm{GL}(V_4) \cong \mathrm{GL}(4,2)$ the elements belonging to class 3A, see [1, page [22]], are of particular relevance to our present concerns. For if $W \in \mathrm{GL}(4,2)$ is in class 3A then it satisfies $W^2 + W + I = 0$, acts fixed-point-free upon $\mathrm{PG}(3,2) = \mathbb{P}V_4$, and generates a Z_3-subgroup $\mathcal{Z} := \langle W \rangle$ of $\mathrm{GL}(4,2)$ whose orbits in $\mathrm{PG}(3,2)$ form a spread $\mathcal{S}_5$ of five lines; see for example [4, Table 3]. By interpreting $\mathfrak{A} := \{0, I, W, W^2\}$ as the Galois field $\mathrm{GF}(4) = \{0, 1, \omega, \omega^2\}$ we may view $V_4 = V(4,2)$ as $V(2,4)$, and the spread $\mathcal{S}_5$ is then seen as a $\mathrm{GF}(2)$ manifestation of the 5-point projective line $\mathrm{PG}(1,4) = \mathbb{P}(V(2,4))$. So, if $\mathcal{G}(\mathcal{S}_5) < \mathrm{GL}(4,2)$ denotes the stabilizer of the spread $\mathcal{S}_5$, then

$$\mathcal{G}(\mathcal{S}_5) \cong \Gamma\,\mathrm{L}(2,4) \cong (\mathrm{Alt}(5) \times Z_3).Z_2, \qquad \text{and so } |\mathcal{G}(\mathcal{S}_5)| = 360. \tag{1.2}$$

The only elements of $\mathrm{GL}(4,2)$ which fix every line of $\mathcal{S}_5$ are I, W and W^2, and so we will refer to the subgroup $\mathcal{Z}(\mathcal{S}_5) := \langle W \rangle$ as the *distinguished* Z_3-*subgroup* of $\mathcal{G}(\mathcal{S}_5)$. Note that those elements of $\mathcal{G}(\mathcal{S}_5)$ which centralize $\langle W \rangle$ form a subgroup $\mathcal{H}(\mathcal{S}_5)$ of index 2 in $\mathcal{G}(\mathcal{S}_5)$:

$$\mathcal{H}(\mathcal{S}_5) \cong \mathrm{GL}(2,4) \cong \mathrm{Alt}(5) \times Z_3, \qquad \text{and so } |\mathcal{H}(\mathcal{S}_5)| = 180. \tag{1.3}$$

Further $\mathcal{H}(\mathcal{S}_5)$ contains a normal subgroup $\mathcal{H}_1(\mathcal{S}_5)$ of order 60:

$$\mathcal{H}_1(\mathcal{S}_5) \cong \mathrm{SL}(2,4) \cong \mathrm{Alt}(5). \tag{1.4}$$

(ii) If an element $W \in \mathrm{GL}(V_6) \cong \mathrm{GL}(6,2)$ belongs to class 3A, then, see [4, Table 5a], it satisfies $W^2 + W + I = 0$, acts fixed-point-free upon $\mathrm{PG}(5,2) = \mathbb{P}V_6$ and generates a Z_3-subgroup $\mathcal{Z} := \langle W \rangle$ of $\mathrm{GL}(6,2)$ whose orbits in $\mathrm{PG}(5,2)$ form a spread $\mathcal{S}_{21}$ of twenty-one lines. By interpreting $\mathfrak{A} := \{0, I, W, W^2\}$ as the Galois field $\mathrm{GF}(4) = \{0, 1, \omega, \omega^2\}$ we may view $V_6 = V(6,2)$ as $V(3,4)$, and the spread $\mathcal{S}_{21}$ is then seen as a $\mathrm{GF}(2)$ manifestation of the 21-point Desarguesian plane $\mathrm{PG}(2,4) = \mathbb{P}(V(3,4))$. Consequently, if $\mathcal{G}(\mathcal{S}_{21}) < \mathrm{GL}(6,2)$ denotes the stabilizer of the spread $\mathcal{S}_{21}$, then

$$\mathcal{G}(\mathcal{S}_{21}) \cong \Gamma\,\mathrm{L}(3,4) \cong \mathrm{GL}(3,4).2, \quad \text{and so } |\mathcal{G}(\mathcal{S}_{21})| = 362,880 = 2^7.3^4.5.7. \tag{1.5}$$

We will refer to the subgroup $\mathcal{Z}(\mathcal{S}_{21}) := \langle W \rangle$ as the *distinguished* Z_3-*subgroup* of $\mathcal{G}(\mathcal{S}_{21})$.

The *normal, or Desarguesian,* spreads $\mathcal{S}_{21}$ in PG(5,2) which we have just constructed are thus in bijective correspondence with those Z_3-subgroups of GL(6,2) which are generated by an element $W \in \mathrm{GL}(6,2)$ of class 3A. Since, see [4, Table 5], $|\text{class 3A}| = 111,104$ it follows that the number N of normal line spreads $\mathcal{S}_{21}$ in PG(5,2) is $\frac{1}{2} \times 111,104 = 55,552$. As a check, we have

$$N = \frac{|\,\mathrm{GL}(6,2)|}{|\mathcal{G}(\mathcal{S}_{21})|} = \frac{2^{15}.3^4.5.7^2.31}{2^7.3^4.5.7} = 2^8.7.31 = 55,552. \checkmark \tag{1.6}$$

Remark 1.2 *Through a given point* m *of* PG(2,4) *there pass five lines, each line containing four further points. In* GF(2) *language the point* m *is the spine* μ *of a book* $\mathcal{B}$ *whose five* PG(3,2) *pages arise from the five* PG(1,4) *lines. Moreover the four points other than* m *on a* PG(1,4) *line equip each* PG(3,2) *page with a quatrain, and so we have a quatrain book* ${}^5\mathcal{B}$*. Consequently* a normal spread in PG(5,2) is an example of a book spread. *Observe that such a book spread* $\mathcal{S}_{21}$ *has a highly unusual feature, namely that it may be viewed as a quatrain book* ${}^5\mathcal{B}$ *in 21 different ways, since any of its lines may serve as the spine. Consequently the order of the stabilizer* $\mathcal{G}({}^5\mathcal{B})$ *of such a quatrain book is*

$$|\mathcal{G}({}^5\mathcal{B})| = |\mathcal{G}(\mathcal{S}_{21})|/21 = 2^7.3^3.5 = 17,280. \tag{1.7}$$

Of course these normal spreads account for just one of the 131044 different kinds of PG(5,2) line spreads in the classification [13]. Incidentally the order 362880 of their stabilizer group dwarfs the size, namely 5760 (see [13, Table I]) of the second largest stabilizer group.

In contrast, the Desarguesian spreads $\mathcal{S}_5$ in PG(3,2), described in *(i)* above, account for all the PG(3,2) line spreads. This fact is well known, but we will spell it out in more detail in the next section.

1.2.2 Our standard spread Σ in PG(3,2)

Given any two skew lines $\kappa_1 = \{a_1, b_1, c_1\}$ and $\kappa_2 = \{a_2, b_2, c_2\}$ in $\mathrm{PG}(3,2) = \mathbb{P}\mathbb{V}_4$ we may define two elements $W, W' \in \mathrm{GL}(V_4) \cong \mathrm{GL}(4,2)$ by requiring them to cyclically permute the points of κ_1 and κ_2 in the manner:

$$W : (a_1 b_1 c_1)(a_2 b_2 c_2), \qquad W' : (a_1 b_1 c_1)(a_2 c_2 b_2). \tag{1.8}$$

Both W and W' are of class 3A, see [1, page [22]], [4, Table 3], and the two Z_3-subgroups $\mathcal{Z} := \langle W \rangle$ and $\mathcal{Z}' := \langle W' \rangle$ of $\mathrm{GL}(V_4)$ are the only Z_3-subgroups having a class 3A generator which fix both of the skew lines κ_1 and κ_2. The $\mathcal{Z}$ orbits in PG(3,2) form a spread $\mathcal{S}_5 := \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ and the $\mathcal{Z}'$ orbits form a spread $\mathcal{S}'_5 := \{\kappa_1, \kappa_2, \kappa'_3, \kappa'_4, \kappa'_5\}$. Here $\{\kappa_3, \kappa_4, \kappa_5\}$ and $\{\kappa'_3, \kappa'_4, \kappa'_5\}$ are a pair of opposite reguli which cover the nine points in PG(3,2) external to $\kappa_1 \cup \kappa_2$. No other lines can be formed from these nine points, and so it follows that every spread in $PG(3,2)$ is of the normal kind constructed in Section 1.2.1(i). Now in $PG(3,2)$ there exist 35 lines and $(35 \times 16)/2 = 280$ skew pairs of lines. Since each skew pair $\{\kappa_1, \kappa_2\}$ lies inside 2 spreads $\mathcal{S}_5$, $\mathcal{S}'_5$ and since each $\mathcal{S}_5$ contains 10 skew pairs, the total number of line spreads in $PG(3,2)$ is $(280 \times 2)/10 = 56$. (As a check: $|\operatorname{GL}(4,2)|/|\mathcal{G}(\Sigma)| = 20160/360 = 56$.)

For our later convenience it will prove useful to have to hand a standard example of a line spread in $PG(3,2)$. To this end, choose a subgroup $\langle A\rangle \cong Z_{15}$ of $\operatorname{GL}(4,2)$ generated by a Singer element $A \in \operatorname{GL}(4,2)$, see for example [4, Table 3]. Without loss of generality we may suppose that A satisfies $A^4 = A + I$. If we set $B := A^6$ and $W := A^{10}$ we then have

$$BW = A = WB, \qquad B^5 = I,\ W^3 = I,\ A^{15} = I. \tag{1.9}$$

Let $Z^a = \{a_1, a_2, a_3, a_4, a_5\}$, where $a_{i+1} = B^i a_1$, be any one of the three orbits of $\langle B\rangle \cong Z_5$ upon $PG(3,2) = \mathbb{P}(V_4)$. Upon defining $b_i = W a_i$ and $c_i = W^2 a_i$ the other two Z_5-orbits are then $Z^b = \{b_1, b_2, b_3, b_4, b_5\}$ and $Z^c = \{c_1, c_2, c_3, c_4, c_5\}$, where $b_{i+1} = B^i b_1$ and $c_{i+1} = B^i c_1$. Since $W^2 + W + I = 0$, observe that

$$\Sigma := \{\kappa_1,\ ...,\kappa_5\}, \qquad \text{where } \kappa_i = \{a_i, b_i, c_i\}, \tag{1.10}$$

is a spread for $PG(3,2)$ which has $\langle W\rangle$ as its distinguished Z_3-subgroup $\mathcal{Z}(\Sigma)$. Note that B, W and $A = BW$ have the effect

$$\begin{aligned}
B:&\quad a_i \longmapsto a_{i+1}, \quad b_i \longmapsto b_{i+1}, \quad c_i \longmapsto c_{i+1},\\
W:&\quad a_i \longmapsto b_i \longmapsto c_i \longmapsto a_i,\\
A:&\quad ... \longmapsto a_i \longmapsto b_{i+1} \longmapsto c_{i+2} \longmapsto a_{i+3} \longmapsto b_{i+4} \longmapsto c_i \longmapsto ... \quad ,
\end{aligned} \tag{1.11}$$

where the index i labelling the five elements of a Z_5-orbit runs through the five values $1, 2, 3, 4, 5 \pmod 5$. Using $A^4 = A + I$ we see that under the action of the subgroup $\langle B\rangle \cong Z_5$ the 35 lines of $PG(3,2)$ form seven Z_5-orbits, one of which is the spread Σ in (1.10), and the other six are given by the following six kinds of linear relations:

$$\begin{aligned}
&a_i + a_{i+1} = b_{i+3}; \quad b_i + b_{i+1} = c_{i+3}; \quad c_i + c_{i+1} = a_{i+3};\\
&a_i + c_{i+1} = a_{i+2}; \quad b_i + a_{i+1} = b_{i+2}; \quad c_i + b_{i+1} = c_{i+2}.
\end{aligned} \tag{1.12}$$

We will adopt Σ as in (1.10) as our *standard spread* in $PG(3,2)$. Its stabilizer $\mathcal{G}(\Sigma) \cong \Gamma\operatorname{L}(2,4) \cong (\operatorname{Alt}(5) \times Z_3).Z_2$, see (1.2), has order 360 and contains the normal subgroup $\mathcal{H}(\Sigma) \cong \operatorname{GL}(2,4) \cong \operatorname{Alt}(5) \times Z_3$, see (1.3), of order 180, which consists of those elements of $\mathcal{G}(\Sigma)$ which centralize $\langle W\rangle$; further the group $\mathcal{H}(\Sigma)$ contains the normal subgroup $\mathcal{H}_1(\Sigma) \cong \operatorname{SL}(2,4) \cong \operatorname{Alt}(5)$, see (1.4), of order 60. Since W fixes each line of Σ, the elements of $\mathcal{G}(\Sigma)$ are in a $3:1$ correspondence with the $5! = 120$ permutations of the five lines $\Sigma = \{\kappa_1, ..., \kappa_5\}$, the three elements of any coset of $\mathcal{Z}(\Sigma)$ in $\mathcal{G}(\Sigma)$ effecting the same permutation of Σ. And the elements of $\mathcal{H}_1(\Sigma)$ are in bijective correspondence with the 60 even permutations of Σ.

We now give examples of particular elements of $\mathcal{G}(\Sigma)$ which enter into some of of our later concerns, and especially those of Section 5.2.

(i) By use of the relations (1.12) we see that there exists an involution $L_{(23)(45)} \in \mathrm{GL}(4,2)$ which keeps the line κ_1 pointwise fixed and effects the following permutations of the points of the other four lines:

$$L_{(23)(45)} : (a_2b_3)(b_2c_3)(c_2a_3)(a_4c_5)(b_4a_5)(c_4b_5). \tag{1.13}$$

Each of the three elements $L_{(23)(45)}, WL_{(23)(45)}$ and $W^2L_{(23)(45)}$ thus effects the *even* permutation $(\kappa_1)(\kappa_2\kappa_3)(\kappa_4\kappa_5)$ of Σ. So all three elements belong to $\mathcal{H}(\Sigma) \cong \mathrm{Alt}(5) \times Z_3$; however, since $WL_{(23)(45)}$ and $W^2L_{(23)(45)}$ are of order 6, only $L_{(23)(45)}$ is an element of $\mathcal{H}_1(\Sigma) \cong \mathrm{Alt}(5)$. It follows that the involution $L_{(12)(45)} : B^2L_{(23)(45)}B^{-2} \in \mathrm{GL}(4,2)$ given by

$$L_{(12)(45)} : (a_1c_2)(b_1a_2)(c_1b_2)(a_4b_5)(b_4c_5)(c_4a_5) \tag{1.14}$$

is also an element of $\mathcal{H}_1(\Sigma)$, namely that which effects the permutation $(\kappa_1\kappa_2)(\kappa_3)(\kappa_4\kappa_5)$ of the lines of the spread Σ.

(ii) The product $T_{(132)} := L_{(23)(45)}L_{(12)(45)}$, with effect

$$T_{(132)} : (a_1a_3c_2)(b_1b_3a_2)(c_1c_3b_2)(a_4c_4b_4)(a_5b_5c_5), \tag{1.15}$$

is in consequence that element of $\mathcal{H}_1(\Sigma)$ which effects the permutation $(\kappa_1\kappa_3\kappa_2)(\kappa_4)(\kappa_5)$ of Σ.

(iii) The involution $N_{(12)} \in \mathrm{GL}(4,2)$ which effects the permutations

$$N_{(12)} : (a_1c_2)(b_1b_2)(c_1a_2)(a_3)(b_3c_3)(c_4)(a_4b_4)(a_5)(b_5c_5). \tag{1.16}$$

is an element of $\mathcal{G}(\Sigma)$ which effects the *odd* permutation $(\kappa_1\kappa_2)(\kappa_3)(\kappa_4)(\kappa_5)$ of the lines of the spread Σ. It follows that $N_{(12)}$, $WN_{(12)}$ and $W^2N_{(12)}$ are elements of $\mathcal{G}(\Sigma)\backslash\mathcal{H}(\Sigma)$. Similar assertions apply in the case of the involution $N_{(13)} := T_{(132)}N_{(12)}T_{(132)}^{-1} (= N_{(12)}T_{(132)}) \in \mathcal{G}(\Sigma)$ with effect

$$N_{(13)} : (a_1a_3)(b_1c_3)(c_1b_3)(a_2b_2)(c_2)(a_4c_4)(b_4)(a_5c_5)(b_5). \tag{1.17}$$

By taking conjugates of $N_{(12)}$ and $N_{(13)}$ by powers of B we obtain altogether ten involutions $N_{(ij)}$, $1 \le i < j \le 5$, such as the involution $N_{(45)} := B^{-2}N_{(12)}B^2$, which effects the permutations

$$N_{(45)} : (a_1)(b_1c_1)(a_2b_2)(c_2)(a_3)(b_3c_3)(a_4c_5)(b_4b_5)(c_4a_5). \tag{1.18}$$

These ten involutions generate a subgroup $\mathcal{G}_1(\Sigma)$ of $\mathcal{G}(\Sigma)$ isomorphic to $\mathrm{Sym}(5)$:

$$\mathcal{G}_1(\Sigma) := \langle N_{(ij)}, 1 \le i < j \le 5\rangle \cong \mathrm{Sym}(5). \tag{1.19}$$

Remark 1.3 *The elements of* $\mathcal{G}_1(\Sigma)$ *are thus in bijective correspondence with the* 5! *permutations of the five lines* $\Sigma = \{\kappa_1, ..., \kappa_5\}$. *It should be further noted that* $\mathcal{G}(\Sigma)$ *contains two other* Sym(5) *subgroups, namely* $W\mathcal{G}_1(\Sigma)W^{-1}$ *and* $W^2\mathcal{G}_1(\Sigma)W^{-2}$, *all three* Sym(5) *subgroups being of index 3 in* $\mathcal{G}(\Sigma)$ *and sharing the common normal subgroup* $\mathcal{H}_1(\Sigma) \cong \mathrm{Alt}(5)$. *In particular, upon defining ten further involutions* $K_{(ij)}$ *by*

$$K_{(ij)} := WN_{(ij)}, \quad 1 \le i < j \le 5, \tag{1.20}$$

we see that $W^{-1}\mathcal{G}_1(\Sigma)W = \langle K_{(ij)}, 1 \le i < j \le 5\rangle \cong \mathrm{Sym}(5)$.

2 The book $\mathcal{B}$

We consider a book $\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ in $\mathrm{PG}(5,2) = \mathbb{P}V_6$ with spine $\mu = \mathbb{P}V_2 = \{u, v, w\}$, $u + v + w = 0$, and pages $\sigma_i = \mathbb{P}V_4^{(i)}$. Let us choose a direct sum decomposition $V_6 = V_4 \oplus V_2$; so projectively we choose a solid $\sigma = \mathbb{P}V_4$ which is skew to μ. Observe that the book $\mathcal{B}$ equips σ with a spread

$$\Sigma := \{\kappa_1, ..., \kappa_5\}, \qquad \text{where } \kappa_i = \sigma \cap \sigma_i. \tag{2.1}$$

Expressing Σ in our standard form, with $\kappa_i = \{a_i, b_i, c_i\}$ as in section 1.2.2, then we have our *standard book* $\mathcal{B}$:

$$\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}, \qquad \text{where } \sigma_i = \langle a_i, b_i, u, v\rangle. \tag{2.2}$$

Let ϖ denote the canonical projection $V_6 \to \overline{V}_4 := V_6/V_2$, which maps $x \in V_6$ to $\varpi x = \overline{x} := x + V_2 \in V_6/V_2$. Then the spread Σ for σ gives rise to a spread $\overline{\Sigma}$ in $\mathbb{P}\overline{V}_4$, namely

$$\overline{\Sigma} = \{\overline{\kappa}_1, \overline{\kappa}_2, \overline{\kappa}_3, \overline{\kappa}_4, \overline{\kappa}_5\}, \qquad \text{where } \overline{\kappa}_i = \{\overline{a_i}, \overline{b_i}, \overline{c_i}\} := \{\varpi a_i, \varpi b_i, \varpi c_i\}. \tag{2.3}$$

The distinguished Z_3-subgroup $\mathcal{Z}(\overline{\Sigma}) < \mathrm{GL}(\overline{V}_4)$ of the spread $\overline{\Sigma}$, see section 1.2.1(i), is thus

$$\mathcal{Z}(\overline{\Sigma}) = \langle \overline{W}\rangle, \qquad \text{where } \overline{W} \text{ is that element of } \mathrm{GL}(\overline{V}_4) \text{ with effect:}$$
$$\overline{W}: \quad \overline{a_i} \mapsto \overline{b_i} \mapsto \overline{c_i} (\mapsto \overline{a_i}), \quad i = 1, 2, 3, 4, 5. \tag{2.4}$$

2.1 Quatrains for our standard book $\mathcal{B}$ in PG(5, 2)

Since each line in PG(3, 2) belongs to eight distinct spreads, there is a choice of eight possible quatrains $Q_1^{(i)}, Q_2^{(i)}, \dots, Q_8^{(i)}$ for each page σ_i of our book $\mathcal{B}$. In displaying these quatrains it helps to adopt abbreviations of the kind

$$\begin{pmatrix} u & w & v \\ 0 & u & u \\ w & 0 & w \\ v & v & 0 \end{pmatrix}^{(i)} := \begin{pmatrix} a_i{+}u & b_i{+}w & c_i{+}v \\ a_i & b_i{+}u & c_i{+}u \\ a_i{+}w & b_i & c_i{+}w \\ a_i{+}v & b_i{+}v & c_i \end{pmatrix}. \tag{2.5}$$

For the page $\sigma_i = \langle a_i, b_i, u, v\rangle$, four of the quatrains are $Q_1^{(i)}, Q_2^{(i)}, Q_3^{(i)}, Q_4^{(i)}$, as given by the respective four arrays

$$\begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ v & w & u \\ w & u & v \end{pmatrix}^{(i)}, \begin{pmatrix} u & w & v \\ 0 & u & u \\ w & 0 & w \\ v & v & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} w & v & u \\ 0 & w & w \\ v & 0 & v \\ u & u & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} v & u & w \\ 0 & v & v \\ u & 0 & u \\ w & w & 0 \end{pmatrix}^{(i)}, \quad (2.6)$$

the four lines of a quatrain being given by adding (a_i, b_i, c_i) to each row of the array. The remaining four quatrains $Q_5^{(i)}, Q_6^{(i)}, Q_7^{(i)}, Q_8^{(i)}$ are then those given by the respective four arrays

$$\begin{pmatrix} 0 & 0 & 0 \\ u & w & v \\ w & v & u \\ v & u & w \end{pmatrix}^{(i)}, \begin{pmatrix} u & v & w \\ 0 & u & u \\ v & 0 & v \\ w & w & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} v & w & u \\ 0 & v & v \\ w & 0 & w \\ u & u & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} w & u & v \\ 0 & w & w \\ u & 0 & u \\ v & v & 0 \end{pmatrix}^{(i)}. \quad (2.7)$$

If $J_{(vw)} \in \mathrm{GL}(6,2)$ denotes that involution which fixes each point a_i, b_i, c_i of $\sigma = \mathbb{P}V_4$ and which interchanges the points $v, w \in \mu$, then observe that

$$J_{(vw)} \text{ effects the interchanges } (Q_1^{(i)} Q_5^{(i)})(Q_2^{(i)} Q_6^{(i)})(Q_3^{(i)} Q_7^{(i)})(Q_4^{(i)} Q_8^{(i)}). \quad (2.8)$$

2.2 Harmony considerations and quatrain books

In the display in (2.6), (2.7) of the eight quatrains $\mathcal{Q}^{(i)} := \{Q_r^{(i)}\}_{1\le r\le 8}$ for the page σ_i, we have ordered the points of each row so that a_i, b_i, c_i appear in the respective columns 1,2,3. Thus if (x_i, y_i, z_i) is any row of the quatrain $Q_r^{(i)}$, then, see before Equation (2.3), we have $\overline{x_i} = \overline{a_i}$, $\overline{y_i} = \overline{b_i}$, $\overline{z_i} = \overline{c_i}$. Consequently the distinguished Z_3-subgroup $\mathcal{Z}(\Sigma_r^{(i)})$ (see section 1.2.1(i)) for the spread $\Sigma_r^{(i)} := \{\mu\} \cup Q_r^{(i)}$ is generated by that element $W_r^{(i)}$ of $\mathrm{GL}(V_4^{(i)})$ which effects the cyclic permutation $(x_i y_i z_i)$ in each row (x_i, y_i, z_i).

Lemma 2.1 *If $r \in \{1,2,3,4\}$ then, for each $i \in \{1,2,3,4,5\}$, $W_r^{(i)}$ effects the permutation (uvw) in the spine μ, while if $r \in \{5,6,7,8\}$ then $W_r^{(i)}$ effects the permutation (uwv) in the spine μ.*

Proof. This is quickly verified. For example, consider the quatrain $Q_6^{(i)}$ in (2.7): then that element of $\mathrm{GL}(V_4^{(i)})$ which effects $a_i{+}u \mapsto b_i{+}v \mapsto c_i{+}w$, and $a_i \mapsto b_i{+}u \mapsto c_i{+}u$ is seen to effect $u \mapsto w \mapsto v$, $a_i{+}v \mapsto b_i \mapsto c_i{+}v$ and $a_i{+}w \mapsto b_i{+}w \mapsto c_i$. ∎

Consequently *the set of forty quatrains $\mathcal{Q} := \{Q_r^{(i)}\}_{1\le i\le 5,\ 1\le r\le 8} = \cup_{i=1}^5 \mathcal{Q}^{(i)}$ for the book $\mathcal{B}$ has the natural $20+20$ splitting*

$$\mathcal{Q} = \mathcal{Q}_+ \cup \mathcal{Q}_- = \{\mathcal{Q}_+^{(i)}\}_{i\in\{1,2,3,4,5\}} \cup \{\mathcal{Q}_-^{(i)}\}_{i\in\{1,2,3,4,5\}}, \quad (2.9)$$

where $\mathcal{Q}_+^{(i)} := \{Q_r^{(i)}\}_{r\in\{1,2,3,4\}}$ are the four quatrains (2.6) and $\mathcal{Q}_-^{(i)} := \{Q_r^{(i)}\}_{r\in\{5,6,7,8\}}$ are the four quatrains (2.7). We will say that the twenty quatrains of the set $\mathcal{Q}_+ := \{Q_r^{(i)}\}_{1\le i\le 5,\ 1\le r\le 4}$ are in *harmony* with each other, as are the twenty quatrains of the set $\mathcal{Q}_- := \{Q_r^{(i)}\}_{1\le i\le 5,\ 5\le r\le 8}$,

Remark 2.2 *Let $\mathcal{L}_{16}^{(i)}$ denote the set of sixteen lines of the page σ_i which are skew to the spine μ. In the $4+4$ splitting $\mathcal{Q}^{(i)} = \mathcal{Q}_+^{(i)} \cup \mathcal{Q}_-^{(i)}$, see (2.6), (2.7), of the eight quatrains for the page σ_i, each line $\lambda \in \mathcal{L}_{16}^{(i)}$ appears in precisely two quatrains, one a member of $\mathcal{Q}_+^{(i)}$ and one a member of $\mathcal{Q}_-^{(i)}$. This ties in with the fact, see Section 1.2.2, that the skew pair of lines $\{\lambda, \mu\}$ lies in two spreads in $\mathrm{PG}(3,2)$ whose distinguished Z_3-subgroups act differently upon the two lines.*

By using harmony considerations *we may classify full Qbooks ${}^5\mathcal{B}$ into three broad harmony types:*
$\mathcal{T}(5,0)$: all five quatrains of ${}^5\mathcal{B}$ are in harmony;
$\mathcal{T}(4,1)$: precisely four quatrains of ${}^5\mathcal{B}$ are in harmony;
$\mathcal{T}(3,2)$: the quatrains of ${}^5\mathcal{B}$ split $(3,2)$ or $(2,3)$ between the sets $\mathcal{Q}_+,\ \mathcal{Q}_-$.

Furthermore, since the involution $J_{(vw)}$ effects the interchange $\mathcal{Q}_+ \rightleftarrows \mathcal{Q}_-$, then without loss of generality we may restrict our attention to Qbooks ${}^5\mathcal{B}$ such that *at least three of its quatrains belong to $\mathcal{Q}_+$*. Moreover all permutations of the five lines $\overline{\kappa}_i$ are effected by the subgroup $\mathcal{G}_1(\overline{\Sigma}) \cong \mathrm{Sym}(5)$ of $\mathcal{G}(\overline{\Sigma}) \cong (\mathrm{Alt}(5) \times Z_3).Z_2$, see (1.19), and so we may restrict attention to those Qbooks ${}^5\mathcal{B}$ for which *the quatrains in its first three pages σ_1, σ_2 and σ_3 all belong to $\mathcal{Q}_+$.*

3 Aspects of the groups $\mathcal{G}(\mathcal{B})$, $\mathcal{G}_0(\mathcal{B})$ and $\mathcal{G}_0({}^2\mathcal{B})$

3.1 The groups $\mathcal{G}(\mathcal{B})$, $\mathcal{G}_0(\mathcal{B})$

When dealing with group theory aspects of our standard book $\mathcal{B}$ we will adopt the decomposition $V_6 = V_4 \oplus V_2$ of Section 2, with $\mu = \mathbb{P}V_2$ and with $\sigma = \mathbb{P}V_4$ equipped with our standard spread $\Sigma = \{\kappa_1, ..., \kappa_5\}$, $\kappa_i = \{a_i, b_i, c_i\}$. If $\mathcal{G}(\mathcal{B}) < \mathrm{GL}(6,2)$ denotes the stabilizer of $\mathcal{B}$ then a general element $A \in \mathcal{G}(\mathcal{B})$ has the block form

$$A = \begin{pmatrix} A_4 & 0 \\ X & A_2 \end{pmatrix}, \quad A_4 \in \mathcal{G}(\Sigma) \cong \Gamma\,\mathrm{L}(2,4),\ A_2 \in \mathrm{GL}(V_2) \cong \mathrm{Sym}(3), \quad (3.1)$$

where X is an arbitrary 2×4 block. Consequently

$$\mathcal{G}(\mathcal{B}) \cong 2^8 : (\Gamma\,\mathrm{L}(2,4) \times \mathrm{GL}(2,2)), \qquad \text{and so } |\mathcal{G}(\mathcal{B})| = 2^{12}.3^3.5 = 552,960. \quad (3.2)$$

It follows that the number of books in PG(5,2) is $|\,\mathrm{GL}(6,2)|/|\mathcal{G}(\mathcal{B})| = 36,456$, in agreement with the number found more directly in Section 1.1. Those elements A in (3.1) such that $A_4 = I_4$ and $A_2 = I_2$ form a subgroup $\mathcal{G}_{256}$ of $\mathcal{G}(\mathcal{B})$ whose non-identity elements are transvections, and so, since we work over GF(2), are involutions: $\mathcal{G}_{256} \cong (Z_2)^8$.

In terms of the direct sum decomposition $V_6 = V_4 \oplus V_2$, we now denote by B_4, W_4 the elements of $\mathcal{G}(\Sigma)$, of respective orders 5, 3, previously denoted B, W in Equations (1.9), (1.11). In particular $\langle W_4 \rangle$ is the distinguished Z_3-subgroup $\mathcal{Z}(\Sigma)$ of the spread Σ. Also we denote by W_2 the element of order 3 in $\mathrm{GL}(V_2)$ with effect

$$W_2: \quad u \mapsto v \mapsto w \ (\mapsto u). \tag{3.3}$$

Let $\mathcal{G}_0(\mathcal{B})$ denote the subgroup of $\mathcal{G}(\mathcal{B})$ which consists of those elements which fix each page of the book $\mathcal{B}$. Now if A in (3.1) fixes each page then A_4 must belong to the Z_3-subgroup $\langle W_4 \rangle = \mathcal{Z}(\Sigma)$. Consequently

$$\mathcal{G}_0(\mathcal{B}) \cong 2^8 : (Z_3 \times \mathrm{GL}(2,2)), \qquad \text{and so } |\mathcal{G}_0(\mathcal{B})| = 2^9.3^2 = 4608. \tag{3.4}$$

We now define elements $W, W^*, B \in \mathrm{GL}(6,2)$, of respective orders 3, 3, 5, by

$$W = W_4 \oplus W_2, \quad W^* = W_4 \oplus (W_2)^{-1}, \quad B = B_4 \oplus I_2. \tag{3.5}$$

Lemma 3.1 *(i) Both W and W^* are elements of $\mathcal{G}_0(\mathcal{B})$, their effects on the eight quatrains $\{Q_r^{(i)}\}_{1\le r\le 8}$ for the page σ_i being as follows:*

$$W: \quad Q_r^{(i)} \text{ is fixed for } r = 1,5,6,7,8 \quad \text{and } Q_2^{(i)} \mapsto Q_3^{(i)} \mapsto Q_4^{(i)} \mapsto Q_2^{(i)};$$
$$W^*: \quad Q_r^{(i)} \text{ is fixed for } r = 1,2,3,4,5 \quad \text{and } Q_6^{(i)} \mapsto Q_7^{(i)} \mapsto Q_8^{(i)} \mapsto Q_6^{(i)}.$$

(ii) B is an element of $\mathcal{G}(\mathcal{B})$ which effects the permutation $(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)$ of the pages of $\mathcal{B}$, and which effects the permutations $(Q_r^{(1)}Q_r^{(2)}Q_r^{(3)}Q_r^{(4)}Q_r^{(5)})$, $1 \le r \le 8$, of the 40 quatrains.

Proof. Immediate, from (2.6), (2.7). ■

3.2 Some subgroups of $\mathcal{G}_0(\mathcal{B})$

3.2.1 Some elementary abelian subgroups

Since $\{a_i, b_i, a_{i+1}, b_{i+1}, u, v\}$ is, for each $i \in \{1,2,3,4,5\} \pmod 5$, a basis for V_6, we may define involutions $J_i, J_i', J_i'' \in \mathrm{GL}(6,2)$ by requiring each of J_i, J_i', J_i'' to keep pointwise fixed the page $\sigma_i = \langle a_i, b_i, u, v\rangle$ and to further satisfy

$$J_i\!: \quad a_{i+1} \rightleftarrows a_{i+1} + u, \ b_{i+1} \rightleftarrows b_{i+1} + v, \quad \text{and hence } c_{i+1} \rightleftarrows c_{i+1} + w,$$
$$J_i'\!: \quad a_{i+1} \rightleftarrows a_{i+1} + v, \ b_{i+1} \rightleftarrows b_{i+1} + w, \quad \text{and hence } c_{i+1} \rightleftarrows c_{i+1} + u,$$
$$J_i''\!: \quad a_{i+1} \rightleftarrows a_{i+1} + w, \ b_{i+1} \rightleftarrows b_{i+1} + u, \quad \text{and hence } c_{i+1} \rightleftarrows c_{i+1} + v. \tag{3.6}$$

The involutions $J_i, J'_i, J''_i \in \mathcal{G}_0(\mathcal{B})$ since, from the relations (1.12), they also preserve the other pages $\sigma_{i-1}, \sigma_{i-2}$, and σ_{i+2}. In particular J_i effects

$$\begin{aligned}
&a_{i-1} \rightleftarrows a_{i-1} + u, \quad b_{i-1} \rightleftarrows b_{i-1} + v, \quad c_{i-1} \rightleftarrows c_{i-1} + w, \\
&a_{i-2} \rightleftarrows a_{i-2} + w, \quad b_{i-2} \rightleftarrows b_{i-2} + u, \quad c_{i-2} \rightleftarrows c_{i-2} + v, \\
&a_{i+2} \rightleftarrows a_{i+2} + w, \quad b_{i+2} \rightleftarrows b_{i+2} + u, \quad c_{i+2} \rightleftarrows c_{i+2} + v. \qquad (3.7)
\end{aligned}$$

Noting that $J''_i = J_i J'_i$, observe that we have five groups $\mathcal{A}_4^{(i)}$, each isomorphic to $(Z_2)^2$:

$$\mathcal{A}_4^{(i)} := \{I, J_i, J'_i, J''_i\} \cong (Z_2)^2, \quad i = 1, 2, 3, 4, 5. \qquad (3.8)$$

By conjugation with the involution $J_{(vw)}$, see Equation (2.8), we may define a further set of fifteen involutions K_i, K'_i, K''_i, $i \in \{1, 2, 3, 4, 5\}$, and a further five groups $\mathcal{C}_4^{(i)} := \{I, K_i, K'_i, K''_i\} \cong (Z_2)^2$:

$$\begin{aligned}
K_i &:= J_{(vw)} J_i J_{(vw)}, \ K'_i := J_{(vw)} J'_i J_{(vw)}, \ K''_i := J_{(vw)} J''_i J_{(vw)}, \\
\mathcal{C}_4^{(i)} &:= J_{(vw)} \mathcal{A}_4^{(i)} J_{(vw)} = \{I, K_i, K'_i, K''_i\} \cong (Z_2)^2, \quad i = 1, 2, 3, 4, 5. \qquad (3.9)
\end{aligned}$$

So K_i, K'_i, K''_i keep pointwise fixed the page σ_i and their action upon the other pages is given by interchanging v and w in (3.6), (3.7). Observe that the involutions J_i, J'_i, J''_i are conjugates of each other by elements of the group $\langle W^* \rangle$, and the involutions K_i, K'_i, K''_i are conjugates of each other by elements of the group $\langle W \rangle$:

$$\begin{aligned}
W^* J_i (W^*)^{-1} = J'_i, &\qquad W^* J'_i (W^*)^{-1} = J''_i, \\
W K_i W^{-1} = K'_i, &\qquad W K'_i W^{-1} = K''_i. \qquad (3.10)
\end{aligned}$$

The next lemma summarizes some relevant properties of the involutions and 2-groups just defined in Equations (3.6) - (3.9).

Lemma 3.2 *(i) The fifteen involutions J_i, J'_i, J''_i, $i \in \{1, 2, 3, 4, 5\}$, are the non-identity elements of an elementary abelian subgroup $\mathcal{A}_{16} \cong (Z_2)^4$ of $\mathcal{G}_0(\mathcal{B})$, and the fifteen involutions K_i, K'_i, K''_i, $i \in \{1, 2, 3, 4, 5\}$, are the non-identity elements of a subgroup $\mathcal{C}_{16} := \langle K_1, K'_1, K_2, K'_2 \rangle \cong (Z_2)^4$ of $\mathcal{G}_0(\mathcal{B})$. Indeed for any $i \neq j$ we have*

$$\mathcal{A}_{16} = \mathcal{A}_4^{(i)} \times \mathcal{A}_4^{(j)}, \qquad \mathcal{C}_{16} = \mathcal{C}_4^{(i)} \times \mathcal{C}_4^{(j)}. \qquad (3.11)$$

Moreover the subgroup $\mathcal{G}_{256} := \mathcal{A}_{16} \times \mathcal{C}_{16} < \mathcal{G}_0(\mathcal{B})$ consists of those 2^8 elements A in (3.1) such that $A_4 = I_4$ and $A_2 = I_2$.

(ii) Each of the quatrains $Q \in \mathcal{Q}_+ = \{Q_r^{(i)}\}_{1 \le i \le 5,\ 1 \le r \le 4}$ is stabilized by $\mathcal{A}_{16}$ and each of the quatrains $Q \in \mathcal{Q}_- = \{Q_r^{(i)}\}_{1 \le i \le 5,\ 5 \le r \le 8}$ is stabilized by $\mathcal{C}_{16}$; moreover, for each $i \in \{1, 2, 3, 4, 5\}$, $\mathcal{A}_{16}$ acts transitively on the four quatrains $\mathcal{Q}_-^{(i)}$ for the page σ_i, and $\mathcal{C}_{16}$ acts transitively on the four quatrains $\mathcal{Q}_+^{(i)}$ for the page σ_i.

(iii) The subgroups $\mathcal{A}_4^{(i)}$ and $\mathcal{C}_4^{(i)}$ leave every point of the page σ_i fixed. Moreover for $j \neq i$ the subgroup $\mathcal{A}_4^{(i)}$ acts transitively on the four quatrains $\mathcal{Q}_-^{(j)}$ for the page σ_j, and the subgroup $\mathcal{C}_4^{(i)}$ acts transitively on the four quatrains $\mathcal{Q}_+^{(j)}$ for the page σ_j.

(iv) For given $i \neq j$ the group $\mathcal{A}_{16}$ acts transitively on the sixteen pairs of quatrains $\{Q_r^{(i)}, Q_s^{(j)}\}_{5\le r,s\le 8}$, and $\mathcal{C}_{16}$ acts transitively on the sixteen pairs of quatrains $\{Q_r^{(i)}, Q_s^{(j)}\}_{1\le r,s\le 4}$.

Proof. A somewhat lengthy, but straightforward, verification. Concerning (3.11) we may verify relations such as $J_1J_2 = J_4'$. Concerning (iv), this follows from (iii) on account of the direct product structures (3.11). ∎

3.2.2 The subgroups $\mathcal{G}_{36}^{(i)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{48}^*$ and $\mathcal{G}_{144}$ of $\mathcal{G}_0(\mathcal{B})$

Since $\mathcal{A}_{16}$ is centralized by W we may define an abelian group $\mathcal{G}_{48}$ by

$$\mathcal{G}_{48} = \langle \mathcal{A}_{16}, W\rangle \cong (Z_2)^4 \times Z_3. \tag{3.12}$$

However, see Equation (3.10), W^* does not commute with the foregoing involutions, and so we may further consider two non-abelian groups, namely

$$\mathcal{G}_{48}^* = \langle \mathcal{A}_{16}, W^*\rangle \cong (Z_2)^4 : Z_3, \quad \mathcal{G}_{144} = \langle \mathcal{G}_{48}, W^*\rangle \cong ((Z_2)^4 : Z_3) \times Z_3. \tag{3.13}$$

The groups $\mathcal{A}_{16}, \mathcal{G}_{48}, \mathcal{G}_{48}^*$ and $\mathcal{G}_{144}$ are all subgroups of $\mathcal{G}_0(\mathcal{B})$.

On account of the relations (3.10), and the direct product structure $\mathcal{A}_{16} = \mathcal{A}_4^{(1)} \times \mathcal{A}_4^{(2)}$, the group $\mathcal{G}_{144} = \langle \mathcal{A}_{16}, W, W^*\rangle$ is generated by the four elements J_1, J_2, W and W^*: $\mathcal{G}_{144} = \langle J_1, J_2, W, W^*\rangle$. The five subgroups $\mathcal{A}_4^{(i)}$ of $\mathcal{A}_{16}$ give rise to five subgroups $\mathcal{G}_{36}^{(i)}$ of $\mathcal{G}_{144}$:

$$\mathcal{G}_{36}^{(i)} := \langle \mathcal{A}_4^{(i)}, W, W^*\rangle = \langle J_i, W, W^*\rangle \cong ((Z_2)^2 : Z_3) \times Z_3, \quad 1 \le i \le 5. \tag{3.14}$$

Lemma 3.3 *(i) Under the action of the group $\mathcal{G}_{144}$ the eight quatrains $\{Q_r^{(i)}\}_{1\le r\le 8}$ for the page σ_i, $1 \le i \le 5$, fall into the following three orbits $\Omega_A^{(i)}, \Omega_B^{(i)}, \Omega_C^{(i)}$, of lengths $1, 3, 4$:*

$$\Omega_A^{(i)} = \{Q_1^{(i)}\}, \qquad \Omega_B^{(i)} = \{Q_2^{(i)}, Q_3^{(i)}, Q_4^{(i)}\}, \qquad \Omega_C^{(i)} = \{Q_5^{(i)}, Q_6^{(i)}, Q_7^{(i)}, Q_8^{(i)}\}. \tag{3.15}$$

(ii) The three quatrains $Q_2^{(i)}, Q_3^{(i)}, Q_4^{(i)}$ constitute a W-orbit, and the subgroup of $\mathcal{G}_{144}$ which stabilizes (any) one of them is $\mathcal{G}_{48}^* = \langle \mathcal{A}_{16}, W^*\rangle \cong 2^4 : 3$.

(iii) The four quatrains $Q_5^{(i)}, Q_6^{(i)}, Q_7^{(i)}, Q_8^{(i)}$ constitute an $\mathcal{A}_4^{(j)}$-orbit for any $j \neq i$. The subgroup of $\mathcal{G}_{144}$ which stabilizes $Q_5^{(i)}$ is the group $\mathcal{G}_{36}^{(i)} := \langle \mathcal{A}_4^{(i)}, W, W^*\rangle \cong (2^2 : 3) \times 3$.

Proof. A straightforward check, using the generators J_1, J_2, W, W^* of $\mathcal{G}_{144}$. ∎

3.3 Harmonious quatrains and normal pentads

If a quatrain book ${}^5\mathcal{B}$ has the quatrain $Q^{(i)}_{r_i}$, $r_i \in \{1, 2, \ldots, 8\}$, in its ith page, and so determines the book spread $\mathcal{S}_{21}({}^5\mathcal{B}) = \{\mu\} \cup Q^{(1)}_{r_1} \cup Q^{(2)}_{r_2} \cup Q^{(3)}_{r_3} \cup Q^{(4)}_{r_4} \cup Q^{(5)}_{r_5}$ in PG(5, 2), then we refer to the ordered pentad of quatrains

$$Q_{r_1r_2r_3r_4r_5} := (Q^{(1)}_{r_1}, Q^{(2)}_{r_2}, Q^{(3)}_{r_3}, Q^{(4)}_{r_4}, Q^{(5)}_{r_5}) \tag{3.16}$$

as the *content* of the Qbook ${}^5\mathcal{B}$, and we use ${}^5\mathcal{B}_{r_1r_2r_3r_4r_5}$ to denote this Qbook. If we write quatrains $Q^{(1)}_{r_1}$ and $Q^{(2)}_{r_2}$ in the first two pages of our book $\mathcal{B}$, and leave the other pages blank, then we will say that we have a *2-quatrain book,* or *2Qbook,* ${}^2\mathcal{B}$ whose content is $Q_{r_1r_2}$, and we use the notation ${}^2\mathcal{B}_{r_1r_2}$ for this 2Qbook. We define in an analogous fashion a *3-quatrain book,* or *3Qbook,* ${}^3\mathcal{B}_{r_1r_2r_3}$ whose content is $Q_{r_1r_2r_3}$ and a *4Qbook* ${}^4\mathcal{B}_{r_1r_2r_3r_4}$ whose content is $Q_{r_1r_2r_3r_4}$.

Lemma 3.4 *The line spread of the Qbook* ${}^5Q_{11111}$ *is a normal spread; so is the line spread of the Qbook* ${}^5Q_{55555}$.

Proof. From Equation (2.6) we see that the element W cycles through the points of each line of each of the quatrains $Q^{(i)}_1$, $1 \le i \le 5$, and so $\langle W \rangle$ serves as the required distinguished Z_3-subgroup. From Equation (2.7) the element W^* serves similarly for the quatrains $Q^{(i)}_5$, $1 \le i \le 5$. ■

A pentad $Q_{r_1r_2r_3r_4r_5}$ will be termed a *normal pentad* of quatrains if it is the content of a Qbook ${}^5\mathcal{B}$ whose book spread $\mathcal{S}_{21}({}^5\mathcal{B})$ is a normal spread. Subsets of a normal pentad $Q_{r_1r_2r_3r_4r_5}$ of sizes 2, 3 and 4 will be termed *normal duads, triads and tetrads.*

Lemma 3.5 *If the quatrains* $Q^{(i)}_r, Q^{(j)}_s$, $i \ne j$, *are in harmony then the pair* $\{Q^{(i)}_r, Q^{(j)}_s\}$ *is a normal duad.*

Proof. This follows immediately from Lemmas 3.2(iv) and 3.4. ■

Lemma 3.6 *A book* $\mathcal{B}$ *supports precisely 16 normal pentads with quatrains* $\in \mathcal{Q}_+$ *and 16 normal pentads with quatrains* $\in \mathcal{Q}_-$.

Proof. Within $\mathcal{Q}_+$ there are $\binom{5}{2} \times 4 \times 4 = 160$ duads. But each normal pentad contains $\binom{5}{2}$ duads. ■

Remark 3.7 *As an arithmetical check, in* PG(5, 2) *there exist* $2^8.7.31$ *normal line spreads, see (1.6). Since each of the 21 lines of a normal spread is the spine of a book* $\mathcal{B}$ *which supports the spread, there are* $N_1 := 2^8.3.7^2.31$ *such books* $\mathcal{B}$. *But in* PG(5, 2) *there exist* $N_2 := 2^3.3.7^2.31$ *books, see Section 1.1. Consequently each book supports* $N_1/N_2 = 2^5$ *normal pentads, in agreement with the* $16 + 16$ *of the lemma.*

3.4 The standard 2Qbook ${}^2\mathcal{B}$ and the group $\mathcal{G}_0({}^2\mathcal{B})$

As explained in Section 2.2, in seeking a classification of all book spreads in PG$(5,2)$ *we may restrict attention to those Qbooks* ${}^5\mathcal{B}$, *based on our standard book* $\mathcal{B}$, *for which the quatrains in its first three pages* σ_1, σ_2 *and* σ_3 *all belong to* $\mathcal{Q}_+$. Since, see Lemma 3.2(iv), $\mathcal{C}_{16}$ acts transitively on the sixteen pairs of quatrains $\{Q_r^{(1)}, Q_s^{(2)}\}_{1\le r,s\le 4}$, we may further confine our attention to those Qbooks ${}^5\mathcal{B}$ for which the quatrains in its first two pages σ_1, σ_2 are $Q_1^{(1)}$ and $Q_1^{(2)}$:

$$Q_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ v & w & u \\ w & u & v \end{pmatrix}^{(1)}, \qquad Q_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ v & w & u \\ w & u & v \end{pmatrix}^{(2)}. \tag{3.17}$$

Thus equipped, the book $\mathcal{B}$ becomes our *standard 2Qbook* ${}^2\mathcal{B}_{11}$, whose content is Q_{11}.

Our aim now is to investigate the different kinds of extensions of this 2Qbook ${}^2\mathcal{B}_{11}$ to a 3Qbook ${}^3\mathcal{B}$, then to a 4Qbook ${}^4\mathcal{B}$, and finally to a full Qbook ${}^5\mathcal{B}$.

Lemma 3.8 *The subgroup of* $\mathcal{G}_0(\mathcal{B})$ *which stabilizes our standard 2Qbook* ${}^2\mathcal{B}_{11}$ *is* $\mathcal{G}_0({}^2\mathcal{B}_{11}) = \mathcal{G}_{144} = \langle \mathcal{A}_{16}, W, W^*\rangle$.

Proof. By Lemma 3.1(i) both $Q_1^{(1)}$ and $Q_1^{(2)}$ are stabilized by W and W^*. By Lemma 3.2(ii), both $Q_1^{(1)}$ and $Q_1^{(2)}$ are stabilized by $\mathcal{A}_{16}$, but not by the 15 involutions in $\mathcal{C}_{16}$. So certainly $\mathcal{G}_{144} \le \mathcal{G}_0({}^2\mathcal{B}_{11})$. However, see Equation (2.8), $J_{(vw)} \notin \mathcal{G}_0({}^2\mathcal{B}_{11})$. So for A in the block form (3.1), in order for A to belong to $\mathcal{G}_0({}^2\mathcal{B}_{11})$ the block X must be restricted to represent the 16 elements of $\mathcal{A}_{16} \cong (Z_2)^4$, the element A_4 must be restricted (see before Eq. (3.4)) to belong to the Z_3-subgroup $\langle W_4\rangle$, and the element A_2 must be restricted to belong to the Z_3-subgroup $\langle W_2\rangle$. Hence $\mathcal{G}_0({}^2\mathcal{B}_{11}) = \mathcal{G}_{144} \cong ((Z_2)^4 : Z_3) \times Z_3$. ■

4 Extending the 2Qbook ${}^2\mathcal{B}$ to a full Qbook ${}^5\mathcal{B}$

As indicated in section 3.4 we wish to investigate the different kinds of extensions of our standard 2Qbook ${}^2\mathcal{B}_{11}$ to a 3Qbook ${}^3\mathcal{B}$, then to a 4Qbook ${}^4\mathcal{B}$, and finally to a full Qbook ${}^5\mathcal{B}$. To this end it will help to first have to hand a list of all the normal pentads consisting of quatrains $\in \mathcal{Q}_+$, and also some material concerning normal tetrads and normal triads.

4.1 Normal pentads and their subsets

Lemma 4.1 *(i) For* $r_i \in \{1,2,3,4\}$ *the ordered pentad*

$$Q_{r_1r_2r_3r_4r_5} := (Q^{(1)}_{r_1}, Q^{(2)}_{r_2}, Q^{(3)}_{r_3}, Q^{(4)}_{r_4}, Q^{(5)}_{r_5}) \tag{4.1}$$

is a normal pentad of quatrains if and only if either $r_i = 1,\ 1 \le i \le 5$, *or if* $r_1r_2r_3r_4r_5$ *takes one of the following fifteen values*

$$\begin{aligned} &12442,\ 21244,\ 42124,\ 44212,\ 24421;\\ &13223,\ 31322,\ 23132,\ 22313,\ 32231;\\ &14334,\ 41433,\ 34143,\ 33414,\ 43341. \end{aligned} \tag{4.2}$$

(ii) Each of the 160 *pairs of quatrains* $\{Q^{(i)}_r, Q^{(j)}_s\}_{1\le r,s\le 4,\ 1\le i<j\le 5}$ *belongs to precisely one of these sixteen normal pentads.*

(iii) The sixteen normal pentads comprise a single $\mathcal{C}_{16}$*-orbit.*

Proof. (i) We already know, see Lemma 3.4, that Q_{11111} is a normal pentad. Upon noting that the pentad Q_{12442} is the image of Q_{11111} under the action of the involution K_1 of Section 3.2.1, the pentad Q_{12442} is also normal. From Lemma 3.1(ii) we see that B sends $Q_{r_1r_2r_3r_4r_5}$ to $Q_{r_2r_3r_4r_5r_1}$; so the three rows of (4.2) correspond to three B-orbits of pentads. But from Lemma 3.1(i) we see that W cyclically permutes the three rows (the fifteen pentads thus forming a single BW-orbit). So it follows that all fifteen pentads given by (4.2) are normal. The sixteen normal pentads thus found exhaust the possibilities, since by Lemma 3.6 no further normal pentads exist with quatrains $\in \mathcal{Q}_+$.

(ii) This is a simple verification; see also Lemma 3.6, Proof.

(iii) This follows from (ii) upon recalling from Lemma 3.2(iv) that for given $i \ne j$ the group $\mathcal{C}_{16}$ acts transitively on the sixteen pairs of quatrains $\{Q^{(i)}_r, Q^{(j)}_s\}_{1\le r,s\le 4}$. ■

Remark 4.2 *The subgroup* $\mathcal{G}_0(^5\mathcal{B}_{11111})$ *of* $\mathcal{G}_0(\mathcal{B})$ *which stabilizes the Qbook* $^5\mathcal{B}_{11111}$ *is* $\mathcal{G}_{144}$ *of order* $2^4.3^2$, *and so it follows that* $\mathcal{G}(^5\mathcal{B}_{11111})$ *has order* $144 \times 5! = 2^7.3^3.5$, *in agreement with Eq. (1.7).*

4.1.1 Different kinds of pentads

First consider pentads of harmony type $\mathcal{T}(5,0)$. We can classify such harmonious pentads $P = Q_{r_1r_2r_3r_4r_5}$ into four kinds $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ as follows:
$\mathcal{P}_0$: $P \in \mathcal{P}_0$ if P is a normal pentad; example $P_0 := Q_{11111}$;
$\mathcal{P}_1$: $P \in \mathcal{P}_1$ if $P \notin \mathcal{P}_0$ but P contains a normal tetrad; example $P_1 := Q_{11112}$;
$\mathcal{P}_2$: $P \in \mathcal{P}_2$ if $P \notin \mathcal{P}_0 \cup \mathcal{P}_1$ but P contains a normal triad; example $P_2 := Q_{11122}$;
$\mathcal{P}_3$: $P \in \mathcal{P}_3$ if P contains no normal triads; example $P_3 := Q_{11232}$.

Lemma 4.3 *If a harmonious pentad P_2 is of type $\mathcal{P}_2$ then it contains precisely two normal triads.*

Proof. Without loss of generality we may assume that $P_2 := Q_{111rs}$ for some $r, s \in \{2, 3, 4\}$. By Lemma 4.1(ii) the harmonious duad $\{Q_r^{(4)}, Q_s^{(5)}\}$ is a subset of precisely one of the normal pentads (4.2), and so is a subset of a unique normal triad $\{Q_1^{(i)}, Q_r^{(4)}, Q_s^{(5)}\}$ for some $i \in \{1, 2, 3\}$. Thus $P_2 = Q_{111rs}$ contains just the two normal triads $\{Q_1^{(1)}, Q_1^{(2)}, Q_1^{(3)}\}$ and $\{Q_1^{(i)}, Q_r^{(4)}, Q_s^{(5)}\}$. (For example, besides $\{Q_1^{(1)}, Q_1^{(2)}, Q_1^{(3)}\}$, the only other normal triad contained in $P_2 = Q_{11142}$ is $\{Q_1^{(1)}, Q_4^{(4)}, Q_2^{(5)}\}$, arising from the first entry in (4.2).) ■

Next consider pentads of harmony type $\mathcal{T}(4,1)$. Such pentads may be similarly classified into three kinds:
$\mathcal{P}_1'$: $P \in \mathcal{P}_1'$ if P contains a normal tetrad; example $P_1' := Q_{11115}$;
$\mathcal{P}_2'$: $P \in \mathcal{P}_2'$ if $P \notin \mathcal{P}_1$ but P contains a normal triad; example $P_2' := Q_{11125}$;
$\mathcal{P}_3'$: $P \in \mathcal{P}_3'$ if P contains no normal triads; example $P_3' := Q_{11235}$.

Finally consider pentads of harmony type $\mathcal{T}(3,2)$. Such pentads may be classified into two kinds:
$\mathcal{P}_2''$: $P \in \mathcal{P}_2''$ if P contains a normal triad; example $P_2'' := Q_{11155}$;
$\mathcal{P}_3''$: $P \in \mathcal{P}_3''$ if P contains no normal triads; example $P_3'' := Q_{11255}$.

In ensuing sections our aim is to show that *there are precisely nine distinct* GL(6, 2)*-orbits of Qbooks, whose representatives may be chosen to be those whose pentads are the foregoing*

$$P_0, P_1, P_2, P_3, P_1', P_2', P_3', P_2'', P_3''. \tag{4.3}$$

4.2 Extending the 2Qbook ${}^2\mathcal{B}$ to a 3Qbook ${}^3\mathcal{B}$

By making a choice of quatrain for the page σ_3 of $\mathcal{B}$ we thereby extend our standard 2Qbook ${}^2\mathcal{B} = {}^2\mathcal{B}_{11}$ to a 3Qbook ${}^3\mathcal{B} = {}^3\mathcal{B}_{11r}$. In fact, see end of Section 2.2, it suffices to confine our choice to one of the quatrains $Q_r^{(3)}$, for $r \in \{1, 2, 3, 4\}$. Bearing in mind the quatrain orbits of $\mathcal{G}_0({}^2\mathcal{B}) = \mathcal{G}_{144}$ in Lemma 3.3, it suffices to consider just two kinds of 3Qbooks, represented by ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$. From Lemma 3.3 we already know the subgroups of $\mathcal{G}_0(\mathcal{B})$ which stabilize these 3Qbooks, namely

$$\mathcal{G}_0({}^3\mathcal{B}_{111}) = \mathcal{G}_{144}; \qquad \mathcal{G}_0({}^3\mathcal{B}_{112}) = \mathcal{G}_{48}^*. \tag{4.4}$$

4.3 Qbooks of type $\mathcal{T}(3,2)$

In the case of Qbooks of harmony type $\mathcal{T}(3,2)$, see section 2.2, we may proceed directly from the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ to full Qbooks.

Theorem 4.4 *There exist in* PG(5, 2) *just two* GL(6, 2)*-orbits of full Qbooks of type* $\mathcal{T}(3,2)$*, with representatives* ${}^5\mathcal{B}_{11155}$ *and* ${}^5\mathcal{B}_{11255}$*. These Qbooks* ${}^5\mathcal{B}_{11155}$ *and* ${}^5\mathcal{B}_{11255}$ *may be distinguished by their having pentads belonging to the respective kinds* $\mathcal{P}_2'', \mathcal{P}_3''$*, see section 4.1.1.*

Proof. By Lemma 3.2 the group $\mathcal{A}_{16}$ fixes in each page σ_i the quatrains $\{Q_r^{(i)}\}_{1\leq r\leq 4}$ and acts transitively on the sixteen pairs of quatrains $\{Q_r^{(4)}, Q_s^{(5)}\}_{5\leq r,s\leq 8}$. So, up to isomorphism, the extensions ${}^5\mathcal{B}_{111rs}$ and ${}^5\mathcal{B}_{112rs}$ of type $\mathcal{T}(3,2)$ to the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ are represented by those with $r = s = 5$. ■

4.4 Qbooks of type $\mathcal{T}(5,0)$

In this section we determine up to isomorphism those extensions of the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ to full Qbooks ${}^5\mathcal{B}_{111rs}$ and ${}^5\mathcal{B}_{112rs}$ which are of type $\mathcal{T}(5,0)$, that is those with $r, s \in \{1,2,3,4\}$ for which the five quatrains form a harmonious pentad.

First we consider extensions of type $\mathcal{T}(5,0)$ of the 3Qbook ${}^3\mathcal{B}_{111}$. Since $\mathcal{G}_0({}^3\mathcal{B}_{111}) = \mathcal{G}_{144}$, see (4.4), the $\mathcal{G}_0({}^3\mathcal{B}_{111})$-orbits of quatrains for the page σ_4 are $\Omega_A^{(4)}, \Omega_B^{(4)}$ and $\Omega_C^{(4)}$, as in Equation (3.15). Up to isomorphism there are thus just two kinds of 4Qbook which extend the 3Qbook ${}^3\mathcal{B}_{111}$ such that the four quatrains are in harmony, represented by the two 4Qbooks ${}^4\mathcal{B}_{1111}$ and ${}^4\mathcal{B}_{1112}$. From Lemma 3.3 it follows that the two subgroups $\mathcal{G}_0({}^4\mathcal{B})$ of $\mathcal{G}_0(\mathcal{B})$ which stabilize these two 4Qbooks are

$$\mathcal{G}_0({}^4\mathcal{B}_{1111}) = \mathcal{G}_{144}; \qquad \mathcal{G}_0({}^4\mathcal{B}_{1112}) = \mathcal{G}_{48}^*. \tag{4.5}$$

Because $\mathcal{G}_0({}^4\mathcal{B}_{1111}) = \mathcal{G}_{144}$ it follows, again from Equation (3.15), that there are at most two non-isomorphic extensions of ${}^4\mathcal{B}_{1111}$ to a full Qbook of type $\mathcal{T}(5,0)$, represented by the two Qbooks ${}^5\mathcal{B}_{11111}$ and ${}^5\mathcal{B}_{11112}$. Concerning extensions of ${}^4\mathcal{B}_{1112}$, consider the element $R \in$ GL(6, 2) which fixes a_5, b_5, c_5 and effects the 3-cycles $(a_4b_4c_4)$, (uwv). It follows that R also effects the 3-cycles $(a_1c_3a_2)$, $(b_1a_3b_2)$, $(c_1b_3c_2)$, and so $R \in \mathcal{G}(\mathcal{B})$. Moreover one sees that R fixes the quatrain $Q_2^{(4)}$ and effects the 3-cycles $(Q_1^{(1)}Q_1^{(3)}Q_1^{(2)})$, $(Q_2^{(5)}Q_3^{(5)}Q_4^{(5)})$, whence R effects the 3-cycle

$$Q_{11122} \mapsto Q_{11123} \mapsto Q_{11124} \mapsto Q_{11122}. \tag{4.6}$$

Because $\mathcal{G}_0({}^4\mathcal{B}_{1112}) = \mathcal{G}_{48}^*$ it follows from Lemma 3.3 and from (4.6) that there are at most two non-isomorphic extensions of ${}^4\mathcal{B}_{1112}$ to a full Qbook of type $\mathcal{T}(5,0)$, represented by the two Qbooks ${}^5\mathcal{B}_{11121}$ and ${}^5\mathcal{B}_{11122}$. But observe that the element B in Lemma 3.1(ii) maps ${}^5\mathcal{B}_{11121}$ to ${}^5\mathcal{B}_{11112}$. *Consequently there are at most three* GL(6, 2)*-orbits of extension of the 3Qbook* ${}^3\mathcal{B}_{111}$ *to a full Qbook of type* $\mathcal{T}(5,0)$, as represented by the three Qbooks

${}^5\mathcal{B}_{11111}$, ${}^5\mathcal{B}_{11112}$, ${}^5\mathcal{B}_{11122}$. In fact these three do represent distinct $\mathrm{GL}(6,2)$-orbits, for they arise from three distinct kinds of harmonious pentads, see section 4.1.1, namely

$$Q_{11111} \in \mathcal{P}_0, \quad Q_{11112} \in \mathcal{P}_1, \quad Q_{11122} \in \mathcal{P}_2. \tag{4.7}$$

Next we look at extensions of the 3Qbook ${}^3\mathcal{B}_{112}$ to a full Qbook ${}^5\mathcal{B}_{112rs}$ with $r, s \in \{1,2,3,4\}$. If $r = s = 1$, then note that the square of the element B in Lemma 3.1(ii) maps ${}^5\mathcal{B}_{11211}$ to the previously considered Qbook ${}^5\mathcal{B}_{11112}$. If $s = 1$, but $r \neq 1$, note that the element B maps ${}^5\mathcal{B}_{112r1}$ to ${}^5\mathcal{B}_{1112r}$, which last is, see (4.6), isomorphic to the previously considered ${}^5\mathcal{B}_{11122}$. If $r = 1$, but $s \neq 1$, then the involution $\bar{N}_{(45)} := J_{(vw)} \circ (N_{(45)} \oplus I_2)$, see (1.18), (2.8), maps the Qbook ${}^5\mathcal{B}_{1121s}$ to one, ${}^5\mathcal{B}_{112s'1}$, also just considered. So to encounter anything new of type $\mathcal{T}(5,0)$ we need to consider the nine extensions ${}^5\mathcal{B}_{112rs}$, $r, s \in \{2,3,4\}$, of ${}^3\mathcal{B}_{112}$. By consulting the list (4.2) of normal pentads we see that eight of these extensions are to Qbooks whose harmonious pentads are of the kinds $\mathcal{P}_1$ and $\mathcal{P}_2$ already encountered in (4.7), and that only one extension, namely ${}^5\mathcal{B}_{11232}$, has a harmonious pentad of kind $\mathcal{P}_3$.

Lemma 4.5 *For $k = 0,1,2,3$ all pentads $\in \mathcal{P}_k$ are isomorphic.*

Proof. If $P_1 \in \mathcal{P}_1$ then let $P_0 \in \mathcal{P}_0$ be the normal pentad which shares four of its quatrains with P_1. By Lemma 4.1(iii) there exists $K \in \mathcal{C}_{16}$ such that $K(P_0) = Q_{11111}$. Hence $K(P_1)$ agrees with Q_{11111} in four of its places, and so for some power B^h of B, see Lemma 3.1(ii), we have $B^h K(P_1) = Q_{1111r}$ for some $r \in \{2,3,4\}$. Hence each $P_1 \in \mathcal{P}_1$ is isomorphic to Q_{11112}. Similar considerations, using appropriate elements of $\mathcal{G}(\mathcal{B})$, allow us to prove that all pentads $\in \mathcal{P}_2$ are isomorphic. Finally recall, see the preamble to the Lemma, that the 3Qbook ${}^3\mathcal{B}_{112}$ has a *unique* extension to a Qbook having a harmonious pentad of kind $\mathcal{P}_3$. Consequently all pentads $\in \mathcal{P}_3$ are isomorphic. ■

Our results for Qbooks of type $\mathcal{T}(5,0)$ are thus as in the next theorem.

Theorem 4.6 *There exist in* $\mathrm{PG}(5,2)$ *just four* $\mathrm{GL}(6,2)$*-orbits of full Qbooks of type* $\mathcal{T}(5,0)$, *with representatives:*

$${}^5\mathcal{B}_{11111}; \quad {}^5\mathcal{B}_{11112}; \quad {}^5\mathcal{B}_{11122}; \quad {}^5\mathcal{B}_{11232}. \tag{4.8}$$

The Qbooks (4.8) may be distinguished by their having pentads belonging to the respective kinds $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, see section 4.1.1.

4.5 Qbooks of type $\mathcal{T}(4,1)$

It follows from Lemma 3.2(ii) that there exists just one $\mathrm{GL}(6,2)$-orbit of Qbooks type $\mathcal{T}(4,1)$ which extend the 4Qbook ${}^4\mathcal{B}_{1111}$, .namely that represented by ${}^5\mathcal{B}_{11115}$. Similarly there exists just one $\mathrm{GL}(6,2)$-orbit of Qbooks type $\mathcal{T}(4,1)$ which extend the 4Qbook ${}^4\mathcal{B}_{1112},.$ as represented by ${}^5\mathcal{B}_{11125}$. These two classes are distinct, since ${}^5\mathcal{B}_{11115}$ and ${}^5\mathcal{B}_{11125}$ have respective pentads of the distinct kinds $\mathcal{P}'_1$ and $\mathcal{P}'_2$. We obtain nothing new by considering extensions ${}^5\mathcal{B}_{1115r}$, $r \in \{1,2,3,4\}$, of the 4Qbook ${}^4\mathcal{B}_{1115}$, for the involution $\bar{N}_{(45)} := J_{(vw)} \circ (N_{(45)} \oplus I_2)$, see (1.18), (2.8), maps the Qbook ${}^5\mathcal{B}_{1115r}$ to one, ${}^5\mathcal{B}_{111r'5}$, just considered.

So to find anything further of type $\mathcal{T}(4,1)$ we need to consider the three extensions of ${}^3\mathcal{B}_{112}$ of the form ${}^5\mathcal{B}_{112r5}$, $r \in \{2,3,4\}$. For the choices $r = 2$ and $r = 4$ the pentad is seen from the list (4.2) to be of kind $\mathcal{P}'_2$. By similar arguments to those employed in the proof of Lemma 4.5, we see that ${}^5\mathcal{B}_{11225}$ and ${}^5\mathcal{B}_{11245}$ are isomorphic to ${}^5\mathcal{B}_{11125}$. On the other hand the Qbook ${}^5\mathcal{B}_{11235}$ has pentad Q_{11235} of kind $\mathcal{P}'_3$.

For Qbooks of type $\mathcal{T}(4,1)$ our results are thus as in the next theorem.

Theorem 4.7 *There exist in* $\mathrm{PG}(5,2)$ *just three* $\mathrm{GL}(6,2)$*-orbits of full Qbooks of type* $\mathcal{T}(4,1)$, *with representatives:*

$$ {}^5\mathcal{B}_{11115} \in \mathcal{P}'_1; \quad {}^5\mathcal{B}_{11125} \in \mathcal{P}'_2; \quad {}^5\mathcal{B}_{11235} \in \mathcal{P}'_3. \tag{4.9}$$

Remark 4.8 *The book spread* $\mathcal{S}_{11115} := \mathcal{S}_{21}({}^5\mathcal{B}_{11115})$ *is unusual in that it can be viewed as a quatrain book in two different ways! To see this, recall from Remark 1.2 that the book spread* $\mathcal{S}_{11111} := \mathcal{S}_{21}({}^5\mathcal{B}_{11111})$ *can be viewed as a quatrain book in 21 ways. In particular it is a quatrain book with spine the line* $\{a_5, b_5, c_5\} \in Q_1^{(5)}$. *Now if we replace the regulus* $\begin{pmatrix} a_5+u & b_5+v & c_5+w \\ a_5+v & b_5+w & c_5+u \\ a_5+w & b_5+u & c_5+v \end{pmatrix}$ *in* σ_5 *by its opposite* $\begin{pmatrix} a_5+u & b_5+w & c_5+v \\ a_5+w & b_5+v & c_5+u \\ a_5+v & b_5+u & c_5+w \end{pmatrix}$ *we thereby convert* $Q_1^{(5)}$ *to* $Q_5^{(5)}$ *and* ${}^5\mathcal{B}_{11111}$ *to* ${}^5\mathcal{B}_{11115}$. *So* $\mathcal{S}_{11115}$ *is a quatrain book with spine* $\{a_5, b_5, c_5\}$ *as well as a quatrain book with spine* $\mu = \{u, v, w\}$. *Consequently take note that* $|\mathcal{G}(\mathcal{S}_{11115})| = 2|\mathcal{G}({}^5\mathcal{B}_{11111})|$.

5 The complete classification of Qbooks in PG(5, 2)

In Theorems 4.4, 4.6, 4.7 we have succeeded in classifying all quatrain books ${}^5\mathcal{B}$ in $\mathrm{PG}(5,2)$: there are precisely nine $\mathrm{GL}(6,2)$-orbits with representatives as displayed in the second columns of the following table. See Sections 5.1 and 5.2 below for details concerning the entries in columns 3, 4 and 5.

Type	Content of ${}^5\mathcal{B}$	Invariant sequence	$\lvert\mathcal{G}({}^5\mathcal{B})\rvert$	$\lvert\mathcal{G}(\mathcal{S}_{21})\rvert$
$\mathcal{T}(5,0)$	$Q_{11111} \in \mathcal{P}_0$	$(0,0,0,0,0,0,0,0,0,63)$	17280	$362880 = 7.5.3^4.2^7$
	$Q_{11112} \in \mathcal{P}_1$	$(0,0,0,0,0,0,0,0,48,15)$	1152	$1152 = 3^2.2^7$
	$Q_{11122} \in \mathcal{P}_2$	$(0,0,0,0,0,0,0,48,0,15)$	384	$384 = 3.2^7$
	$Q_{11232} \in \mathcal{P}_3$	$(48,0,0,0,0,0,0,0,0,15)$	5760	$5760 = 5.3^2.2^7$
$\mathcal{T}(4,1)$	$Q_{11115} \in \mathcal{P}'_1$	$(0,0,0,0,0,0,0,0,36,27)$	864	$1728 = 3^3.2^6$
	$Q_{11125} \in \mathcal{P}'_2$	$(0,0,0,0,0,0,0,36,12,15)$	72	$72 = 3^2.2^3$
	$Q_{11235} \in \mathcal{P}'_3$	$(12,0,0,0,0,0,0,36,0,15)$	288	$288 = 3^2.2^5$
$\mathcal{T}(3,2)$	$Q_{11155} \in \mathcal{P}''_2$	$(0,0,0,0,0,0,0,27,18,18)$	108	$108 = 3^3.2^2$
	$Q_{11255} \in \mathcal{P}''_3$	$(3,0,0,0,0,0,0,36,9,15)$	36	$36 = 3^2.2^2$

(5.1)

5.1 Invariant sequences

Any line spread $\mathcal{S}_{21}$ in $\mathrm{PG}(5,2)$ determines a certain *invariant sequence* $\mathcal{I}(\mathcal{S}_{21})$ as described below.

Lemma 5.1 *Let $\mathcal{S}_{21}$ be a line spread in $\mathrm{PG}(5,2)$, and let H be any hyperplane of $\mathrm{PG}(5,2)$. Then precisely five lines of $\mathcal{S}_{21}$ lie inside H.*

Proof. The 32 points of H^c account for 16 lines of $\mathcal{S}_{21}$ which meet H in a point. The remaining $31-16=15$ points of H must therefore support the remaining $21-16=5$ lines of $\mathcal{S}_{21}$. ■

Thus $\mathcal{S}_{21}$ gives rise to an induced partial spread $\mathcal{S}_5(H)$ in each hyperplane H of $\mathrm{PG}(5,2)$. Now all partial line spreads in $\mathrm{PG}(4,2)$ have been classified in [5, Table B.1]. In particular there exist in $\mathrm{PG}(4,2)$ ten projectively distinct kinds of partial spreads of size 5. Of the 63 partial spreads $\mathcal{S}_5(H)$ determined by the spread $\mathcal{S}_{21}$ suppose that precisely N_x belong to class Vx.1, x = a, b, ... , j; see [5, Table B.1]. Then we will say that the sequence $(N_a, N_b, \dots, N_j)$ is the *invariant sequence* $\mathcal{I}(\mathcal{S}_{21})$ of the spread $\mathcal{S}_{21}$.

Clearly spreads $\mathcal{S}_{21}$, $\mathcal{S}'_{21}$ in $\mathrm{PG}(5,2)$ which have different invariant sequences will be non-isomorphic. In the case of a book spread determined by a Qbook ${}^5\mathcal{B}$ then the 15 hyperplanes through the spine μ of $\mathcal{B}$ contribute $(0,0,0,0,0,0,0,0,0,15)$ to the invariant sequence. In order to determine the full invariant sequence one needs to determine the contributions of the 48 hyperplanes $\mathcal{H}_{48}$ which meet μ in a point. In some unpublished research in 2004 R. Shaw succeeded in doing this by first finding the orbit structure of these 48 hyperplanes under the action of various relevant groups. His 2004 results are listed in column 3 in the table. These invariants are related to the invariants calculated by computer in [13], namely the number n_m of 3-flats containing m spread lines is calculated in [13] for $m=3,4,5$. In particular $N_j = 3n_5$ and $N_i = 3n_4$. (To see that $N_i = 3n_4$, observe that if σ is a solid which contains 4 lines $\mathcal{S}_4$ of the spread $\mathcal{S}_{21}$ then the three hyperplanes which contain σ are $H_r := \langle \sigma, \lambda_r \rangle$, where λ_1, λ_2 and λ_3 denote the three lines of

$\mathcal{S}_{21} \setminus \mathcal{S}_4$ which meet σ in a point. Since the induced partial spread $\mathcal{S}_5(H_r)$ in each of these 3 hyperplanes is thus of class Vi.1 it follows that $N_i = 3n_4$.)

5.2 The nine stabilizer groups $\mathcal{G}(^5\mathcal{B})$

Here we briefly sketch some of the main features of the nine stabilizer groups $\mathcal{G}(^5\mathcal{B})$, and in particular deal with their orders as given in column 4 of the table (5.1). Of course a quatrain book $^5\mathcal{B}$ lies on a GL(6, 2)-orbit $\Omega(^5\mathcal{B})$ of length $|\Omega(^5\mathcal{B})| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}(^5\mathcal{B})|$, and so one modest check on the accuracy of the entries in column 4 is that the nine orbit lengths sum to the total number of Qbooks in PG(5, 2), namely $1,194,590,208 = 2^{18}.3.7^2.31$ (see Section 1.1). In column 5 of the table (5.1) we have also listed the order of the stability group $\mathcal{G}(\mathcal{S}_{21})$ of the associated spread $\mathcal{S}_{21}(^5\mathcal{B})$. Recall from Remarks 4.8, 1.2 that $|\mathcal{G}(^5\mathcal{B}_{11115})| = |\mathcal{G}(\mathcal{S}_{11115})|/2$ and $|\mathcal{G}(^5\mathcal{B}_{11111})| = |\mathcal{G}(\mathcal{S}_{11111})|/21 = 17280$; however $|\mathcal{G}(^5\mathcal{B})| = |\mathcal{G}(\mathcal{S}_{21})|$ holds for the other seven entries. As a more detailed check on the accuracy of our results the nine stabilizer orders given in column 5 of our table are seen to be the same as those found by computer in [13] (see the nine entries with $n_5 \geq 5$ in [13, Table III]). Further, see Section 6 below for details of other computer-aided checks on the accuracy of our results.

Recall that $\mathcal{G}_0(^5\mathcal{B})$ denotes the subgroup of $\mathcal{G}(^5\mathcal{B})$ which consists of those elements which fix each page of the Qbook $^5\mathcal{B}$. Now, by appeal to our results in Lemmas 3.1, 3.2, 3.3, for each of the nine Qbooks $^5\mathcal{B}$ in the table (5.1) the subgroup $\mathcal{G}_0(^5\mathcal{B})$ is seen to be as given in the following table:

Qbook $^5\mathcal{B}$	$\mathcal{G}_0(^5\mathcal{B})$
$^5\mathcal{B}_{11111}$	$\mathcal{G}_{144} = \langle \mathcal{A}_{16}, W, W^* \rangle$
$^5\mathcal{B}_{11112}$, $^5\mathcal{B}_{11122}$, $^5\mathcal{B}_{11232}$	$\mathcal{G}^*_{48} = \langle \mathcal{A}_{16}, W^* \rangle$
$^5\mathcal{B}_{11115}$	$\mathcal{G}^{(5)}_{36} = \langle \mathcal{A}^{(5)}_4, W, W^* \rangle$
$^5\mathcal{B}_{11125}$, $^5\mathcal{B}_{11235}$	$\mathcal{A}^{(5)}_{12} = \langle \mathcal{A}^{(5)}_4, W^* \rangle$
$^5\mathcal{B}_{11155}$	$\langle W, W^* \rangle \cong Z_3 \times Z_3$
$^5\mathcal{B}_{11255}$	$\langle W^* \rangle \cong Z_3$

(5.2)

In particular see Lemma 3.3 for the results in the table (5.2) for the Qbooks $^5\mathcal{B}_{11112}$, $^5\mathcal{B}_{11122}$, $^5\mathcal{B}_{11232}$. In the case of the Qbook $^5\mathcal{B}_{11155}$, we know, see (4.4), that $\mathcal{G}_0(^3\mathcal{B}_{111}) = \langle \mathcal{A}_{16}, W, W^* \rangle$. But, see Lemma 3.2(iv), only the identity element I_4 of $\mathcal{A}_{16}$ preserves the pair of quatrains $Q^{(4)}_5, Q^{(5)}_5$, and so $\mathcal{G}_0(^5\mathcal{B}_{11155}) = \langle W, W^* \rangle \cong Z_3 \times Z_3$. On the other hand $W \notin \mathcal{G}_0(^5\mathcal{B}_{11255})$ since, see Lemma 3.1(i), W does not preserve $Q^{(3)}_2$.

We now seek to extend $\mathcal{G}_0(^5\mathcal{B})$ to $\mathcal{G}(^5\mathcal{B})$ by finding those further elements of $\mathcal{G}(^5\mathcal{B})$ which do not preserve each page of $\mathcal{B}$. Since $\mathcal{G}_1(\Sigma) \cong \mathrm{Sym}(5)$, see (1.19), any permutation of the five pages $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ can be achieved by the element $A_4 \oplus I_2 \in \mathcal{G}(\mathcal{B})$ for a suitable choice of $A_4 \in \mathcal{G}_1(\Sigma)$ in Equation (3.1). In particular any simple interchange $(\sigma_i \sigma_j)$ can be effected by the involution $N_{(ij)} \oplus I_2$. However the involution $N_{(ij)} \oplus I_2$ effects $\mathcal{Q}_+ \rightleftarrows \mathcal{Q}_-$ in (2.9), and so instead we will in the following make use of the involutions

$$\bar{N}_{(ij)} = J_{(vw)} \circ (N_{(ij)} \oplus I_2) \in \mathcal{G}(\mathcal{B}), \tag{5.3}$$

which map each quatrain into a harmonious image, and which also generate a Sym(5) subgroup, say $\mathcal{G}_1(\mathcal{B})$, of $\mathcal{G}(\mathcal{B})$.

For each ${}^5\mathcal{B}_{r_1r_2r_3r_4r_5}$ in (5.2) our task is to find A_4, X and A_2 such that A in (3.1) preserves the ordered pentad of quatrains $Q_{r_1r_2r_3r_4r_5}$. In the following we set $\mathcal{G}_{r_1r_2r_3r_4r_5} := \mathcal{G}({}^5\mathcal{B}_{r_1r_2r_3r_4r_5})$ and $\mathcal{G}^0_{r_1r_2r_3r_4r_5} := \mathcal{G}_0({}^5\mathcal{B}_{r_1r_2r_3r_4r_5})$.

5.2.1 Qbooks of harmony type $\mathcal{T}(5,0)$

Concerning ${}^5\mathcal{B}_{11111}$ we know that ${\mathcal{G}^0}_{11111} = \mathcal{G}_{144}$. Since the quatrains $Q_1^{(i)}$ are all preserved by the involutions $\bar{N}_{(ij)}$, the group ${\mathcal{G}^0}_{11111}$ extends to $\mathcal{G}_{11111}$ by using elements of the Sym(5) subgroup of $\mathcal{G}(\mathcal{B})$ generated by the $\bar{N}_{(ij)}$. So $|\mathcal{G}_{11111}| = 5! \times 144 = 17280$. (*Aliter:* this result follows of course from (1.7) since the book spread $\mathcal{S}_{21}({}^5\mathcal{B}_{11111})$ is a normal spread.)

Concerning the other three Qbooks ${}^5\mathcal{B}$ of harmony type $\mathcal{T}(5,0)$ we know, see (5.2), that they all have $\mathcal{G}_0({}^5\mathcal{B}) = \mathcal{G}^*_{48} = \langle \mathcal{A}_{16}, W^* \rangle$. In the case of ${}^5\mathcal{B}_{11112}$ observe that the six involutions $\bar{N}_{(ij)}$, $1 \le i < j \le 4$, which generate a Sym(4) subgroup of $\mathcal{G}(\mathcal{B})$, all stabilize the ordered pentad of quatrains Q_{11112}. Consequently $|\mathcal{G}_{11112}| = |\,\mathrm{Sym}(4)| \times |\mathcal{G}^*_{48}| = 1152$.

In the case of ${}^5\mathcal{B}_{11122}$, the involution $\bar{K}_{(13)} := J_{(vw)} \circ (K_{(13)} \oplus I_2)$, where $K_{(13)} = W_4 N_{(13)}$ as in (1.20), preserves the ordered pentad Q_{11122}. For, upon using (1.17), we see that $\bar{K}_{(13)}$ stabilizes the three quatrains $Q_1^{(2)}, Q_2^{(4)}, Q_2^{(5)}$ and also effects the interchange $Q_1^{(1)} \rightleftarrows Q_1^{(3)}$. So $\bar{K}_{(13)} \in \mathcal{G}_{11122}$. In contrast one can check that no involution exists which effects the interchange $Q_1^{(1)} \rightleftarrows Q_1^{(2)}$ and which also preserves the pair of quatrains $\{Q_2^{(4)}, Q_2^{(5)}\}$. However there does exist an involution which effects the interchanges $Q_1^{(1)} \rightleftarrows Q_2^{(5)}$, $Q_1^{(3)} \rightleftarrows Q_2^{(4)}$ and which preserves the quatrain $Q_1^{(2)}$, namely the involution $M_{(15)(34)}$ which keeps pointwise fixed the page $\sigma_2 = \langle a_2, b_2, u, v \rangle$ and which also has the following effect

$$\begin{aligned} M_{(15)(34)} :& a_1 \rightleftarrows b_5 + u, \quad b_1 \rightleftarrows c_5 + u, \quad c_1 \rightleftarrows a_5 \\ & a_3 \rightleftarrows b_4 + u, \quad b_3 \rightleftarrows c_4 + u, \quad c_3 \rightleftarrows a_4. \end{aligned} \tag{5.4}$$

Observe that the subgroup $\langle \bar{K}_{(13)}, M_{(15)(34)} \rangle$ of $\mathcal{G}_{11122}$ contains a Z_4 subgroup generated by the product $\bar{K}_{(13)} M_{(15)(34)}$ which effects the 4-cycle $(Q_1^{(1)} Q_2^{(5)} Q_1^{(3)} Q_2^{(4)})$, and so $\langle \bar{K}_{(13)}, M_{(15)(34)} \rangle$ is isomorphic to the dihedral group D_8. We conclude that the stabilizer $\mathcal{G}_{11122}$ of the quatrain book ${}^5\mathcal{B}_{11122}$ is the group $\langle \mathcal{A}_{16}, W^*, \bar{K}_{(13)}, M_{(15)(34)} \rangle$ of order $8 \times |\mathcal{G}^*_{48}| = 384$.

After dealing with ${}^5\mathcal{B}_{11122}$ it came as quite a surprise to discover that the Qbook ${}^5\mathcal{B}_{11232}$ has a much higher degree of symmetry. In particular it is the only Qbook in our table (5.1) other than ${}^5\mathcal{B}_{11111}$ having a stabilizer group which contains a Z_5 subgroups. To see this last, observe that an element $F \in \mathrm{GL}(6,2)$ of order 5 exists satisfying

$$
\begin{aligned}
F : a_1 \mapsto a_2 \mapsto a_3 + u \mapsto a_4 + w \mapsto a_5 + u \mapsto a_1, \\
b_1 \mapsto b_2 \mapsto b_3 + w \mapsto b_4 + v \mapsto b_5 + w \mapsto b_1, \\
c_1 \mapsto c_2 \mapsto c_3 + v \mapsto c_4 + u \mapsto c_5 + v \mapsto c_1
\end{aligned}
\tag{5.5}
$$

and which is then seen to effect the cyclic permutation $(Q_1^{(1)} Q_1^{(2)} Q_2^{(3)} Q_3^{(4)} Q_2^{(5)})$ of the quatrains of ${}^5\mathcal{B}_{11232}$. Moreover, from Eq. (1.16), we see that the involution $\bar{N}_{(12)}$ effects the interchange $Q_1^{(1)} \rightleftarrows Q_1^{(2)}$ while preserving the other three quatrains of ${}^5\mathcal{B}_{11232}$. Consequently F and $\bar{N}_{(12)}$ generate a Sym(5) subgroup which extends $\mathcal{G}^0{}_{11232} = \mathcal{G}^*_{48}$ to $\mathcal{G}_{11232}$, and so $\mathcal{G}_{11232}$ has order $5! \times 48 = 5760$.

5.2.2 Qbooks of harmony type $\mathcal{T}(4,1)$

In the case of ${}^5\mathcal{B}_{11115}$ the group $\mathcal{G}^0{}_{11115} = \mathcal{G}^{(5)}_{36} = \langle \mathcal{A}_4^{(5)}, W, W^* \rangle$ can be extended to $\mathcal{G}_{11115}$ by elements of that Sym(4) subgroup of $\mathcal{G}_{11111}$ which fixes σ_5. So $|\mathcal{G}_{11115}| = 4! \times 36 = 864$.

In the case of ${}^5\mathcal{B}_{11125}$ we see that the group $\mathcal{G}^0{}_{11125} = \mathcal{A}^{(5)}_{12} = \langle \mathcal{A}_4^{(5)}, W^* \rangle$ can be extended to $\mathcal{G}_{11125}$ by elements of that Sym(3) subgroup of $\mathcal{G}_{11111}$ which fixes κ_4 and κ_5. So $|\mathcal{G}_{11125}| = 3! \times 12 = 72$.

In the case of ${}^5\mathcal{B}_{11235}$ we see that $\mathcal{G}^0{}_{11235} = \mathcal{A}^{(5)}_{12} = \langle \mathcal{A}_4^{(5)}, W^* \rangle$ can be extended to $\mathcal{G}_{11235}$ by elements of that Sym(4) subgroup of $\mathcal{G}_{11232}$ which fix σ_5. For example, the involution $\bar{N}_{(34)}$ effects $Q_2^{(3)} \rightleftarrows Q_3^{(4)}$ and preserves $Q_1^{(1)}$, $Q_1^{(2)}$ and $Q_2^{(5)}$. This is checked to be the case after using Eq. (1.16) to see that $N_{(34)} := B^2 N_{(12)} B^{-2}$ effects the permutations

$$
N_{(34)} : (a_3 c_4)(b_3 b_4)(c_3 a_4)(b_5 c_5)(a_1 b_1)(b_2 c_2). \tag{5.6}
$$

Consequently $|\mathcal{G}_{11235}| = 4! \times 12 = 288$.

5.2.3 Qbooks of harmony type $\mathcal{T}(3,2)$

In the case of ${}^5\mathcal{B}_{11155}$ the group $\mathcal{G}^0{}_{11155} = \langle W, W^* \rangle$ can be extended to $\mathcal{G}_{11155}$ by elements of that $\mathrm{Sym}(3) \times Z_2$ subgroup of $\mathcal{G}_{11111}$ which preserves the partition $\{\sigma_1, \sigma_2, \sigma_3\} \cup \{\sigma_4, \sigma_5\}$. So $|\mathcal{G}_{11155}| = 3! \times 2 \times 9 = 108$. In detail, we find that $\mathcal{G}_{11155} = \langle \bar{L}_{(12)(45)}, \bar{T}_{(132)}, \bar{N}_{45}, W, W^* \rangle$, where $\bar{L}_{(12)(45)} := L_{(12)(45)} \oplus I_2$ and $\bar{T}_{(132)} := T_{(132)} \oplus I_2$.

Finally, in the case of ${}^5\mathcal{B}_{11255}$ the group $\mathcal{G}^0{}_{11255} = \langle W^* \rangle$ can be extended to $\mathcal{G}_{11255}$ by elements of that $\mathrm{Sym}(3) \times Z_2$ subgroup of $\mathcal{G}_{11232}$ which preserves the partition $\{\sigma_1, \sigma_2, \sigma_3\} \cup \{\sigma_4, \sigma_5\}$. So $|\mathcal{G}_{11255}| = 3! \times 2 \times 3 = 36$. In detail, we see that $\mathcal{G}_{11255} = \langle \bar{N}_{(12)}, M_{(23)}, \bar{N}_{(45)}, W^* \rangle$, where $M_{(23)} := F \bar{N}_{(12)} F^{-1}$ and $\langle \bar{N}_{(12)}, M_{(23)}, \bar{N}_{(45)} \rangle \cong \mathrm{Sym}(3) \times Z_2$.

6 Computer-aided check of the main results

Two independent computer-aided checks have been carried out on the results in Table 5.1.

Firstly, in 2004, T.P. McDonough verified using GAP [3] that there were exactly nine different kinds of quatrain books in PG(5, 2), in response to R. Shaw's 2004 proof of this, as previously noted in Section 1. One observes initially that, since $\mathcal{G} = \mathrm{PGL}(6, 2)$ acts transitively on the lines of PG(5, 2), every such spread is equivalent to one with spine μ and, since PGL(4, 2) is transitive on line spreads in PG(3, 2), every such spread is equivalent to one whose pages are the pages of the standard book $\mathcal{B}$. Now observe that two quatrain books in $\mathcal{B}$ are $\mathcal{G}$-equivalent if, and only if, they are $\mathcal{G}(\mathcal{B})$-equivalent. The GAP program thus determines the $\mathcal{G}(\mathcal{B})$-orbits of quatrain books in $\mathcal{B}$. Representatives of the nine orbits are the pentads of quatrains $Q^{(1)}_{j_1}Q^{(2)}_{j_2}Q^{(3)}_{j_3}Q^{(4)}_{j_4}Q^{(5)}_{j_5}$ where the quintuples $j_1j_2j_3j_4j_5$ are listed in the first row of table 6.1. The second row of the table lists the orders of the stabilizers of the quatrain books in $\mathcal{G}(\mathcal{B})$. The third row of the table lists the orders of the stabilizers of the quatrain books in $\mathcal{G}$. The GAP program is available from the authors.

11111	11112	11115	11122	11125	11155	11242	11245	11255
17280	1152	864	384	72	108	5760	288	36
362880	1152	1728	384	72	108	5760	288	36

(6.1)

Secondly the classification of all line spreads in PG(5, 2) [13] shows that up to isomorphism there are only 9 spreads for which there are at least five 3-dimensional subspaces containing 5 spread lines. However it is not clear that they are all book spreads. So S. Topalova recently carried through some new computer-aided work, briefly described below, which concentrated solely on book spreads. Moreover some intermediate theoretical results, proved in the present paper, were also checked.

Without loss of generality we may fix the first 6 lines of the spread, namely the spine, the lines of the first quatrain, and one line of the second quatrain. We choose the remaining 15 spread lines from a set D of 102 lines (out of all 651 lines of the projective space) that are skew to each of the fixed 6 ones. We construct the spread by backtrack search adding to the set of these 6 lines the remaining 3 lines of the second quatrain, then the 4 lines of the third quatrain, of the fourth and of the fifth one. The lines of D are ordered lexicographically, and each line we choose is greater than the previous line of the same quatrain or if it is the first line of a quatrain, it is greater than the first line of the previous quatrain. We obtain $2048 = (2.1.1).(8.2.1.1).(4.2.1.1).(4.2.1.1)$ spreads, and from them we see that:

1. There are 2 choices for the second line of the second quatrain and only

one choice for each of the other two lines of the second quatrain — this is in agreement with the fact that each skew pair of lines (namely the spine and the first line of the second quatrain) is in two spreads in $PG(3,2)$ — section 1.2.2 and with the fact that the spine and one more line fix exactly 2 quatrains (one from $\mathcal{Q}_+$ and one from $\mathcal{Q}_-$ (2.9))

2. There are 8 choices for the first line of the third quatrain. This is in agreement with the following facts: There are 2 possibilities for choosing $\{a_3, b_3, c_3\}$ because each skew pair of lines (namely $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$) is in two spreads in PG$(3,2)$ — section 1.2.2 . Then there are 4 ways of choosing the smallest line of a quatrain because there are 8 quatrains, but one and the same line is in one quatrain from (2.6) and in one quatrain from (2.7).

3. Similar to 1.: there are 2 different choices for the second line of the third quatrain and a unique choice for the other two lines.

4. There are 4 choices for the first line of the fourth (fifth) quatrain — $\{a_i, b_i, c_i\}$ are already fixed so these are the 4 ways of choosing the smallest line of a quatrain (see 2.) and then there are 2 different choices for the second line and a unique choice for the other two lines.

There are 18 automorphisms which fix the first six lines of the spread. To check for isomorphism we use the same technique as in [13], i.e. apply automorphisms of PG$(5,2)$ which map the spread lines to the fixed six lines in all possible ways, and then these 18 automorphisms. We find out that there are 9 nonisomorphic book spreads.

During the isomorphism check we also determine the automorphism groups which stabilize the spreads and their subgroups which preserve the spine. The orders of these groups are the same as those which are obtained theoretically and presented in columns 5 and 4 of Table 5.1.

Concerning the possibility of a future computer-aided study of book spreads in PG$(7,2)$, it would seem, after some initial investigations, that although they are only a small part of all line spreads, their number is too big for a full computer-aided classification to be possible. Thus book spreads in PG$(7,2)$ with certain additional properties ought to be considered, and the knowledge of the structure of PG$(5,2)$ book spreads and their stabilizers gained in the present paper will presumably be very helpful.

References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[2] A. Blokhuis, A. E. Brouwer, H. A. Wilbrink, Blocking sets in $\mathrm{PG}(2,p)$ for small p, and partial spreads in $\mathrm{PG}(3,7)$, *Advances in Geometry,* Special Issue (2003), 245-253.

[3] GAP. Groups, Algorithms and Programming, Version 4. The GAP Group, Lehrstuhl D für Mathematik, RWTH Aachen, Germany and School of Mathematical and Computational Sciences, University of St. Andrews, Scotland. http://www-gap.dcs.st-and.ac.uk/ gap/.

[4] N.A. Gordon, T.M. Jarvis and R. Shaw, Aspects of the linear groups $\mathrm{GL}(n,2)$, $n<7$, *J. Combin. Math. Combin. Comput.* **53** (2005), 13-31.

[5] N. A. Gordon, R. Shaw and L. H. Soicher, Classification of partial spreads in $\mathrm{PG}(4,2)$, 63 pp., Research Report (2004), available from:
http://www.hull.ac.uk/maths/people/rs/staffdetails.html .
See also the $\mathrm{PG}(4,2)$ entry on the site:
http://www.maths.qmul.ac.uk/~leonard/partialspreads/

[6] N. Hamada, On the p-rank of the incidence matrix of a balanced or partially balanced incomplete block design and its applications to error-correcting codes, *Hiroshima Math. J.* **3** (1973), 153-226.

[7] M. Harada, C. Lam and V. D. Tonchev, Symmetric $(4,4)$-nets and generalized Hadamard matrices over groups of order 4, *Des. Codes Cryptogr.* **34** (2005), 71-87.

[8] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions,* Clarendon, Oxford, 1985.

[9] V. Jha, N. L. Johnson, The classification of spreads in $\mathrm{PG}(3,q)$ admitting linear groups of order $q(q+1)$, II. Even order, *Advances in Geometry,* Special Issue (2003), 271-313.

[10] V. Jha, N. L. Johnson, The classification of spreads in $\mathrm{PG}(3,q)$ admitting linear groups of order $q(q+1)$, I. Odd order, *J. Geometry* **81** (2004), 46-80.

[11] N. L. Johnson, Spreads in $\mathrm{PG}(3,q)$ admitting several homology groups of order $q+1$, *Note di Matematica* **24**, n. 2 (2005), 9-39.

[12] N. L. Johnson, *Combinatorics of Spreads and Parallelisms*, Series: Chapman & Hall Pure and Applied Mathematics, CRC Press, 2010.

[13] Z. Mateva and S. Topalova. Line spreads of $\mathrm{PG}(5,2)$, *J. Combin. Des.* **17** (2009), 90-102.

[14] V.C. Mavron, T.P. McDonough and V.D. Tonchev, On affine designs and Hadamard designs with line spreads, Discrete Math. 308 (2008), 2742-2750.

[15] A. Rahilly, On the line structure of designs, *Discrete Math.* **92** (1991), 291-303.

[16] R. Shaw, Configurations of planes in PG$(5, 2)$, *Discrete Math.* **208/209** (1999), 529-546.

[17] L. Soicher, Computation of Partial Spreads, web preprint, http://www.maths.qmul.ac.uk/$\sim$ leonard/partialspreads, 2000.

[18] E.R. van Dam, Classification of spreads of PG$((3, 4) \setminus$ PG$(3, 2)$, *Des. Codes Cryptogr.* **3** (1993), 193-198.