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Trivectors and cubics: $\mathrm{PG}(5,2)$ aspects

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Abstract. The space $\mathrm{Alt}(\times^3 V_6)$ of alternating trilinear forms on $V_6 = V(6,2)$ is naturally isomorphic to the space $\wedge^3(V_6^*)$ of trivectors based on the dual space V_6^*. Under the natural action of the group $\mathrm{GL}(6,2)$ the nonzero elements of $\mathrm{Alt}(\times^3 V_6) \cong \wedge^3(V_6^*)$ are shown to fall into five distinct orbits. In consequence, the cubic hypersurfaces in $\mathrm{PG}(5,2)$ are classified into five large families. For $T \in \mathrm{Alt}(\times^3 V_6)$ let $\mathcal{L}_T$ denote the set of T-singular lines, consisting that is of those projective lines $\langle a, b\rangle$ in $\mathrm{PG}(5,2) = \mathbb{P}V_6$ such that $T(a,b,x) = 0$ for all $x \in V_6$. A description is given of the set $\mathcal{L}_T$ for a representative T of each of the five $\mathrm{GL}(6,2)$-orbits. In particular, for one of the orbits $\mathcal{L}_T$ is a Desarguesian line-spread in $\mathrm{PG}(5,2)$.

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1. Alternating trilinear forms over $\mathrm{GF}(2)$

1.1. Introduction

Throughout we work over GF(2), and so we may identify the nonzero elements of a vector space $V_{n+1} := V(n+1,2)$ with the points S_0 of the associated projective space $PV_{n+1} = \mathrm{PG}(n,2)$. Similarly nonzero elements of the dual vector space V_{n+1}^* will be identified with the points of the dual projective space $\mathbb{P}V_{n+1}^* = \mathrm{PG}(n,2)^*$. We use $\prec u, v, \ldots \succ$ for the vector subspace spanned by vectors $u, v, \ldots$, and $\langle u, v, \ldots \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$. The vector space $F(S_0)$ of all functions $S_0 \to \mathrm{GF}(2)$ is of dimension $|S_0| = 2^{n+1} - 1$. Given a choice of coordinates $x_1, x_2, \ldots, x_{n+1}$ in V_{n+1}, there are $\binom{n+1}{r}$ monomials $\{x_{i_1} x_{i_2} \cdots x_{i_r}\}_{1 \le i_1 < \ldots < i_r \le n+1}$ of (reduced) degree r. The $\Sigma_{r=1}^{n+1}\binom{n+1}{r} = |S_0|$ monomials of degree $r \in \{1, 2, \ldots, n+1\}$ are linearly independent and form a basis for $F(S_0)$, and so a point-set ψ of $\mathrm{PG}(n,2)$ has equation $Q(x) = 0$ for some uniquely determined polynomial Q

of minimal degree on V_{n+1} and satisfying $Q(0) = 0$. The (reduced) degree $d = \deg Q$ of Q is the *polynomial degree* of the point-set ψ. Note that if $F_r = F_r(S_0),\ r > 0$, denotes the subspace of $F(S_0)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \ldots, x_{n+1})$ with $\deg f \leq r$ and $f(0) = 0$, then the subspaces F_r are nested:

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} = F(S_0). \tag{1}$$

In the following theorem, $\mathrm{Alt}(\times^d V_{n+1})$ denotes the vector space of alternating multilinear forms $\times^d V_{n+1} \to \mathrm{GF}(2)$. Also if $\deg Q = d$ then $Q + F_{d-1}$ denotes a general element of the quotient space F_d/F_{d-1}, and the alternating form $T_Q \in \mathrm{Alt}(\times^d V_{n+1})$ is obtained by completely polarizing Q; see [13, Eq. (1.2)]. Further t_Q is defined to be that element of $\wedge^d V^*_{n+1}$ such that if Q is the product $f_1 f_2 \ldots f_d$ of d linear forms $f_i \in V^*_{n+1}$ then $t_Q = f_1 \wedge f_2 \wedge \cdots \wedge f_d$.

Theorem 1. *There exist natural linear isomorphisms* $F_d/F_{d-1} \cong \wedge^d V^*_{n+1} \cong \mathrm{Alt}(\times^d V_{n+1})$, *giving rise, for* $Q \in F_d$, *to bijective correspondences*

$$Q + F_{d-1} \longleftrightarrow t_Q \longleftrightarrow T_Q.$$

See [13, Theorem 1.6] for a proof of this theorem; see also [**?** , Theorem B]. Concerning the second isomorphism, each $t \in \wedge^d V^*_{n+1}$ giving rise to an element $T \in \mathrm{Alt}(\times^d V_{n+1})$ by way of

$$T(v_1, \ldots, v_d) =< t | v_1 \wedge v_2 \wedge \ldots \wedge v_d > . \tag{2}$$

Here $< \cdot | \cdot >$ is given by the standard determinantal pairing

$$< f_1 \wedge f_2 \wedge \cdots \wedge f_d | v_1 \wedge v_2 \wedge \cdots \wedge v_d >= \det[f_i(v_j)].$$

1.2. Cubics and alternating trilinear forms

From now onwards, we will be dealing with the particular case $d = 3$ *of Theorem 1.* Since the space F_2 has dimension $\binom{n+1}{1} + \binom{n+1}{2} = \binom{n+2}{2}$, note that $2^{(n+2)(n+1)/2}$ different cubic polynomials Q give rise to the same alternating trilinear form $T_Q \in \mathrm{Alt}(\times^3 V_{n+1})$, the latter being given, for $v_i \in V_{n+1}$, by

$$T_Q(v_1, v_2, v_3) = \sum_{i=1}^{3} Q(v_i) + \sum_{1 \leq i < j \leq 3} Q(v_i + v_j) + Q(v_1 + v_2 + v_3). \tag{3}$$

If $Q \in F_3$ is given in coordinates by $Q(x) = \Sigma_{i<j<k} t_{ijk} x_i x_j x_k + Q',\ Q' \in F_2$, then observe that T_Q takes the explicit coordinate form

$$T_Q(x, y, z) = \sum_{i<j<k} t_{ijk} |xyz|_{ijk}, \qquad \text{where } |xyz|_{ijk} := \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}. \tag{4}$$

Let $\hat{A} \in \mathrm{GL}(V^*_{n+1})$ denote the contragredient $(A^t)^{-1}$ of an element $A \in \mathrm{GL}(V_{n+1})$: so $(\hat{A} f)(Ax) = f(x)$ for all $f \in V^*_{n+1}$, $x \in V_{n+1}$. The natural action of $A \in \mathrm{GL}(V_{n+1})$ upon $\mathrm{Alt}(\times^3 V_{n+1})$ is by way of Λ_A where

$$(\Lambda_A T)(x, y, z) = T(A^{-1}x, A^{-1}y, A^{-1}z), \qquad T \in \mathrm{Alt}(\times^3 V_{n+1}) \tag{5}$$

and upon $\wedge^3 V_{n+1}^*$ is by way of $\wedge^3 \hat{A}$, with effect on decomposable trivectors

$$(\wedge^3 \hat{A}) f_1 \wedge f_2 \wedge f_3 = \hat{A} f_1 \wedge \hat{A} f_2 \wedge \hat{A} f_3, \qquad f_i \in V_{n+1}^*. \tag{6}$$

Under the isomorphic correspondence $t \longleftrightarrow T$ of Theorem 1 and equation (2) observe that $(\wedge^3 \hat{A}) t \longleftrightarrow \Lambda_A T$. The subgroup of $\mathrm{GL}(V_{n+1})$ which stabilizes an element $T \in \mathrm{Alt}(\times^3 V_{n+1})$, and hence also the isomorphic element $t \in \wedge^3 V_{n+1}^*$, will be denoted $\mathcal{G}_T$, and also $\mathcal{G}_t$.

For a given choice of an element $T \in \mathrm{Alt}(\times^3 V_{n+1})$, a point $a \in \mathrm{PG}(n,2)$ is termed *T-singular* if it satisfies

$$T(a,x,y) = 0 \quad \text{for all } x, y \in V_{n+1}, \tag{7}$$

and the *radical* $\mathrm{rad}\, T$ of T is defined to be the flat in $\mathrm{PG}(n,2)$ consisting of all the T-singular points. Also a line $\langle a, b \rangle$ in $\mathrm{PG}(n,2)$ is termed *T-singular* whenever

$$T(a,b,x) = 0 \quad \text{for all } x \in V_{n+1}, \tag{8}$$

and the set of all T-singular lines in $\mathrm{PG}(n,2)$ will be denoted $\mathcal{L}_T$. Further, for $r > 1$, an r-flat will be termed *T-singular* if it contains one or more singular lines, and *T-non-singular* if it contains no singular lines. Under the isomorphic correspondence $t \longleftrightarrow T$, we also write $\mathrm{rad}\, t = \mathrm{rad}\, T$ and $\mathcal{L}_t = \mathcal{L}_T$.

1.3. Specialization to $\mathrm{PG}(5,2)$

We now specialize further and confine attention to projective dimension $n = 5$. However before embarking upon $d = 3$ considerations in $\mathrm{PG}(5,2) = \mathbb{P}V_6$, it may help to first remind ourselves of certain well-known $d = 2$ results. Upon polarization, a quadratic form $Q \in F_2$ gives rise to an element $B \in \mathrm{Alt}(\times^2 V_6)$, and so to an isomorphic element $b \in \wedge^2 V_6^*$. Now the classification of the 2^{15} bivectors $b \in \wedge^2 V_6^*$ into $\mathrm{GL}(6,2)$-orbits is well known: the nonzero elements of $\wedge^2 V_6^*$ fall into three $\mathrm{GL}(6,2)$-orbits, as represented by the (dual) bivectors

$$b_1 = f_{12}, \quad b_2 = f_{12} + f_{34}, \quad b_3 = f_{12} + f_{34} + f_{56}, \quad \text{where } f_{ij} := f_i \wedge f_j. \tag{9}$$

If for a given $B \in \mathrm{Alt}(\times^2 V_6)$ we define $\mathrm{rad}\, B = \mathrm{rad}\, b$ to be the flat consisting of all the points $a \in \mathrm{PG}(5,2)$ which satisfy $B(a,x) = 0$ for all $x \in V_6$, then the main interest is in the third of the foregoing orbits, since it is the only one having $\mathrm{rad}\, B = \emptyset$. In this case, $B_3 \in \mathrm{Alt}(\times^2 V_6)$ is non-degenerate and has stabilizer $\mathcal{G}_{B_3} \cong \mathrm{Sp}(6,2)$. In this $d = 2$ situation it is a simple task to classify the quadratic forms. In particular, the 64 quadratic forms Q such that $B_Q = B_3$ fall into just two $\mathrm{Sp}(6,2)$-orbits, comprising a family $\mathcal{H}$ of 36 hyperbolic quadrics and a family $\mathcal{E}$ of 28 elliptic quadrics; see for example [5, Lemma 20.7.2], [9, Section 2.2]. Consequently, the non-degenerate quadrics in $\mathrm{PG}(5,2)$ fall into two $\mathrm{GL}(6,2)$-orbits of lengths $36|\delta_3|$ and $28|\delta_3|$, where $|\delta_3| = |\mathrm{GL}(6,2)|/|\mathrm{Sp}(6,2)| = 13888$.

Returning now to $d = 3$ considerations, it is no doubt a daunting prospect to contemplate carrying out a study of all the $2^{41} - 2^{21}$ cubic hypersurfaces ψ in $\mathrm{PG}(5,2)$. Nevertheless, a first step is the appreciably simpler task of classifying the 2^{20} elements $T \in \mathrm{Alt}(\times^3 V_6) \cong \wedge^3 V_6^*$ which arise, as in (3), from the polarization of cubic forms Q on V_6. This simpler task is accomplished in Sect. 2, where we determine the decomposition of $\mathrm{Alt}(\times^3 V_6) \cong \wedge^3 V_6^*$ into distinct $\mathrm{GL}(6,2)$-orbits.

Before proceeding further, it is perhaps worth mentioning that as well as the linear isomorphism $t \mapsto T$ of $\wedge^3 V_6^*$ with $\mathrm{Alt}(\times^3 V_6)$, we also have the Poincaré isomorphism $t \mapsto \perp t$ of $\wedge^3 V_6^*$ with $\wedge^3 V_6$. For a general account of the Poincaré isomorphisms see, for example, [6, Section 9.6]. Fortunately, we are working over $\mathrm{GF}(2)$ and so can take advantage of certain simplifying aspects. For since the space $\wedge^6 V_6$ has a unique nonzero element, the Poincaré isomorphisms.$\wedge^r V_6^* \cong \wedge^{6-r} V_6$ are unique. Moreover, since $-1 = 1$, we are free from sign intricacies. In particular, if $\{f_1, \ldots, f_6\}$ is the basis $\mathcal{B}^*$ for V_6^* which is dual to a basis $\mathcal{B} = \{e_1, \ldots, e_6\}$ for V_6, then the isomorphism $\perp \colon \wedge^3 V_6^* \to \wedge^3 V_6$ has the following simple form on basis trivectors:

$$\perp f_{ijk} = e_{lmn}, \qquad \text{where } f_{ijk} := f_i \wedge f_j \wedge f_k, \quad e_{lmn} := e_l \wedge e_m \wedge e_n, \tag{10}$$

this holding for any permutation $ijklmn$ of 123456. The previous definitions of T-singular points and lines can be rephrased in terms of $\perp t$ as follows: a point $a \in \mathrm{PG}(5,2)$ is T-singular if it satisfies $(\perp t) \wedge a = 0$ and a line $\langle a, b\rangle$ is T-singular whenever $(\perp t) \wedge a \wedge b = 0$.

2. The five orbits $\Delta_1, \ldots, \Delta_5$ of trivectors

In [13], the author studied five particular cubic hypersurfaces $Q(x) = 0$ in $\mathrm{PG}(5,2)$ which arose from partial spreads of planes. The five alternating forms $T = T_Q \in \mathrm{Alt}(\times^3 V_6)$ determined by these five cubics Q gave rise to five sets $\mathcal{L}_T$ of T-singular lines, and these are seen to be projectively distinct; indeed the five cardinalities $|\mathcal{L}_T|$ are distinct. Consequently, the nonzero elements of $\mathrm{Alt}(\times^3 V_6) \cong \wedge^3 V_6^*$ belong to *at least five different* $\mathrm{GL}(6,2)$*-orbits* $\Delta_1, \ldots, \Delta_5$. These orbits are described in Tables 1 and 2 below, and in the subsequent Sect. 2.1 – 2.5. When dealing with a representative t_i of the orbit Δ_i, we write $\mathcal{L}_i$ for $\mathcal{L}_{t_i} (= \mathcal{L}_{T_i})$ and $\mathcal{G}_i$ for the stabilizer $\mathcal{G}_{t_i} (= \mathcal{G}_{T_i})$. The numbering in the tables differs from that in [13]: in the order in which the cubics appear in [13, Sections 2.2 – 2.6] the cardinalities $|\mathcal{L}_T|$ are 203, 49, 21, 35 and 91.

TABLE 1 the five orbits $\Delta_1, \ldots, \Delta_5$ of trivectors $\in \wedge^3 V_6^*$

Orbit	Representative $t_i \in \Delta_i$	$\lvert\mathcal{L}_i\rvert$	$\mathrm{rad}\, t_i$
Δ_1	$t_1 = f_{123}$	203	$\langle e_4, e_5, e_6\rangle$
Δ_2	$t_2 = f_{123} + f_{145}$	91	$\langle e_6\rangle$
Δ_3	$t_3 = f_{123} + f_{456}$	49	$\emptyset$
Δ_4	$t_4 = f_{234} + f_{135} + f_{126}$	35	$\emptyset$
Δ_5	$t_5 = f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345}$	21	$\emptyset$

Table 2 the five stabilizers $\mathcal{G}_1, \ldots, \mathcal{G}_5$

Orbit	Stabilizer $\mathcal{G}_i$	$\lvert\mathcal{G}_i\rvert$	$\lvert\Delta_i\rvert$
Δ_1	$2^9 : (\mathrm{GL}(3,2) \times \mathrm{GL}(3,2))$	$2^{15}.3^2.7^2$	1395
Δ_2	$2^9 : \mathrm{Sp}(4,2)$	$2^{13}.3^2.5$	54684
Δ_3	$(\mathrm{GL}(3,2) \times \mathrm{GL}(3,2)).2$	$2^7.3^2.7^2$	357120
Δ_4	$2^8 : \mathrm{GL}(3,2)$	$2^{11}.3.7$	468720
Δ_5	$\mathrm{SL}(3,4).2$	$2^7.3^3.5.7$	166656

Of course, the orbit lengths in the fourth column of Table 2 follow from the stabilizer orders in the third column: $|\Delta_i| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}_i|$, where

$$|\,\mathrm{GL}(6,2)| = 2^{15}.3^4.5.7^2.31 = 20158709760. \tag{11}$$

Remark 2. By (10), the five $\mathrm{GL}(6,2)$-orbits of trivectors $t \in \wedge^3 V_6^*$ in Table 1 give rise to five $\mathrm{GL}(6,2)$-orbits of trivectors $\perp t \in \wedge^3 V_6$ with representatives

$$\begin{aligned} \perp t_1 &= e_{456}, \qquad \perp t_2 = e_{456} + e_{236}, \\ \perp t_3 &= e_{456} + e_{123}, \qquad \perp t_4 = e_{156} + e_{246} + e_{345}, \\ \perp t_5 &= e_{156} + e_{246} + e_{345} + e_{234} + e_{135} + e_{126}. \end{aligned} \tag{12}$$

2.1. The orbit Δ_1

For the trivector $t_1 = f_{123}$, we have $T_1(a, y, z) = |ayz|_{123}$, see (4), whence $\operatorname{rad} t_1 = \langle e_4, e_5, e_6\rangle$. The T_1-singular lines $\mathcal{L}_1$ consist of the 7 lines of the plane $\operatorname{rad} t_1$ together with the $7 \times 28 = 196$ lines which meet $\operatorname{rad} t_1$ in a point; so $|\mathcal{L}_1| = 203$. In terms of $V_6 = \prec e_1, e_2, e_3 \succ \oplus \prec e_4, e_5, e_6 \succ$ the stabilizer $\mathcal{G}_1$ consists of all elements $A \in \mathrm{GL}(6,2)$ of the $(3+3)$-block form

$$A = \begin{pmatrix} A_1 & O \\ B & A_2 \end{pmatrix}, \quad A_i \in \mathrm{GL}(3,2), B \in (\mathbb{F}_2)^{3\times 3}. \tag{13}$$

So $\mathcal{G}_1 \cong 2^9 : (\mathrm{GL}(3,2) \times \mathrm{GL}(3,2))$, of order $|\mathcal{G}_1| = 2^{15}.3^2.7^2$, whence

$$|\Delta_1| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}_1| = 3^2.5.31 = 1395. \tag{14}$$

2.2. The orbit Δ_2

Since for the trivector $t_2 = f_{123} + f_{145}$ we have

$$T_2(a, y, z) = |ayz|_{123} + |ayz|_{145}, \tag{15}$$

it follows that $a = e_6$ is the sole T_2-singular point: $\operatorname{rad} t_2 = \{e_6\}$. (Equivalently $a = e_6$ is the only point satisfying $(\perp t_2) \wedge a = 0$.) Further, observe that t_2 determines a unique point in $\mathrm{PG}(5,2)^*$, namely f_1, since f_1 is the only non-zero element $f \in V_6^*$ satisfying $t_2 \wedge f = 0$. So not only does t_2 determine the $\mathcal{G}_2$-fixed point e_6 in $\mathrm{PG}(5,2)$, but also determines the $\mathcal{G}_2$-invariant hyperplane $H = \langle e_2, e_3, e_4, e_5, e_6\rangle$, with equation $f_1(x) = 0$. The lines of $\mathrm{PG}(5,2)$ can be classed into three kinds. First there are the 31 lines which contain e_6, all therefore being T_2-singular. Secondly there are the lines which meet H in a point other than e_6; from (15) we see that none of these are T_2-singular. Thirdly consider lines $L \subset H$ which do not contain e_6. For such a line $L = \langle a, b\rangle$ we have, since $a_1 = b_1 = 0$,

$$T(a,b,x) = B(a,b)x_1, \qquad \text{where } B(a,b) := T(a,b,e_1). \tag{16}$$

Upon restriction to the solid $D = \langle e_2, e_3, e_4, e_5\rangle$ the alternating bilinear form B is non-degenerate and so D is thereby equipped with $\mathrm{Sp}(4,2)$-geometry. Consequently precisely 15 lines $L = \langle a, b\rangle$ in D are self-polar, satisfying $B(a,b) = 0$, and hence by (16) are T_2-singular. These 15 self-polar lines $L \subset D$ give rise to 15 planes $P = \langle e_6, L\rangle$, each line of P being T_2-singular. These 15 planes $P = \langle e_6, L\rangle$ thus contribute $15 \times 4 = 60$ lines of the third kind which are T_2-singular. Hence $|\mathcal{L}_2| = 31 + 0 + 60 = 91$.

In terms of the basis $\mathcal{B} = \{e_1, \ldots, e_6\}$, it follows from the foregoing that elements $A \in \mathrm{GL}(V_6)$ which fix t_2 are those with matrices of the $(1+4+1)$-block form

$$A = \begin{pmatrix} 1 & O & 0 \\ a^t & C & O \\ \alpha & b & 1 \end{pmatrix}, \qquad a, b \in \mathrm{GF}(2)^4,\ \alpha \in \mathrm{GF}(2), \tag{17}$$

with the 4×4 matrix C any element of $\mathrm{Sp}(4,2)$. So $\mathcal{G}_2 \cong 2^9 : \mathrm{Sp}(4,2)$, of order $|\mathcal{G}_2| = 2^9.720 = 2^{13}.3^2.5$, whence

$$|\Delta_2| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}_2| = 2^2.3^2.7^2.31 = 54684. \tag{18}$$

2.3. The orbit Δ_3

If $t_3 = f_{123} + f_{456}$, and so $\perp t_3 = e_{456} + e_{123}$, then a line L is T_3-singular if and only if L meets each of the planes $\langle e_1, e_2, e_3\rangle$ and $\langle e_4, e_5, e_6\rangle$ in a point. So $\mathrm{rad}\, T_3 = \emptyset$ and $|\mathcal{L}_3| = 7 \times 7 = 49$. The stabilizer $\mathcal{G}_3$ clearly contains all elements $A_1 \oplus A_2$, $A_i \in \mathrm{GL}(3,2)$, acting upon $V_6 = \prec e_1, e_2, e_3 \succ \oplus \prec e_4, e_5, e_6 \succ$. But also $\mathcal{G}_3$ contains the involution $e_1 \rightleftarrows e_4, e_2 \rightleftarrows e_5, e_3 \rightleftarrows e_6$. So $\mathcal{G}_3 \cong (\mathrm{GL}(3,2) \times \mathrm{GL}(3,2)).2$, of order $|\mathcal{G}_3| = 2^7.3^2.7^2$, whence

$$|\Delta_3| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}_3| = 2^8.3^2.5.31 = 357120. \tag{19}$$

2.4. The orbit Δ_4

For the trivector $t_4 = f_{234} + f_{135} + f_{126}$ we have

$$T_4(a,y,z) = |ayz|_{234} + |ayz|_{135} + |ayz|_{126}, \tag{20}$$

from which it quickly follows that $\mathrm{rad}\, T_4 = \emptyset$. Observe that $T_4(e_i, e_j, \cdot) = 0$ for $ij = 45, 46, 56$. *Every line in the plane $Y := \langle e_4, e_5, e_6\rangle$ is therefore T_4-singular.* Such a plane will be termed *totally singular.* No other plane is totally T_4-singular and so any element $A \in \mathcal{G}_4$ must preserve $Y = \langle e_4, e_5, e_6\rangle$; also the contragredient $\hat{A}$ must preserve the annihilator $Y^{\mathrm{O}} = \langle f_1, f_2, f_3\rangle$ of Y. Any element $A \in \mathcal{G}_4$ thus has a matrix of the $(3+3)$-block form (21). In fact we can show that $\mathcal{G}_4$ contains a copy of $\mathrm{GL}(3,2)$. For consider the element $\hat{C} \in \mathrm{GL}(V_6^*)$, of order 7, with effect

$$\hat{C} : f_1 \mapsto f_2,\ f_2 \mapsto f_3,\ f_3 \mapsto f_1 + f_2, \qquad f_4 \mapsto f_5,\ f_5 \mapsto f_6,\ f_6 \mapsto f_4 + f_5.$$

Note that $\wedge^3\hat{C}$ effects $f_{126} \mapsto f_{234} + f_{235}$, $f_{135} \mapsto f_{126}$, $f_{234} \mapsto f_{135} + f_{235}$, whence $(\wedge^3\hat{C})t_4 = t_4$ and so $C \in \mathcal{G}_4$. Consider also the involution $\hat{K} \in \mathrm{GL}(V_6^*)$ with effect

$$\hat{K} : f_1 \rightleftarrows f_2,\ f_3 \mapsto f_3, \qquad f_4 \rightleftarrows f_5,\ f_6 \mapsto f_6.$$

Since $\wedge^3\hat{K}$ effects $f_{126} \mapsto f_{126}$ and $f_{135} \rightleftarrows f_{234}$ then also $K \in \mathcal{G}_4$. So $\mathcal{G}_4$ contains the subgroup $\langle C, K\rangle \cong \mathrm{GL}(3, 2)$.

We claim that $\mathcal{G}_4$ consists precisely of all elements of $\mathrm{GL}(V_6)$ whose matrices (with respect to the basis $\mathcal{B}$) are of the $(3 + 3)$-block form

$$A = \begin{pmatrix} A_1 & O \\ B & A_1 \end{pmatrix}, \quad A_1 \in \mathrm{GL}(3, 2), B \in (\mathbb{F}_2)^{3\times 3},\ \mathrm{tr}\, B = 0. \tag{21}$$

To justify this claim first consider the involution $L_B \in \mathrm{GL}(V_6)$ obtained by setting $A_1 = I$ in (21). Now the contragredient $\hat{L}_B$ of L_B fixes f_i for $i = 1, 2, 3$ and effects $f_4 \mapsto f_4 + \Sigma_{j=1}^3 b_{1j} f_j$, $f_5 \mapsto f_5 + \Sigma_{j=1}^3 b_{2j} f_j$, $f_6 \mapsto f_6 + \Sigma_{j=1}^3 b_{3j} f_j$. So $\wedge^3\hat{L}_B$ effects

$$f_{234} \mapsto f_{234} + b_{11} f_{123}, \quad f_{135} \mapsto f_{135} + b_{22} f_{123}, \quad f_{126} \mapsto f_{126} + b_{33} f_{123},$$

and so preserves t_4 if and only if $\mathrm{tr}\, B = 0$. Hence $\mathcal{G}_4$ contains all the elements of the form (21). To see that $\mathcal{G}_4$ is no larger, it suffices to carry out the straightforward check that if $A \in \mathrm{GL}(V_6)$ has matrix $\begin{pmatrix} C & O \\ O & I \end{pmatrix}$ then $\wedge^3 A$ preserves $\perp t_4 = e_{156} + e_{246} + e_{345}$ if and only if $C = I$. Consequently, $\mathcal{G}_4 \cong 2^8 : \mathrm{GL}(3, 2)$, of order $|\mathcal{G}_4| = 2^{11}.3.7$, whence

$$|\Delta_4| = 2^4.3^2.5.31 = 468720. \tag{22}$$

In order to describe the T_4-singular lines $\mathcal{L}_4$ it helps to make use of the involution $J := L_I$, having matrix $\begin{pmatrix} I & O \\ I & I \end{pmatrix}$, which fixes each point of the plane Y and which effects the interchanges

$$J : e_1 \rightleftarrows e_1 + e_4,\ e_2 \rightleftarrows e_2 + e_5,\ e_3 \rightleftarrows e_3 + e_6.$$

Take note that J is the only element $\neq I$ of $\mathrm{GL}(V_6)$ which centralizes $\mathcal{G}_4$. We claim that $\mathcal{L}_4$ comprises:

(i) the 7 lines of the totally singular plane Y;

(ii) the 28 lines of the form $\langle a, Ja\rangle, a \notin Y$.

Since $\mathcal{G}_4$ is transitive on the 56 points $a \notin Y$ to prove our claim it suffices to check that $\langle e_1, e_4\rangle$ is the only T_4-singular line L which contains the point e_1. This is seen to be the case, since $T_4(e_1, b, x) = b_2x_6 + b_6x_2 + b_3x_5 + b_5x_3$ and so $T_4(e_1, b, x) = 0$ for all x if and only if $b \in \langle e_1, e_4\rangle$. So $|\mathcal{L}_4| = 7 + 28 = 35$.

2.5. The orbit Δ_5

The trivector $t_5 \in \Delta_5$ was studied in some detail in [16], where it was shown that $\mathcal{L}_5$ is a Desarguesian line-spread in $\mathrm{PG}(5, 2)$ (whence $\mathrm{rad}\, t_5 = \emptyset$), the 21 lines of the spread $\mathcal{L}_5$ being the orbits in $\mathrm{PG}(5,2)$ of the subgroup $\langle R\rangle \cong Z_3$ of $\mathrm{GL}(V_6)$, where

$$R : e_1 \mapsto e_4 \mapsto e_1 + e_4, \quad e_2 \mapsto e_5 \mapsto e_2 + e_5, \quad e_3 \mapsto e_6 \mapsto e_3 + e_6. \tag{23}$$

The spread $\mathcal{L}_5$ has for its stabilizer a subgroup $\mathcal{G}(\mathcal{L}_5) \cong \Gamma\,\mathrm{L}(3,4)$ of $\mathrm{GL}(V_6)$, and the stabilizer of t_5 was shown in [16] to be a subgroup $\mathcal{G}_5 \cong \mathrm{SL}(3,4).2$ of index 3 in $\mathcal{G}(\mathcal{L}_5)$. So $|\mathcal{G}_5| = 2^7.3^3, 5.7$, whence

$$|\Delta_5| = |\,\mathrm{GL}(6,2)|/|\mathcal{G}_5| = 2^8.3.7.31 = 166656. \tag{24}$$

Remark 3. It was shown in [16] that the same Desarguesian line-spread $\mathcal{L}_5$ occurs as the set of singular lines of precisely three elements $t_5, t_5', t_5'' \in \wedge^3 V_6^*$, namely

$$\begin{aligned} t_5 &= f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345},\\ t_5' &= f_{156} + f_{246} + f_{345} + f_{123} + f_{456},\\ t_5'' &= f_{234} + f_{135} + f_{126} + f_{123} + f_{456.}, \end{aligned} \tag{25}$$

So t_5' or t_5'' could replace t_5 in Table 1 as representative of the orbit Δ_5.

2.6. The main theorem

The number of nonzero elements of $\mathrm{Alt}(\times^3 V_6) \cong \wedge^3 V_6^*$ is $|\mathbb{P}(\wedge^3 V_6^*)| = 2^{20} - 1 = 1048575$. But note from Table 2 that $\sum_{i=1}^{5} |\Delta_i| = 1048575$. Consequently, we have proved the following theorem.

Theorem 4. *The nonzero elements of* $\mathrm{Alt}(\times^3 V_6) \cong \wedge^3 V_6^*$ *belong to precisely five* $\mathrm{GL}(6,2)$*-orbits* $\Delta_1, \ldots, \Delta_5$ *as set out in Tables 1 and 2 and as described in the subsequent Sect. 2.1 – 2.5.*

2.7. A conjecture

Reverting temporarily to general projective dimension n, we put forward the following conjecture.

Conjecture A. For any $n > 2$ and any $T \in \mathrm{Alt}(\times^3 V_{n+1})$ the set $\mathcal{L}_T$ of T-singular lines in $\mathrm{PG}(n,2)$ is non-empty.

We have seen that the conjecture holds in the case $n = 5$. Using the canonical forms t_1 and t_2 in Table 1 it is easy to check that the conjecture also holds for $n = 3$ and $n = 4$. The conjecture can be re-phrased in terms of a property of the linear map $C_T : \wedge^2 V_{n+1} \longrightarrow V_{n+1}^*$ whose effect on decomposable elements $a \wedge b \in \wedge^2 V_{n+1}$ is defined by

$$C_T : a \wedge b \longmapsto f, \quad \text{where} \quad f(x) = T(a,b,x). \tag{26}$$

(In terms of $\perp t$ this last reads $\perp f = (\perp t) \wedge a \wedge b$.) For observe that a line $L = \langle a, b\rangle$ is T-singular whenever its Grassmann image $a \wedge b$ belongs to $\ker(C_T)$. So Conjecture A is equivalent to the following conjecture.

Conjecture B. For any $n > 2$ and any $T \in \mathrm{Alt}(\times^3 V_{n+1})$ the Grassmannian $\mathcal{G}_{1,n,2} \subset \mathbb{P}(\wedge^2 V_{n+1})$ has non-empty intersection with $\mathbb{P}(\ker C_T)$.

Since $\dim(\operatorname{im} C_T) + \dim(\ker C_T) = \dim \wedge^2 V_{n+1} = \binom{n+1}{2}$, and since $\operatorname{im} C_T$ has dimension $\leq n+1$, it follows that the subspace $\ker C_T$ of $\wedge^2 V_{n+1}$ has dimension $\geq d_n$ where $d_n = \binom{n}{2} - 1$. In cases where $\dim(\ker C_T) > d_n$, that is when $\mathbb{P}(\ker C_T)$ is an r-flat for some $r \geq d_n$, then certainly $\mathcal{G}_{1,n,2} \cap \mathbb{P}(\ker C_T)$ is nonempty. For the Grassmannian $\mathcal{G}_{1,n,2}$ is known, see [4,14], to have polynomial degree d_n, and consequently, see [8], any r-flat with $r \geq d_n$ meets $\mathcal{G}_{1,n,2}$ in a nonzero (in fact odd) number of points. Unfortunately this argument fails in the cases where $\operatorname{im} C_T = V^*_{n+1}$, and so $\dim(\ker C_T) = d_n$, since, see [2], there exist $(d_n - 1)$-flats in $\mathbb{P}(\wedge^2 V_{n+1})$ which are external to $\mathcal{G}_{1,n,2}$.

3. The five families of cubics in $\mathrm{PG}(5, 2)$

It follows from Theorem 4 that the $2^{21}(2^{20} - 1)$ cubic hypersurfaces $\psi : Q = 0$ in $\mathrm{PG}(5, 2)$ can be classified into five large families $\mathcal{F}_1, \ldots, \mathcal{F}_5$, with Q, or ψ, belonging to $\mathcal{F}_i$ if $t_Q \in \Delta_i$. Ideally, for each family $\mathcal{F}_i$ one would like to classify the 2^{21} cubics Q such that $t_Q = t_i \in \Delta_i$ into $\mathcal{G}_{t_i}$-orbits. However — in contrast to the $d = 2$ result that there are just the two $\mathrm{Sp}(6, 2)$-orbits $\mathcal{H}$ and $\mathcal{E}$ of quadratic forms which arise from the bivector t_3 in Eq. (25) — in our present $d = 3$ situation there are very many $\mathcal{G}_i$-orbits of cubic forms. Indeed it will probably need computer assistance to achieve a complete classification of all the cubic hypersurfaces in $\mathrm{PG}(5, 2)$.

Many of the cubic hypersurfaces $\psi : Q = 0$ belonging to a family $\mathcal{F}_i$ will no doubt have as their stabilizer $\mathcal{G}_Q (= \mathcal{G}_\psi)$ a fairly small subgroup of $\mathcal{G}_i$, and so perhaps do not justify much attention. It would seem more rewarding to seek out and study those cubic hypersurfaces in a family $\mathcal{F}_i$ which have appreciable symmetry, with $\mathcal{G}_\psi$ being a sizeable subgroup of $\mathcal{G}_i$. See [13, Sections 2.2 – 2.6] for consideration of one such cubic hypersurface in the case of each of the five families $\mathcal{F}_i$.

In the study of a particular cubic $Q \in F_i$ one confidently expects that the properties of its trivector $t_Q \in \Delta_i$, as described in Sect. 2, will play a significant role. For example, consider the case where the cubic hypersurface $Q = 0$ is the 35-set $\psi \subset \mathrm{PG}(5, 2)$ supporting a *non-maximal* partial spread Σ_5 of five planes in $\mathrm{PG}(5, 2)$. Here t_Q is seen to belong to the family $\mathcal{F}_4$, and so, see Sect. 2.4, there exists a unique totally singular plane P. As noted in [13, §2.5], this plane is in fact one of the planes of Σ_5, being the *privileged plane* of Σ_5, see [11, Theorem 4.3] and [12, Theorem 4.2]), with the property that each of the seven internal planes $\notin \Sigma_5$ of ψ meets P in a line and meets each of the four other planes $\in \Sigma_5$ in a point.

As another example, consider the case where the cubic hypersurface $\psi \subset \mathrm{PG}(5, 2)$ is the 35-set supporting a *maximal* partial spread $\Sigma_5 = \{P_i\}_{i \in \{1,2,3,4,5\}}$ of five planes in $\mathrm{PG}(5, 2)$. Now such a set ψ also supports a second maximal partial spread $\Sigma'_5 = \{P'_i\}_{i \in \{1,2,3,4,5\}}$ of planes such that $L_i := P_i \cap P'_i$ is a line for $i = 1, \ldots, 5$ and $P_i \cap P'_j$ is a point for $i \neq j$. The hypersurface ψ is accordingly called a *double-five of planes;* see [10] and [11] for more details, including a proof that ψ has a cubic equation $Q = 0$. Here t_Q can be seen to belong to the family $\mathcal{F}_2$, and so, see Sect. 2.2, there exists a unique T_Q-singular point a and a unique invariant hyperplane H, with $a \in H$. So one confidently expects that a and H will enter significantly in any study of a double-five ψ. Indeed one sees that H is the *even,* or *invariant,* hyperplane of ψ, see [11, Theorem 1.3], such that $H \cap \psi$ is a parabolic quadric $\mathcal{P}_4 \subset H$, with the five lines L_i forming a 1-spread for $\mathcal{P}_4$, Further the singular point a is the nucleus of the parabolic quadric $\mathcal{P}_4$.

Despite the great variety of cubic forms Q in $\mathrm{PG}(5, 2)$, those that belong to the same family $\mathcal{F}_i$ will share certain features arising from the fact that their completely polarized forms T_Q belong to the same orbit Δ_i. In particular, in the case of the family $\mathcal{F}_5$, all of its $N = 2^{21}|\Delta_5| = 349502963712$ members Q have a Desarguesian line-spread for its set $\mathcal{L}_{T_Q}$ of singular lines. Further, as pointed out in [16, Section 6(i)], for each of these N cubic hypersurfaces ψ any of the 630 solids D which contains just one line of the line-spread $\mathcal{L}_{T_\psi}$ possesses a distinguished point n_D with the property that all planes $P \subset D$ which contain n_D meet ψ in an odd number of points.

One particular choice of cubic hypersurface belonging to the family $\mathcal{F}_5$ is that of a Segre variety $\psi = \mathcal{S}_{1,2}$; see [15, Eq.(1.10)] for its cubic equation $Q = 0$. The cubic Q has appreciable symmetry, its stabilizer $\mathcal{G}_Q$ being isomorphic to $\mathrm{GL}(2, 2) \times \mathrm{GL}(3, 2)$ and so of order $6 \times 168 = 1008$. However there is at least one other member of $\mathcal{F}_5$ that has more symmetry. For consider that hypersurface ψ in $\mathrm{PG}(5, 2)$ whose equation is $Q(x) = 0$ where Q is *precisely*

$$Q(x) = x_2x_3x_4 + x_1x_3x_5 + x_1x_2x_6 + x_1x_5x_6 + x_2x_4x_6 + x_3x_4x_5, \tag{27}$$

with (in the basis $\mathcal{B}$) *no linear or quadratic terms.* One finds that ψ is a 45-set in $\mathrm{PG}(5, 2)$ consisting of the points on a certain fifteen of the lines of the Desarguesian line-spread $\mathcal{L} := \mathcal{L}_{T_Q}$, these fifteen lines in fact arising from the complement of a 6-arc in $\mathrm{PG}(2, 4)$, whence $\mathcal{G}_Q \cong 3.\,\mathrm{Sym}(6)$ is of order 2160.

4. Postscript

Shortly after the foregoing work was completed the author discovered the paper [1] which is in many ways very much more general than the present paper. For in it the authors, Cohen and Helminck, deal with a vector space E of dimension 7 as compared to dimension 6 of the present paper, and moreover succeed in determining the $\mathrm{GL}(E)$-orbits in $\mathrm{Alt}(\times^3 E)$ for very many choices of base field $\mathbb{F}$; in particular, see [1, Table 1], for any perfect field $\mathbb{F}$, and so for any algebraically closed field and for any finite field. The author was at first perturbed on reading, see [1, page 2], that for any finite field $\mathbb{F}$ "… the number of projective $\mathrm{GL}(n, \mathbb{F})$-orbits on $\wedge^3\mathbb{F}^n$ is 4 if $n = 6$ …" However, upon contacting Cohen, it emerged that the 4 here is a misprint for 5, the 5 canonical forms for $f \in \mathrm{Alt}(\times^3 E), \dim E = 6$, being the forms f_1, f_2, f_3, f_4 and $f_{10,\lambda}$ in [1, Table 1]. In the case $\mathbb{F} = \mathrm{GF}(2)$ these $\dim E = 6$ results in [1, Table 1] agree with the results in Table 1 of the present paper, with $f_i = t_i,\ i = 1, 2, 3, 4$ and

$f_{10,1} = t_5$. (Incidentally there is also a misprint in [1, Table 2]: the denominator in the result for $|G/G_{f_1}|$ should be $(q^2 - 1)$ and not $(q-1)^2$.)

Despite it being much more specialized than [1], it is hoped that the present paper nevertheless satisfies a need. For the results, in Sect. 2 above, for the five GL(6, 2)-orbits in $\mathrm{Alt}(\times^3 V_{6,2})$ are more detailed, and more easily accessible, than those in [1]. In particular the present paper contains a full description of the sets $\mathcal{L}_T$ of T-singular lines for each of the five orbits. Further the specialized nature of the present paper stems from an interest in cubic hypersurfaces in PG(n, 2): see Sect. 3 above, and also [13, Section 2] and [15]. But the tie-in of cubics with trivectors is by way of the isomorphism $F_3/F_2 \cong \wedge^3 V^*_{n+1}$ of Theorem 1, *and this isomorphism is peculiar to the field* GF(2).

Fortuitously, the discovery of a misprint in [1] proved fruitful! — since Cohen quickly produced a simple proof that if $V(n+1, \mathbb{F})$ has even dimension then Conjecture A in Sect. 2.7 holds for any field $\mathbb{F}$. Moreover it holds in a powerful manner, in that through any projective point in PG(n, $\mathbb{F}$) there passes at least one singular line. Furthermore, the case of an odd-dimensional space V was taken up by Jan Draisma, a colleague of Cohen's at Eindhoven, who, by use of a beautiful equivariant map from alternating trilinear forms T on $V(2m+1, \mathbb{F}), m > 1$, to polynomials of degree $m-1$, proved that the conjecture holds for any quasi-algebraically field $\mathbb{F}$ (and in particular for any finite field $\mathbb{F}$); see [3, Theorem 1.1]. However, see [3, Example 1.2], the conjecture does not hold for a certain exceptional alternating trilinear form T on a real space $V(7, \mathbb{R})$, the form T being related to the composition algebra $\mathbb{O}$ of the real octonions.

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