Trivectors yielding spreads in $\mathrm{PG}(5,2)$

Ron Shaw

Abstract

Given an alternating trilinear form $T \in \mathrm{Alt}(\times^3 V_6)$ on $V_6 = V(6,2)$ let $\mathcal{L}_T$ denote the set of those lines $\langle a, b\rangle$ in $\mathrm{PG}(5,2) = \mathbb{P}V_6$ which are T-singular, satisfying, that is, $T(a,b,x) = 0$ for all $x \in \mathrm{PG}(5,2)$. If $\mathcal{L}_{21}$ is a Desarguesian line-spread in $\mathrm{PG}(5,2)$ it is shown that $\mathcal{L}_T = \mathcal{L}_{21}$ for precisely three choices T_1, T_2, T_3 of T, which moreover satisfy $T_1 + T_2 + T_3 = 0$. For $T \in \mathcal{T} := \{T_1, T_2, T_3\}$ the $\mathcal{G}_T$-orbits of flats in $\mathrm{PG}(5,2)$ are determined, where $\mathcal{G}_T \cong \mathrm{SL}(3,4).2$ denotes the stabilizer of T under the action of $\mathrm{GL}(6,2)$. Further, for a representative U of each $\mathcal{G}_T$-orbit, the T-associate $U^\#$ is also determined, where by definition

$$U^\# = \{v \in \mathrm{PG}(5,2) \,|\, T(u_1, u_2, v) = 0 \text{ for all } u_1, u_2 \in U\}.$$

Keywords trivectors over GF(2), Desarguesian spread, cubic hypersurfaces

MSC 2000: 51E20, 15A75

1 Preliminaries: alternating trilinear forms over $\mathrm{GF}(2)$ and cubics

For most of the time we will be working over $\mathrm{GF}(2)$, and so we may then identify the nonzero elements of a vector space $V(n+1,2) = V_{n+1}$ with the points S_0 of the associated projective space $PV_{n+1} = \mathrm{PG}(n,2)$. Similarly nonzero elements of the dual vector space V^*_{n+1} will be identified with the points of the dual projective space $\mathbb{P}V^*_{n+1} = \mathrm{PG}(n,2)^*$. We use $\prec u, v, \ \ldots \succ$ for the vector subspace spanned by vectors $u, v, \ldots$, and $\langle u, v, \ \ldots \ \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$. The vector space $F(S_0)$ of all functions $S_0 \to \mathrm{GF}(2)$ is of dimension $|S_0| = 2^{n+1} - 1$. Given a choice of coordinates $x_1, x_2, \ldots, x_{n+1}$ in V_{n+1}, there are $\binom{n+1}{r}$ monomials $\{x_{i_1} x_{i_2} \ldots x_{i_r}\}_{1 \le i_1 < \ldots < i_r \le n+1}$ of (reduced) degree r. So altogether we have $\Sigma_{r=1}^{n+1} \binom{n+1}{r} = |S_0|$ linearly independent monomials, and so these form a basis for $F(S_0)$. Given a point-set $\psi \subset \mathrm{PG}(n,2)$ it follows that it has equation $Q(x) = 0$ for some *uniquely determined* polynomial Q of minimal degree on V_{n+1} and satisfying $Q(0) = 0$. Briefly stated, *every point-set* ψ *of* $\mathrm{PG}(n,2)$ *is a hypersurface.* The (reduced) degree $d = \deg Q$ of Q is the *polynomial degree* of the point-set ψ.

Note that if $F_r = F_r(S_0)$, $r > 0$, denotes the subspace of $F(S_0)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \ ..., x_{n+1})$ with $\deg f \leq r$ and $f(0) = 0$, then the subspaces F_r are nested:

$$F_1 \subset F_2 \subset \ ... \ \subset F_n \subset F_{n+1} = F(S_0), \tag{1}$$

with $\dim F_r = \Sigma_{i=1}^{r}\binom{n+1}{i}$.

In fact *we will chiefly be concerned with the case* $(n, d) = (5, 3)$, that is with cubic hypersurfaces ψ in $\mathrm{PG}(5,2)$, and so with the subspaces F_1, F_2, F_3 of functions on $\mathrm{PG}(5,2)$, these being of dimensions $6, 21, 41$, respectively.

1.1 Alternating trilinear forms $\times^3 V_6 \to \mathrm{GF}(2)$

Given a cubic hypersurface ψ in $\mathrm{PG}(5,2)$, of equation $Q(x) = 0, \ x \in V_6$, where $\deg Q = 3$, then points, lines, ... , which lie inside ψ will be called *internal* points, lines, ... , and those which lie in ψ^c are termed *external.* Further a projective flat U of $\mathrm{PG}(5,2)$ will be termed ψ-odd whenever $|U \cap \psi|$ is odd and ψ-even whenever $|U \cap \psi|$ is even. If a particular ψ has been agreed then we omit the "ψ-". Upon completely polarizing the cubic polynomial Q we obtain an alternating trilinear form $T_Q : \times^3 V_6 \to \mathrm{GF}(2)$. where $T_Q (= T_\psi)$ is given, for $v_i \in V_6$, by

$$T_Q(v_1, v_2, v_3) = \textstyle\sum_{i=1}^{3} Q(v_i) + \sum_{1 \leq i < j \leq 3} Q(v_i + v_j) + Q(v_1 + v_2 + v_3). \tag{2}$$

It follows immediately from (2) that the following lemma holds.

Lemma 1 *(i) If the points v_1, v_2, v_3 are dependent then $T_Q(v_1, v_2, v_3) = 0$.*

(ii) If the points v_1, v_2, v_3 are independent, generating a projective plane P, then

$$T_Q(v_1, v_2, v_3) = \textstyle\sum_{v \in P} Q(v), \tag{3}$$

and $T_Q(v_1, v_2, v_3) = 0$, or $= 1$, according as the plane P is ψ-odd, or ψ-even.

Let $\mathrm{Alt}(\times^3 V_6)$ denote the vector space, of dimension $\binom{6}{3} = 20$, consisting of all the alternating trilinear forms $\times^3 V_6 \to \mathrm{GF}(2)$. Observe that each line $L = \langle a, b \rangle$ of $\mathrm{PG}(5,2)$ determines, for a given $T \in \mathrm{Alt}(\times^3 V_6)$, an associated linear form $f_L \in V_6^*$ defined by

$$f_L(x) = T(a, b, x), \quad x \in V_6. \tag{4}$$

The form f_L is well-defined: since T is alternating and trilinear, any choice of independent points $a, b \in L$ yields the same linear form. Given any nonzero element $T \in \mathrm{Alt}(\times^3 V_6)$ we make the following definitions.

Definition 2 *(i) A point $a \in \mathrm{PG}(5,2)$ is* T-singular *if it satisfies*

$$T(a, x, y) = 0 \qquad \textit{for all } x, y \in V_6. \tag{5}$$

The radical $\mathrm{rad}\, T$ *of T is the flat consisting of all the T-singular points.*

(ii) A line $\langle a, b\rangle$ in $\mathrm{PG}(5,2)$ is *T-singular* if

$$T(a,b,x)=0 \qquad \text{for all } x\in V_6. \tag{6}$$

The set of all T-singular lines in $\mathrm{PG}(5,2)$ will be denoted $\mathcal{L}_T$. So a line $L\in\mathcal{L}_T$ whenever f_L is the zero form.

(iii) For $r\geq 1$, an r-flat will be termed *T-singular* if it contains one or more singular lines, and *T-non-singular* if it contains no singular lines.

It is important to note that many different polynomials Q give rise to the same alternating form $T=T_Q$. *Relative to a choice of coordinates* $x_1, x_2, \dots, x_6$ in V_6 we have a direct sum decomposition $F_3=Y_3\oplus F_2$ where Y_3 is the 20-dimensional subspace spanned by the 20 monomials $\{x_ix_jx_k\}_{1\leq i<j<k\leq 6}$. Then for $0\neq Q\in Y_3$ we have $T_{Q+Q'}=T_Q$ for any element $Q'\in F_2$. So $(|F_2|=)\ 2^{21}$ different cubic polynomials Q share the same alternating form T. In the direct sum decomposition $F_3=Y_3\oplus F_2$ it should be stressed that there is no basis-independent choice of a complementary subspace Y_3 to the subspace F_2 of F_3. In contrast, we now deal with some natural linear isomorphisms surrounding T_Q and the property $T_{Q+Q'}=T_Q$, for $Q'\in F_2$.

1.2 The linear isomorphisms $F_3/F_2\cong\wedge^3V_6^*\cong\mathrm{Alt}(\times^3V_6)$

In the next theorem $Q+F_2$ is a general element of the quotient space F_3/F_2 and T_Q is as previously in (2). Also t_Q is defined to be that element of $\wedge^3V_6^*$ such that if Q is the product $f_1f_2f_3$ of 3 linear forms $f_i\in V_6^*$ then $t_Q=f_1\wedge f_2\wedge f_3$.

Theorem 3 *There exist natural linear isomorphisms*

$$F_3/F_2\cong\wedge^3V_6^*\cong\mathrm{Alt}(\times^3V_6), \tag{7}$$

giving rise, for $Q\in F_3$, to the bijective correspondences

$$Q+F_2\longleftrightarrow t_Q\longleftrightarrow T_Q. \tag{8}$$

Proof. This is the particular case $d=3,\ n=5$ of a theorem in [7, Theorem 1.6]; *cf.* also [5, Theorem B]. ■

Remark 4 *The linear isomorphism $\theta:\wedge^3V_6^*\to F_3/F_2$ is such that*

$$\theta(f_1\wedge f_2\wedge f_3)=f_1f_2f_3+F_2, \qquad f_i\in V_6^*. \tag{9}$$

Concerning the isomorphism of $t\in\wedge^3V_6^*$ with $T\in\mathrm{Alt}(\times^3V_6)$, if $\mathcal{B}^*=\{f_1,f_2,\dots,f_6\}$ is the basis for V_6^* dual to the basis $\mathcal{B}=\{e_1,e_2,\dots,e_6\}$ for V_6 then

$$t=\textstyle\sum_{1\leq i<j<k\leq 6}T(e_i,e_j,e_k)f_{ijk}, \quad \text{where } f_{ijk}:=f_i\wedge f_j\wedge f_k. \tag{10}$$

Equivalently expressed each $t\in\wedge^3V_6^*$ gives rise to an element $T\in\mathrm{Alt}(\times^3V_6)$ by way of

$$T(v_1,v_2,v_3)=<t|v_1\wedge v_2\wedge v_3>, \tag{11}$$

where $<\cdot|\cdot>$ is the standard determinantal pairing of $\wedge^3V_6^*$ with $\wedge^3V_6$, given on decomposable trivectors by $<f_1\wedge f_2\wedge f_3|v_1\wedge v_2\wedge v_3>=\det[f_i(v_j)]$.

From the foregoing it follows that if $Q \in F_3$ has the coordinate form $Q(x) = \Sigma_{i<j<k} c_{ijk} x_i x_j x_k + Q'$, $Q' \in F_2$, then

$$T_Q(x,y,z) = \sum_{1 \le i<j<k \le 6} c_{ijk} \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}, \quad \text{where } c_{ijk} = T_Q(e_i, e_j, e_k). \tag{12}$$

The natural action of $A \in \mathrm{GL}(V_6)$ upon $\mathrm{Alt}(\times^3 V_6)$ is by way of Λ_A where

$$(\Lambda_A T)(x,y,z) = T(A^{-1}x, A^{-1}y, A^{-1}z), \qquad T \in \mathrm{Alt}(\times^3 V_6). \tag{13}$$

If $\hat{A} \in \mathrm{GL}(V_6^*)$ denotes the contragredient of an element $A \in \mathrm{GL}(V_6)$, and so $(\hat{A}f)(Ax) = f(x)$ for all $f \in V_6^*,\ x \in V_6$, then the natural action of $A \in \mathrm{GL}(V_6)$ upon $\wedge^3 V_6^*$ is by way of $\wedge^3 \hat{A}$, with effect on decomposable trivectors

$$(\wedge^3 \hat{A}) f_1 \wedge f_2 \wedge f_3 = \hat{A} f_1 \wedge \hat{A} f_2 \wedge \hat{A} f_3, \qquad f_i \in V_6^*. \tag{14}$$

Under the isomorphic correspondence $T \longleftrightarrow t$ of Theorem 3 and Equations (10), (11) observe that $\Lambda_A T \longleftrightarrow (\wedge^3 \hat{A}) t$. Let $\mathcal{G}_\psi < \mathrm{GL}(6,2) = \mathrm{GL}(V_6)$ denote the stabilizer of the set ψ. We denote by $\mathcal{G}_T$ (or by $\mathcal{G}_t$) the subgroup of $\mathrm{GL}(6,2)$ which stabilizes T (or equivalently t):

$$\mathcal{G}_T = \{A \in \mathrm{GL}(6,2) | \Lambda_A T = T\} = \{A \in \mathrm{GL}(6,2) | (\wedge^3 \hat{A}) t = t\}. \tag{15}$$

Of course if ψ has cubic equation $Q(x) = 0$ then $\mathcal{G}_\psi$ is a subgroup of $\mathcal{G}_{T_Q}$.

2 The spread $\mathcal{L}_{21}$ and its trio of trivectors

2.1 The line-spread $\mathcal{L}_{21}$ in $\mathrm{PG}(5,2)$

Detailed information concerning the conjugacy classes of the group $\mathrm{GL}(6,2)$ can be found in [1], [2]. For our present purposes we consider a subgroup $\mathcal{Z} = \langle R \rangle \cong Z_3$ of $\mathrm{GL}(V_6) = \mathrm{GL}(6,2)$ generated by an element $R \in \mathrm{GL}(6,2)$ belonging to class 3A; see [1, Table 5a] or [2, Table 5a]. Such an element R is of the form $R = R_2 \oplus R_2 \oplus R_2$ with respect to a direct sum decomposition $V_6 = V_2 \oplus V_2 \oplus V_2$, where $R_2 \in \mathrm{GL}(V_2)$ is of order 3. Given a basis $\mathcal{B} = \{e_1, e_2, \ \dots\ , e_6\}$ for V_6 we will choose R to be that element of $\mathrm{GL}(6,2)$ given by

$$R: e_1 \mapsto e_4 \mapsto e_1 + e_4, \quad e_2 \mapsto e_5 \mapsto e_2 + e_5, \quad e_3 \mapsto e_6 \mapsto e_3 + e_6. \tag{16}$$

Also of use will also be the involution $J = J_2 \oplus J_2 \oplus J_2$ of class 2C in $\mathrm{GL}(6,2)$ given by

$$J: e_1 \rightleftarrows e_4, \quad e_2 \rightleftarrows e_5, \quad e_3 \rightleftarrows e_6. \tag{17}$$

Since $JRJ^{-1} = R^2$, note that J is an element of the normalizer $N(\mathcal{Z})$ of $\mathcal{Z}$ in $\mathrm{GL}(6,2)$.

The minimal polynomial $x^2 + x + 1$ of R being irreducible, we may identify $\{0, I, R, R^2\}$ with the field $\mathrm{GF}(4) = \{0, 1, \rho, \rho^2\}$ and so we may view $V(6,2)$ as

$V_{3,4} := V(3,4)$. Since $J(Rx) = R^2(Jx)$ it follows that J viewed as a mapping j of $V_{3,4}$ is σ-linear, where $\sigma : \mathrm{GF}(4) \to \mathrm{GF}(4)$ denotes the field automorphism $\alpha \mapsto \alpha^\sigma := \alpha^2$. The centralizer $C(\mathcal{Z})$ and normalizer $N(\mathcal{Z})$ of $\mathcal{Z}$ are consequently subgroups of $\mathrm{GL}(6,2)$ such that

$$C(\mathcal{Z}) \cong \mathrm{GL}(3,4), \quad N(\mathcal{Z}) = C(\mathcal{Z}).\langle J\rangle \cong \Gamma\,\mathrm{L}(3,4) = \mathrm{GL}(3,4).\langle j\rangle. \tag{18}$$

(See for example [2, Lemma 3.1 and Section 7].) In fact $N(\mathcal{Z}) \cong \Gamma\,\mathrm{L}(3,4)$ is a maximal subgroup of $\mathrm{GL}(6,2)$. (For a list of all the maximal subgroups of $\mathrm{GL}(6,2)$ consult [10].)

The collineation R is fixed-point-free on $\mathrm{PG}(5,2)$, permuting the 63 points of $\mathrm{PG}(5,2)$ in 21 cycles of length 3. Since $R^2 + R + I = 0$, each of these cycles constitutes a projective line, and so R determines a spread, say $\mathcal{L}_{21}$, of 21 lines in $\mathrm{PG}(5,2)$. Viewing $V(6,2)$ as $V(3,4)$ these 21 lines in $\mathrm{PG}(5,2)$ arise from the 21 Points of the Desarguesian plane $\mathrm{PG}(2,4) = \mathbb{P}(V(3,4))$. The full collineation group of the plane $\mathrm{PG}(2,4)$ is $\mathrm{P}\,\Gamma\,\mathrm{L}(3,4) = \Gamma\,\mathrm{L}(3,4)/\langle \rho I_3\rangle$ and the full symmetry group $\mathcal{G}_{\mathcal{L}} = \mathcal{G}_{\mathcal{L}_{21}} < \mathrm{GL}(6,2)$ of the spread $\mathcal{L}_{21}$ is

$$\mathcal{G}_{\mathcal{L}} = N(\mathcal{Z}) \cong \Gamma\,\mathrm{L}(3,4). \tag{19}$$

Notation 5 *At times we may be switching frequently between* $\mathrm{PG}(2,4)$ *and* $\mathrm{PG}(5,2)$ *concerns, and so we will call the 1-flats in* $\mathrm{PG}(2,4)$ Lines *to distinguish them from the lines in* $\mathrm{PG}(5,2)$; *similarly we will call the 0-flats in* $\mathrm{PG}(2,4)$ Points.

2.2 $\mathcal{G}_{\mathcal{L}}$-orbits of flats in $\mathrm{PG}(5,2)$

We denote by $\Phi^{(r)}_N$ a $\mathcal{G}_{\mathcal{L}}$-orbit of r-flats in $\mathrm{PG}(5,2)$ of length N. These orbits are readily determined. First there are the $\mathcal{G}_{\mathcal{L}}$-orbits $\Phi^{(1)}_{21} = \mathcal{L}_{21}$ and $\Phi^{(3)}_{21}$, arising respectively from the 21 Points and 21 Lines of $\mathrm{PG}(2,4)$. Take note that, since each Line in $\mathrm{PG}(2,4)$ contains five Points of $\mathrm{PG}(2,4)$, *each solid* $D \in \Phi^{(3)}_{21}$ *contains five lines* $L \in \Phi^{(1)}_{21}$, *these lines therefore constituting a spread for* D. Since two Lines of $\mathrm{PG}(2,4)$ meet in a Point, two solids $D, D' \in \Phi^{(3)}_{21}$ meet in a line $\in \Phi^{(1)}_{21}$, and so the 30 lines in D which do not belong to $\Phi^{(1)}_{21}$ are distinct from the 30 lines in D' which do not belong to $\Phi^{(1)}_{21}$. The 21 solids $D \in \Phi^{(3)}_{21}$ thus account for all of the $(21 \times 30 =)$ 630 lines $\notin \Phi^{(1)}_{21}$, these lines forming a $\mathcal{G}_{\mathcal{L}}$-orbit $\Phi^{(1)}_{630}$. Dually, since a Point of $\mathrm{PG}(2,4)$ lies on five Lines of $\mathrm{PG}(2,4)$, *each line* $L \in \Phi^{(1)}_{21}$ *lies inside 5 solids* $\in \Phi^{(3)}_{21}$. Each of the 21 lines $L \in \Phi^{(1)}_{21}$ lies inside a further 30 solids and these $(21 \times 30 =)$ 630 solids $D \notin \Phi^{(3)}_{21}$ form a $\mathcal{G}_{\mathcal{L}}$-orbit $\Phi^{(3)}_{630}$, with *each solid* $D \in \Phi^{(3)}_{630}$ *containing precisely one line* $L \in \mathcal{L}_{21}$.

Remark 6 *The* $\mathcal{G}_{\mathcal{L}}$*-orbit* $\Phi^{(1)}_{630}$ *could also have been arrived at as follows. Each of the* 21 *Lines contains* $\binom{5}{3} = 10$ *subsets of three colLinear Points, and each of these* 210 *subsets* $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ *gives rise to a regulus* $\mathcal{R} = \{L_1, L_2, L_3\}$ *of three lines* $L_i \in \mathcal{L}_{21}$. *The three lines of the opposite regulus* $\mathcal{R}^{opp}$ *are thus lines* $\notin \mathcal{L}_{21}$, *and so such lines form a* $\mathcal{G}_{\mathcal{L}}$*-orbit of length* $210 \times 3 = 630$.

Concerning the planes, through each line $\in \mathcal{L}_{21}$ pass 15 planes, and the $21 \times 15 = 315$ planes P in $\mathrm{PG}(5,2)$ which contain a line $L \in \mathcal{L}_{21}$ form a $\mathcal{G}_{\mathcal{L}}$-orbit $\Phi^{(2)}_{315}$. The remaining $1395 - 315 = 1080$ planes in $\mathrm{PG}(5,2)$ are of the form $\langle a, b, c\rangle$ where, viewing a, b, c as elements of $V(3,4)$, $\{a, b, c\}$ is a basis for $V(3,4)$. Since $\mathrm{GL}(3,4)$ is transitive on bases, these 1080 planes $\notin \Phi^{(2)}_{315}$ form a single $\mathcal{G}_{\mathcal{L}}$-orbit $\Phi^{(2)}_{1080.}$.

For future reference take note of the following (dual pairs of) incidence relations involving flats in $\mathrm{PG}(5,2)$ belonging to the aforementioned $\mathcal{G}_{\mathcal{L}}$-orbits:
(ia) Each solid $\in \Phi^{(3)}_{21}$ contains 5 lines $\in \Phi^{(1)}_{21}$, 30 lines $\in \Phi^{(1)}_{630}$, and 15 planes $\in \Phi^{(2)}_{315}$;
(ib) Each line $\in \Phi^{(1)}_{21}$ lies inside 5 solids $\in \Phi^{(3)}_{21}$, 30 solids $\in \Phi^{(3)}_{630}$, and 15 planes $\in \Phi^{(2)}_{315}$;
(iia) Each solid $\in \Phi^{(3)}_{630}$ contains 1 line $\in \Phi^{(1)}_{21}$, 34 lines $\in \Phi^{(1)}_{630}$, 3 planes $\in \Phi^{(2)}_{315}$, and 12 planes $\in \Phi^{(2)}_{1080}$;
(iib) Each line $\in \Phi^{(1)}_{630}$ lies inside 1 solid $\in \Phi^{(3)}_{21}$, 34 solids $\in \Phi^{(3)}_{630}$, 3 planes $\in \Phi^{(2)}_{315}$, and 12 planes $\in \Phi^{(2)}_{1080}$;
(iiia) Each plane $\in \Phi^{(2)}_{315}$ contains 1 line $\in \Phi^{(1)}_{21}$ and 6 lines $\in \Phi^{(1)}_{630}$;
(iiib) Each plane $\in \Phi^{(2)}_{315}$ lies inside 1 solid $\in \Phi^{(3)}_{21}$ and 6 solids $\in \Phi^{(3)}_{630}$;
(iva) Each plane $\in \Phi^{(2)}_{1080}$ contains 7 lines $\in \Phi^{(1)}_{630}$;
(ivb) Each plane $\in \Phi^{(2)}_{1080}$ lies inside 7 solids $\in \Phi^{(3)}_{630}$.

Notation 7 *For $D \in \Phi^{(3)}_{630}$ the unique line $L \in \Phi^{(1)}_{21}$ contained in D will be denoted L_D. For $L \in \Phi^{(1)}_{630}$ the unique solid $D \in \Phi^{(3)}_{21}$ which contains L will be denoted D_L. If $P \in \Phi^{(2)}_{315}$ then L_P and D_P will denote the unique line $\in \Phi^{(1)}_{21}$ and the unique solid $\in \Phi^{(3)}_{21}$ such that $L_P \subset P \subset D_P$. Further for a point $a \in \mathrm{PG}(5,2)$ the unique line $\in \Phi^{(1)}_{21}$ containing a will be denoted L_a and, dually, for a hyperplane H the unique solid $\in \Phi^{(3)}_{21}$ contained in H will be denoted D_H.*

2.3 Trivectors $t \in \wedge^3 V_6^*$ such that $\mathcal{L}_t = \mathcal{L}_{21}$

The set $\mathcal{L}_T$ of all T-singular lines in $\mathrm{PG}(5,2)$, see Definition 2(ii), will also be denoted $\mathcal{L}_t$ when we deal with the isomorph $t \in \wedge^3 V_6^*$ of $T \in \mathrm{Alt}(\times^3 V_6)$. We pose the question: *do there exist elements $T \in \mathrm{Alt}(\times^3 V_6)$ such that the set $\mathcal{L}_T$ of T-singular lines coincides with the Desarguesian line-spread $\mathcal{L}_{21}$?* Stemming from the fact that $\mathrm{Alt}(\times^3 V_{3,4})$ is one-dimensional, and hence possesses precisely three nonzero elements τ_1, τ_2, τ_3, we now proceed to show that there exist precisely three elements $T_1, T_2, T_3 \in \mathrm{Alt}(\times^3 V_6)$ such that $\mathcal{L}_{T_i} = \mathcal{L}_{21},\ i = 1, 2, 3$.

To this end let τ_1, τ_2, τ_3 denote the three elements of $\mathrm{Alt}(V_{3,4})$ whose values on the basis $\{e_1, e_2, e_3\}$ for $V_{3,4}$ are as follows:

$$\tau_1(e_1, e_2, e_3) = 1, \quad \tau_2(e_1, e_2, e_3) = \rho^2, \quad \tau_3(e_1, e_2, e_3) = \rho, \tag{20}$$

and let the further vectors of the basis $\mathcal{B} = \{e_1, e_2, \dots, e_6\}$ for V_6 be, as in Section 2.1, $e_4 = \rho e_1,\ e_5 = \rho e_2,\ e_6 = \rho e_3$. We now define a trio $\mathcal{T} = \{T_1, T_2, T_3\}$ of elements of $\mathrm{Alt}(\times^3 V_6)$ by

$$T_i(x,y,z) = \mathrm{Tr}(\tau_i(x,y,z)), \quad i = 1,2,3. \tag{21}$$

Here Tr is the usual trace mapping $\mathrm{GF}(4) \to \mathrm{GF}(2)$ given by $\mathrm{Tr}(\alpha) = \alpha + \alpha^2$, $\alpha \in \mathrm{GF}(4)$; see for example [3]. Since Tr is a GF(2)-linear map, the definition (21) succeeds in defining elements $T_i \in \mathrm{Alt}(\times^3 V_6)$. Also if a, b belong to the same 1-dimensional subspace of $V_{3,4}$ then $\tau_i(a,b,z) = 0$ for all z; consequently each projective line $\langle a,b\rangle \in \mathcal{L}_{21}$ in $\mathrm{PG}(5,2)$ is T-singular for $T \in \mathcal{T}$. Furthermore if a line $\langle a,b\rangle$ in $\mathrm{PG}(5,2)$ is not in $\mathcal{L}_{21} = \Phi_{21}^{(1)}$ then $\tau_i(a,b,z) \neq 0$ for some z and so no line $\langle a,b\rangle \in \Phi_{630}^{(1)}$ is T-singular. So indeed $\mathcal{L}_{T_i} = \mathcal{L}_{21}$, $i = 1,2,3$. Since all lines containing a T-singular point are T-singular, note that there are no T_i-singular points: $\mathrm{rad}\, T_i = \emptyset$.

If $\mathcal{B}^* = \{f_1, f_2, \dots, f_6\}$ is the basis for V_6^* dual to the basis $\mathcal{B} = \{e_1, e_2, \dots, e_6\}$ for V_6 then the three (dual) trivectors $t_i \in \wedge^3 V_6^*$ determined by (21) are, upon using $\mathrm{Tr}(1) = 0$ and $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^2) = 1$, seen to be as follows:

$$\begin{aligned} t_1 &= f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345}, \\ t_2 &= f_{156} + f_{246} + f_{345} + f_{123} + f_{456}, \\ t_3 &= f_{234} + f_{135} + f_{126} + f_{123} + f_{456}, \end{aligned} \tag{22}$$

where $f_{ijk} := f_i \wedge f_j \wedge f_k$. For example: $T_1(e_1,e_2,e_3) = 0$ since $\mathrm{Tr}(\tau_1(e_1,e_2,e_3)) = \mathrm{Tr}(1) = 0$, and $T_1(e_1,e_2,e_6) = 1$ since $\tau_1(e_1,e_2,\rho e_3) = \rho$ and $\mathrm{Tr}(\rho) = 1$. Observe from (22) that $t_1 + t_2 + t_3 = 0$, and hence $T_1 + T_2 + T_3 = 0$ — but of course this follows from (21) since the τ_i in (20) satisfy $\tau_1 + \tau_2 + \tau_3 = 0$.

Various other properties follow immediately from the definition (21) of T_1, T_2, T_3. In particular if $T \in \mathcal{T} = \{T_1, T_2, T_3\}$ then, for all $x, y, z \in V_6$, we have

$$\begin{aligned} &\text{(i)} \quad T(x, Rx, z) = 0 = T(x, R^2x, z), \\ &\text{(ii)} \quad T(Rx, y, z) = T(x, Ry, z) = T(x, y, Rz), \\ &\text{(iii)} \quad T(R^2x, y, z) = T(x, R^2y, z) = T(x, y, R^2z), \\ &\text{(iv)} \quad T(Rx, Ry, Rz) = T(x, Ry, R^2z) = T(x, y, z). \end{aligned} \tag{23}$$

For example, property (iv) of (23) holds since $\tau(\rho x, \rho y, \rho z) = \tau(x, \rho y, \rho^2 z) = \rho^3 \tau(x,y,z) = \tau(x,y,z)$. Also, since $\tau_1(\rho^2 x, y, z) = \tau_2(x,y,z)$ and $\tau_1(\rho x, y, z) = \tau_3(x,y,z)$, take note that the trilinear forms T_1, T_2, T_3 are related as follows:

$$T_2(x,y,z) = T_1(R^2x, y, z), \qquad T_3(x,y,z) = T_1(Rx, y, z), \qquad x, y, z \in V_6. \tag{24}$$

For any $T \in \mathrm{Alt}(\times^3 V_6)$ Equation (23)(i) states that each line of $\mathcal{L}_{21}$ is T-singular. Now it is easy to see that properties (23)(ii)-(iv) follow from (23)(i), Moreover, using the basis $\mathcal{B}$, the only solutions T of Equations (23)(i)-(iv) are those displayed in (22). *Hence $\mathcal{L}_T = \mathcal{L}_{21}$ for precisely the three elements $T \in \mathcal{T}$.*

Information concerning the stabilizer subgroups $\mathcal{G}_{T_i} < \mathrm{GL}(6,2)$ can be quickly gleaned from the stabilizer subgroups $\mathcal{G}_{\tau_i} < \Gamma\,\mathrm{L}(3,4)$. On the one hand, under the natural action Λ_a of $a \in \Gamma\mathrm{L}(3,4)$ upon $\tau \in \mathrm{Alt}(V_{3,4})$ given by

$$(\Lambda_a\tau)(x,y,z) = \tau(a^{-1}x, a^{-1}y, a^{-1}z), \quad \text{if } a \in \mathrm{GL}(3,4), \tag{25}$$

$$(\Lambda_a\tau)(x,y,z) = \tau(a^{-1}x, a^{-1}y, a^{-1}z)^{\sigma}, \quad \text{if } a \in \Gamma\mathrm{L}(3,4) \setminus \mathrm{GL}(3,4), \tag{26}$$

one easily sees that $\Gamma\mathrm{L}(3,4)$ acts transitively upon the three trilinear forms τ_i in (20). For example if $w \in \mathrm{GL}(3,4)$ is defined on the basis $\{e_1, e_2, e_3\}$ by

$$we_1 = \rho e_1, \quad we_2 = e_2, \quad we_3 = e_3, \tag{27}$$

then Λ_w effects the cyclic permutation $(\tau_1\tau_2\tau_3)$. On the other hand each τ_i is of course stabilized by Λ_a for all $a \in \mathrm{SL}(3,4)$. To see that $\mathcal{G}_{\tau_i}$ is bigger than $\mathrm{SL}(3,4)$ consider the three σ-linear mappings k_1, k_2, k_3 on $V_{3,4}$ defined as follows: let all three k_i keep e_2 and e_3 fixed but let their effect upon e_1 be:

$$k_1e_1 = e_1; \quad k_2e_1 = \rho^2e_1; \quad k_3e_1 = \rho e_1. \tag{28}$$

Observe that $(k_i)^2 = I,\ i = 1,2,3$. It then follows from (26) that Λ_{k_i} fixes τ_i but interchanges the two $\tau_j,\ j \neq i$:

$$\Lambda_{k_1} : \tau_2 \rightleftarrows \tau_3, \quad \Lambda_{k_2} : \tau_1 \rightleftarrows \tau_3, \quad \Lambda_{k_3} : \tau_1 \rightleftarrows \tau_2. \tag{29}$$

So the three stabilizers

$$\mathcal{G}_{\tau_i} = \mathrm{SL}(3,4).\langle k_i \rangle \cong \mathrm{SL}(3,4).2, \quad i = 1,2,3, \tag{30}$$

are distinct subgroups of $\Gamma\mathrm{L}(3,4)$, each of index 3. They are isomorphic, being conjugate subgroups of $\Gamma\mathrm{L}(3,4)$ under conjugation by w and w^2.

It follows from the foregoing that $\mathcal{G}_{\mathcal{L}} \cong \Gamma\mathrm{L}(3,4)$ acts transitively upon $\mathcal{T} = \{T_1, T_2, T_3\}$, with Λ_W effecting the cyclic permutation $(T_1T_2T_3)$, where $W \in \mathcal{G}_{\mathcal{L}}$ is the element of $\mathrm{GL}(6,2)$ determined by w in (27):

$$We_1 = e_4, \quad We_4 = e_1 + e_4, \quad We_i = e_i,\ i = 2,3,5,6. \tag{31}$$

Further the stabilizer of T_i is a subgroup $\mathcal{G}_{T_i} \cong \mathrm{SL}(3,4).\langle k_i \rangle$ of $\mathcal{G}_{\mathcal{L}}$ of index 3.

We summarize our findings this section in the next theorem.

Theorem 8 *There exist precisely three elements $T_i \in \mathrm{Alt}(\times^3 V_6),\ i = 1,2,3$, such that $\mathcal{L}_{T_i} = \mathcal{L}_{21}$, these three trilinear forms being related as in (24). With respect to the basis $\mathcal{B}$ for V_6 the corresponding three (dual) trivectors t_1, t_2, t_3 are as displayed explicitly in Eq. (22). The stabilizers $\mathcal{G}_{T_1}, \mathcal{G}_{T_2}, \mathcal{G}_{T_3}$ of T_1, T_2, T_3 are conjugate subgroups of $\mathcal{G}_{\mathcal{L}} \cong \Gamma\mathrm{L}(3,4)$, of index 3, with the structure*

$$\mathcal{G}_{T_i} \cong \mathrm{SL}(3,4).2, \quad i = 1,2,3. \tag{32}$$

The trio $\{t_1, t_2, t_3\}$ constitutes a projective line in $\mathrm{PG}(19,2) = \mathbb{P}(\wedge^3 V_6^*)$ upon which $\mathcal{G}_{\mathcal{L}}$ acts transitively.

3 The T-associate $U^{\#}$ of a flat U in $\mathrm{PG}(5,2)$

In this section we deal with a general element $T \in \mathrm{Alt}(\times^3 V_6)$, and in subsequent sections return to elements T of the trio $\mathcal{T} = \{T_1, T_2, T_3\}$.

Definition 9 *Given $T \in \mathrm{Alt}(\times^3 V_6)$ the* T-associate *$U^{\#}$ of an r-flat U in* $\mathrm{PG}(5,2)$ *is*

$$U^{\#} = \{v \in \mathrm{PG}(5,2) \,|\, T(u_1, u_2, v) = 0 \textit{ for all } u_1, u_2 \in U\}. \tag{33}$$

Since T is trilinear, $U^{\#}$ is an s-flat for some s. If U is a point note that $U^{\#} = \mathrm{PG}(5,2)$. For a line $U = L$ note that $L^{\#}$ is the flat with equation $f_L(x) = 0$; so $L^{\#}$ is a hyperplane whenever f_L is not the zero form, and the T-singular lines $\mathcal{L}_T$ are those lines L such that $L^{\#} = \mathrm{PG}(5,2)$. Since $U^{\#}$ is uniquely determined, for the given T, by U it follows that $\mathcal{G}_U \leq \mathcal{G}_{U^{\#}}$, where $\mathcal{G}_U$ denotes that subgroup of $\mathcal{G}_T$ which stabilizes the flat U. Further, if W is also a flat, then from $W \subset U$ it follows that $U^{\#} \subseteq W^{\#}$.

When dealing with a particular hypersurface ψ in $\mathrm{PG}(5,2)$, with cubic equation $Q(x) = 0$, we may, by taking $T = T_Q$ in (33), also refer to $U^{\#}$ as the ψ-associate of U. Then Definition 9 is the special case $d = 3,\ n = 5$ of the general definition, see [7], of the ψ-associate of a flat in $\mathrm{PG}(n,2)$ with respect to a given subset ψ of polynomial degree d.

It may help to get our bearings if we first remind ourselves of corresponding concerns in a familiar situation, namely when dealing instead with an element $B \in \mathrm{Alt}(\times^2 V_6)$, or with its isomorphic dual bivector $b \in \wedge^2 V_6^*$. Leaving aside $b = 0$, the latter fall into three $\mathrm{GL}(6,2)$-orbits, as represented by the bivectors

$$b_1 = f_{12}, \quad b_2 = f_{12} + f_{34}, \quad b_3 = f_{12} + f_{34} + f_{56}, \quad \text{where } f_{ij} := f_i \wedge f_j, \tag{34}$$

having respective ranks 2, 4 and 6. For a given $B \in \mathrm{Alt}(\times^2 V_6)$ each flat U in $\mathrm{PG}(5,2)$ determines a flat $U^{\perp}$, the orthogonal flat of U, defined by

$$U^{\perp} = \{v \in \mathrm{PG}(5,2) \,|\, B(u,v) = 0 \text{ for all } u \in U\}. \tag{35}$$

This $d = 2$ definition of $U^{\perp}$ is the progenitor of the $d = 3$ definition of the T-associate $U^{\#}$ of a flat U in Eq. (33). A flat U is said to be singular if there exists $u \in U$ such that $B(u,v) = 0$ for all $v \in U$, and is said to be isotropic if $U \subseteq U^{\perp}$, that is if $B(u,v) = 0$ for all $u, v \in U$.

Consider the case of $b = b_3$ in (34) for which the alternating bilinear form B is non-degenerate, having as stabilizer $\mathcal{G}_b = \mathcal{G}_B \cong \mathrm{Sp}(6,2)$. In this case $U^{\perp}$ satisfies various simple properties, including:

$$\begin{aligned} &\text{(i)}\ \dim U + \dim U^{\perp} = 4, \qquad \text{(ii)}\ (U^{\perp})^{\perp} = U, \\ &\text{(iii)}\ \langle U, U^{\perp} \rangle = \mathrm{PG}(5,2), \quad \text{whenever } U \text{ is non-singular.} \end{aligned} \tag{36}$$

It follows from (i) that $P^{\perp} = P$ if P is an isotropic plane. In this non-degenerate case the classification of flats into $\mathcal{G}_B$-orbits is straightforward. One finds that there are 315 isotropic and 336 non-isotropic lines in $\mathrm{PG}(5,2)$, forming just two $\mathcal{G}_B$-orbits which we denote $\Theta_{315}^{(1)}$ and $\Theta_{336}^{(1)}$. Also there are 135 isotropic and 1260 non-isotropic planes, forming just two $\mathcal{G}_B$-orbits which we denote $\Theta_{135}^{(2)}$ and $\Theta_{1260}^{(2)}$. Further there are two $\mathcal{G}_B$-orbits $\Theta_{315}^{(3)}$ and $\Theta_{336}^{(3)}$ of solids (3-flats), with a solid $L^{\perp} \in \Theta_{315}^{(3)}$, or $\in \Theta_{336}^{(3)}$, according as $L \in \Theta_{315}^{(1)}$, or $\in \Theta_{336}^{(1)}$.

Returning to an element $T \in \mathrm{Alt}(\times^3 V_6)$ we will in later sections be concerned with $\mathcal{G}_T$-orbits and T-associates of planes, and we will then have need of the following definition and lemma.

Definition 10 *For a given $T \in \mathrm{Alt}(\times^3 V_6)$ a plane $P = \langle a,b,c\rangle$ in $\mathrm{PG}(5,2)$ is said to be a $T=0$ plane or a $T=1$ plane according as $T(a,b,c)=0$ or $T(a,b,c)=1$.*

This definition stands up, for since T is alternating and trilinear all choices of independent points $a,b,c \in P$ yield the same value for $T(a,b,c)$. Observe that the preceding definition (33) of the *T-associate* $U^{\#}$ of an r-flat U in $\mathrm{PG}(5,2)$ can now be rephrased as follows (granted that $r \geq 2$ in (ii)):

$$\begin{aligned}&\text{(i) if } v \in U^c, \text{ then } v \in U^{\#} \iff \langle L, v\rangle \text{ is a } T=0 \text{ plane for all lines } L \subset U;\\ &\text{(ii) if } v \in U, \text{ then } v \in U^{\#} \iff \text{all planes in } U \text{ containing } v \text{ are } T=0 \text{ planes.}\end{aligned} \tag{37}$$

Lemma 11 *For any $T \in \mathrm{Alt}(\times^3 V_6)$ the following results hold for the T-associate $P^{\#}$ of a plane $P = \langle a,b,c\rangle$ in $\mathrm{PG}(5,2)$:*
(i) $P^{\#} = \langle b,c\rangle^{\#} \cap \langle a,c\rangle^{\#} \cap \langle a,b\rangle^{\#}$.
(ii) If P is T-singular then P is a $T=0$ plane. Moreover $P^{\#} \supset P$, and if P contains just one singular line then $P^{\#}$ is a solid $D \supset P$.
(iii) If P is T-non-singular then $P^{\#}$ is a plane. Further
  (a) if P is a non-singular $T=0$ plane then $P^{\#} = P$,
  (b) if P is a $T=1$ plane then P is non-singular and $P^{\#} \cap P = \emptyset$.

Proof. (i) Since T is alternating and trilinear, if x satisfies the three equations $T(b,c,x)=0,\ T(a,c,x)=0$ and $T(a,b,x)=0$ then x satisfies $T(u_1,u_2,x)=0$ for all $u_1,u_2 \in P$, and so $x \in P^{\#}$.

(ii) We may suppose here that $P = \langle a,b,c\rangle$ where the line $\langle a,b\rangle$ is T-singular. Then $T(a,b,x)=0$ for all x, and in particular for all $x \in P$. Thus P is a $T=0$ plane. So $T(u_1,u_2,x)=0$ for all $u_1,u_2,x \in P$, and hence each point $x \in P$ belongs to $P^{\#}$. The inclusion $P \subset P^{\#}$ is a proper inclusion since, by (i), $P^{\#}$ is the intersection $\langle b,c\rangle^{\#} \cap \langle a,c\rangle^{\#}$ of at most two hyperplanes, and so is an r-flat for some $r \geq 5-2=3$. If $P = \langle a,b,c\rangle$ contains just the one singular line $\langle a,b\rangle$ then by (i) $P^{\#}$ consists of points $x \in \mathrm{PG}(5,2)$ satisfying the two independent linear conditions $T(b,c,x)=0,\ T(a,c,x)=0$ and so $P^{\#}$ is a $(5-2=)$ 3-flat D.

(iii) Since P contains no T-singular lines the three linear forms $f_{\langle b,c\rangle}(x),\ f_{\langle a,c\rangle}(x)$ and $f_{\langle a,b\rangle}(x)$ are (see [7, Theorem 1.18(i)]) independent, whence, by (i), $P^{\#}$ is a 2-flat.

(iiia) Knowing that $T(u_1, u_2, v) = 0$ for any independent points $u_1, u_2, v \in P$, the plane $P^\#$ must be P.

(iiib) By (ii) P is non-singular plane and so by (iii) $P^\#$ is a plane. Since we are given that $T(u_1, u_2, v) = 1$ for any independent points $u_1, u_2, v \in P$, no point $v \in P$ can belong to $P^\#$. ■

4 $\mathcal{G}_T$-orbits of flats in $\mathrm{PG}(5, 2)$

For $T \in \mathrm{Alt}(\times^3 V_6)$ such that $\mathcal{L}_T = \mathcal{L}_{21}$ we now determine the $\mathcal{G}_T$-orbits of flats in $\mathrm{PG}(5, 2)$, denoting by $\Theta_N^{(r)}$ a $\mathcal{G}_T$-orbit of r-flats of length N. Then, in the next section, for a representative U of each orbit we determine its T-associate $U^\#$.

Given that T is one of the trio $\mathcal{T} = \{T_1, T_2, T_3\}$ as defined in Equation (21), recall from Theorem 8 that $\mathcal{G}_T \cong \mathrm{SL}(3, 4).2$ is of index 3 in $\mathcal{G}_\mathcal{L} \cong \Gamma\,\mathrm{L}(3, 4)$. Now in Section 2.2 we found the $\mathcal{G}_\mathcal{L}$-orbits of lines, planes and solids in $\mathrm{PG}(5, 2)$ to be as follows:

$$\Phi_{21}^{(1)}(= \mathcal{L}_{21}),\ \Phi_{630}^{(1)}; \qquad \Phi_{315}^{(2)},\ \Phi_{1080}^{(2)}; \qquad \Phi_{21}^{(3)}, \Phi_{630}^{(3)}\ . \tag{38}$$

With the exception of the orbit $\Phi_{1080}^{(2)}$ one readily sees that these $\mathcal{G}_\mathcal{L}$-orbits are also $\mathcal{G}_T$-orbits; that is $\Theta_N^{(r)} = \Phi_N^{(r)}$ for the five $\mathcal{G}_\mathcal{L}$-orbits other than $\Phi_{1080}^{(2)}$. Upon recalling Definition 2 observe that $\Theta_{21}^{(1)}$ comprises the T-singular lines and $\Theta_{630}^{(1)}$ the T-non-singular lines. Also, see (iiia), (iva) in Section 2.2, $\Theta_{315}^{(2)}$ consists of all of the T-singular planes. By (ia), (iia) in Section 2.2, all solids are singular, with a solid D containing either one or five singular lines according as $D \in \Phi_{630}^{(3)}$ or $D \in \Phi_{21}^{(3)}$. In the following, the solids $\in \Phi_{21}^{(3)}$, which contain a spread of five singular lines, will be termed (T-)*super-singular.* This term is further justified by the next lemma.

Lemma 12 *A solid D is T-super-singular if and only if $T(a, b, c) = 0$ for all $a, b, c \in D$; equivalently, if and only if every plane in D is a $T = 0$ plane.*

Proof. If D is super-singular then every plane in D is singular and so, Lemma 11(ii), is a $T = 0$ plane. But if D is not super-singular, and so $D \in \Phi_{630}^{(3)}$, then there exist $T = 1$ planes in D; in fact, see Section 5.2 below, D contains precisely eight $T = 1$ planes. ■

Let us now consider the T-non-singular planes, that is those 1080 planes which do not contain a line $L \in \mathcal{L}_{21}$. In Section 2.2 we saw that these 1080 planes form a single $\mathcal{G}_\mathcal{L}$-orbit $\Phi_{1080}^{(2)}$ upon observing that (i) such planes are of the form $P = \langle a, b, c\rangle$ where, viewing a, b, c as elements of $V(3, 4)$, $\{a, b, c\}$ is a basis for $V_{3,4}$, and (ii) $\mathrm{GL}(3, 4)$ is transitive on bases. However a basis $\{a, b, c\}$ for $V_{3,4}$ lies on the same $\mathrm{SL}(3, 4)$-orbit as the basis $\{\alpha a, \beta b, \gamma c\}$ if and only if the elements $\alpha, \beta, \gamma \in \mathrm{GF}(4)$ satisfy $\alpha\beta\gamma = 1$. Consequently the bases for $V_{3,4}$ fall into three $\mathrm{SL}(3, 4)$-orbits, each of length 360, with representatives

$$\mathcal{B}_1 = \{e_1, e_2, e_3\}, \quad \mathcal{B}_2 = \{\rho e_1, e_2, e_3\}, \quad \mathcal{B}_3 = \{\rho^2 e_1, e_2, e_3\}. \tag{39}$$

It follows that the 1084 T-non-singular planes in $\mathrm{PG}(5,2)$ fall into three $\mathcal{G}_T^0$-orbits, each of length 360, with representatives

$$P_1 = \langle e_1, e_2, e_3\rangle, \quad P_2 = \langle e_4, e_2, e_3\rangle, \quad P_3 = \langle e_1 + e_4, e_2, e_3\rangle, \tag{40}$$

where $\mathcal{G}_T^0$ denotes the common subgroup $\cong \mathrm{SL}(3,4)$ of each of the groups $\mathcal{G}_{T_i} \cong \mathrm{SL}(3,4).2,\ i = 1,2,3$. Observe that for $i = 1,2,3$ the plane P_i is a $T_i = 0$ plane but is a $T_j = 1$ plane for $j \neq i$. We will denote the $\mathcal{G}_T^0$-orbit of non-singular planes which contains the plane P_i in (40) by $\Pi_{360}(i)$. Thus $\Pi_{360}(i)$ consists of those non-singular planes which are $T_i = 0$ planes (and which are then necessarily $T_j = 1$ planes for $j \neq i$).

Lemma 13 *For $T_i \in \mathcal{T}$ the 1080 T_i-non-singular planes in* $\mathrm{PG}(5,2)$ *fall into two $\mathcal{G}_{T_i}$-orbits $\Theta_{360}^{(2)}(i)$ and $\Theta_{720}^{(2)}(i)$, where*

$$\Theta_{360}^{(2)}(i) = \Pi_{360}(i), \qquad \Theta_{720}^{(2)}(i) = \Pi_{360}(j) \cup \Pi_{360}(k), \quad \{i,j,k\} = \{1,2,3\}. \tag{41}$$

Proof. Recall from Section 2 that $\mathcal{G}_{\mathcal{T}_1} = \mathrm{SL}(3,4).\langle k_1\rangle \cong \mathrm{SL}(3,4).2$ where k_1 is the σ-linear mapping of $V_{3,4}$ which, see Equation (28), fixes each element e_i of the basis $\mathcal{B}_1$. Now k_1 preserves $\mathcal{B}_1$ but, since $k_1(\rho e_1) = \rho^2 e_1$, note that k_1 effects the interchange $\mathcal{B}_2 \rightleftarrows \mathcal{B}_3$. For $i = 1,2,3$ the bases for $V_{3,4}$ thus fall into two $\mathcal{G}_{\mathcal{T}_i}$-orbits, one of length 360 with representative $\mathcal{B}_i$ and one of length 720, represented by either of the bases $\mathcal{B}_j$, $j \neq i$. Consequently for $T_i \in \mathcal{T}$ the 1080 T-non-singular planes fall into two $\mathcal{G}_{T_i}$-orbits $\Theta_{360}^{(2)}(i)$ and $\Theta_{720}^{(2)}(i)$ as in (41). ■

The next theorem summarizes our results concerning $\mathcal{G}_T$-orbits of flats.

Theorem 14 *Let $\mathcal{T} = \{T_1, T_2, T_3\}$ be the trio of elements $T \in \mathrm{Alt}(\times^3 V_6)$ such that $\mathcal{L}_T = \mathcal{L}_{21}$, as given in Equation (21). Then for $T \in \mathcal{T}$ the $\mathcal{G}_T$-orbits of lines, planes and solids in* $\mathrm{PG}(5,2)$ *are as follows.*

(i) There is one orbit $\Theta_{21}^{(1)} = \mathcal{L}_{21}$ of T-singular lines and one orbit $\Theta_{630}^{(1)}$ of T-non-singular lines.

(ii) There is one orbit $\Theta_{315}^{(2)}$ of T-singular planes and two orbits $\Theta_{360}^{(2)}$, $\Theta_{720}^{(2)}$ of T-non-singular planes, the orbit $\Theta_{360}^{(2)}$ consisting of the non-singular $T = 0$ planes and the orbit $\Theta_{720}^{(2)}$ consisting of the $T = 1$ planes.

(iii) There are two orbits $\Theta_{21}^{(3)}, \Theta_{630}^{(3)}$ of solids; a solid $D \in \Theta_{21}^{(3)}$ is super-singular, in that it contains a spread of five singular lines, while a solid $D \in \Theta_{630}^{(3)}$ contains just one singular line.

5 The T-associates of flats in $\mathrm{PG}(5,2)$

For $T \in \mathcal{T} = \{T_1, T_2, T_3\}$ we now carry out our plan of determining the T-associate $U^{\#}$ of a flat U for each $\mathcal{G}_T$-orbit of flats in $\mathrm{PG}(5,2)$.

5.1 The T-associates of planes in $\mathrm{PG}(5,2)$

With the one exception of the orbit $\Phi^{(2)}_{1080}$ recall that the $\mathcal{G}_{\mathcal{L}}$-orbits of flats are also $\mathcal{G}_{T_i}$-orbits for each $i = 1, 2, 3$. As we will shortly see, our previous results allow us to deal very quickly with the T-associates of these non-exceptional flats. However, see Lemma 13, in the case of the T-non-singular planes $\Phi^{(2)}_{1080}$ the $\mathcal{G}_{T_i}$-orbits differ from the $\mathcal{G}_{T_j}$-orbits for $i \neq j$. Since the determination of the T-associates of such planes involves some interesting intricacies we will deal first with these planes.

The non-singular planes form the single $\mathcal{G}_{\mathcal{L}}$-orbit

$$\Phi^{(2)}_{1080} = \Pi_{360}(1) \cup \Pi_{360}(2) \cup \Pi_{360}(3)$$

where, as in Section 4, $\Pi_{360}(i)$ is the $\mathcal{G}^0_T$-orbit which consists of those non-singular planes which are $T_i = 0$ planes, and hence which are $T_j = 1$ planes for $j \neq i$. For each $i = 1, 2, 3$ there are, see Lemma 13, two $\mathcal{G}_{T_i}$-orbits $\Theta^{(2)}_{360}(i)$ and $\Theta^{(2)}_{720}(i)$ of non singular planes; in particular the two $\mathcal{G}_{T_1}$-orbits are $\Theta^{(2)}_{360}(1) = \Pi_{360}(1)$ and $\Theta^{(2)}_{720}(1) = \Pi_{360}(2) \cup \Pi_{360}(3)$.

We denote the T_i-associate of a plane P by $P^{\#_i}$. Also $(P^{\#_i})^{\#_i} := P^{\#_i\#_i}$.

Theorem 15 *The following hold for a non-singular plane P:*

$$\begin{aligned} &\text{(i) if } P \in \Pi_{360}(1) \text{ then} \quad P^{\#_1} = P, \quad P^{\#_2} = R(P), \quad P^{\#_3} = R^{-1}(P), \\ &\text{(ii) if } P \in \Pi_{360}(2) \text{ then} \quad P^{\#_1} = R^{-1}(P), \quad P^{\#_2} = P, \quad P^{\#_3} = R(P), \\ &\text{(iii) if } P \in \Pi_{360}(3) \text{ then} \quad P^{\#_1} = R(P), \quad P^{\#_2} = R^{-1}(P), \quad P^{\#_3} = P. \end{aligned} \tag{42}$$

(iv) If $P \in \Theta^{(2)}_{720}(i) = \Pi_{360}(j) \cup \Pi_{360}(k)$, where ijk is a permutation of 123, *then $P^{\#_i\#_i\#_i} = P$ and $P, P^{\#_i}, P^{\#_i\#_i}$ are disjoint planes $\in \Theta^{(2)}_{720}(i)$.*

Proof. (i) By Lemma 11(iii), since P is a $T_1 = 0$ plane it is T_1-self-associate: $P^{\#_1} = P$. Given $P = \langle a, b, c\rangle \in \Pi_{360}(1)$ we now show that $P^{\#_2} = R(P) = \langle Ra, Rb, Rc\rangle$ and $P^{\#_3} = R^2(P) = \langle R^2a, R^2b, R^2c\rangle$. From eq. (24) $T_2(Ra, b, c) = T_1(a, b, c) = 0$; but since also $T_2(Ra, a, b) = 0$ and $T_2(Ra, a, c) = 0$ it follows that $T_2(Ra, u, v) = 0$ for all $u, v \in P$, whence $Ra \in P^{\#_2}$. Similarly $Rb \in P^{\#_2}$ and $Rc \in P^{\#_2}$. So $R(P) \subseteq P^{\#_2}$. But, by Lemma 11(iii), $P^{\#_2}$ is a plane, and so $P^{\#_2} = R(P)$. The proof that $P^{\#_3} = R^2(P)$ is entirely similar since $T_3(R^2a, b, c) = T_1(a, b, c) = 0$. The results (42)(ii),(iii) similarly follow.

(iv) Since $R^3 = I$, and since $R \in \mathcal{G}_{T_i}$, it follows from (42) that $P^{\#_i\#_i\#_i} = P$, and that $P^{\#_i}, P^{\#_i\#_i}$ belong to the same $\mathcal{G}_{T_i}$-orbit $\Theta^{(2)}_{720}(i)$ as P. Since P here is a $T_i = 1$ plane, it follows from Lemma 11(iii) that $P^{\#_i}$ is disjoint from P, that $P^{\#_i\#_i}$ is disjoint from $P^{\#_i}$ and that $P(= P^{\#_i\#_i\#_i})$ is disjoint from $P^{\#_i\#_i}$. ■

Remark 16 *The three disjoint planes may be viewed as the three internal planes of a Segre variety $\mathcal{S}_{1,2}$ whose other family of flats are the seven lines $L_p = \{p, Rp, R^2p\},\ p \in P$, which meet each of the planes in a point.*

The results for all planes can now be stated as in the next theorem.

Theorem 17 *For $T \in \mathcal{T}$ the T-associates of planes in* $\mathrm{PG}(5,2)$ *are as follows.*

(i) If $P \in \Theta^{(2)}_{315}$ then $P^{\#} = D_P \in \Theta^{(3)}_{21}$.

(ii) If $P \in \Theta^{(2)}_{360}$ then $P^{\#} = P$.

(iii) If $P \in \Theta^{(2)}_{720}$ then $P^{\#}$ is a plane $\in \Theta^{(2)}_{720}$. Further $P^{\#\#\#} = P$ and the three planes $P, P^{\#}, P^{\#\#}$ are disjoint. See Theorem 15 for more details of how $P^{\#}$ is related to P.

Proof. (i) If $D_P \in \Theta^{(3)}_{21}$ is as in Notation 7 then, see Lemma 12, $T(a,b,c) = 0$ for all $a,b,c \in D_P$, whence $P^{\#} \supseteq D_P$. But by Lemma 11(iv) $P^{\#}$ is a solid, and so $P^{\#} = D_P$.

(ii) By Lemma 11(iiia) $P^{\#} = P$.

(iii) See Theorem 15. ■

Remark 18 *According to part (iii) of the theorem, if P is any $T = 1$ plane then $P, P^{\#}, P^{\#\#}$ are pairwise disjoint planes such that $P^{\#\#\#} = P$. It is worth stressing that this property of $T = 1$ planes can not be attributed solely to the fact that $T \in \mathrm{Alt}(\times^d V_6)$ for $d = 3$; see [7, Remark 2.7].*

5.2 The T-associates of lines, solids and hyperplanes

Concerning lines we have the following:
(i) if $L \in \Theta^{(1)}_{21} = \mathcal{L}_{21}$ then $L^{\#} = \mathrm{PG}(5,2)$;
(ii) if $L \in \Theta^{(1)}_{630}$ then $L^{\#}$ is the hyperplane $f_L(x) = 0$.

Concerning solids it helps to first introduce the *T-nucleus n_D of a solid $D \in \Phi^{(3)}_{630}$*. The fifteen planes inside a solid $D \in \Phi^{(3)}_{630}$ comprise the three singular planes which contain the singular line L_D of D together with twelve non-singular planes which meet L_D in a point. Following on from part (ii) of the Theorem 14, for each $T_i \in \mathcal{T}$ the twelve non-singular planes in D consist of a 4-set, say $\mathcal{F}_i(D)$, of $T_i = 0$ planes and an 8-set of $T_i = 1$ planes. All four planes $P \in \mathcal{F}_i(D)$ meet the singular line L_D of D in the same point, say a_i: for if $D = \langle L_D, c, d\rangle$ then, since $T_i(p, c+\lambda Rp, d+\mu Rp) = T_i(p,c,d)$ for $p \in L_D$ and $\lambda, \mu \in \mathrm{GF}(2)$, all four non-singular planes which meet L_D in the point p have the same T_i-value. Thus the singular line of any solid $D \in \Phi^{(3)}_{630}$, can be expressed $L_D = \{a_1, a_2, a_3\}$ where the point a_i is such that the four non-singular planes $P \subset D$ which meet L_D in a_i are $T_j = 0$ planes for $j = i$ and $T_j = 1$ planes for $j \neq i$.

The three singular planes which contain each of the points $a_i \in L_D$ are $T_i = 0$ planes for each $i = 1,2,3$. Consequently, for a given $T \in \mathcal{T} = \{T_1, T_2, T_3\}$, each solid $D \in \Phi^{(3)}_{630}$ contains a unique point $n_D = n_D(T) \in L_D$ with the property that the seven planes $P \subset D$ which pass through n_D are all $T = 0$ planes (and such that the eight planes $P \subset D$ which do not contain n_D are all $T = 1$ planes). We will term this point $n_D = n_D(T)$ the *T-nucleus* of the solid $D \in \Phi^{(3)}_{630}$. The T-nucleus n_D thus satisfies

$$T(x, y, n_D) = 0 \quad \text{for all } x, y \in D. \tag{43}$$

Remark 19 *In a dual vein, for a given $T \in \mathcal{T}$, each line $L \in \Phi_{630}^{(1)}$ is contained in a unique hyperplane $H_L = H_L(T)$ with the property that the seven planes $P \supset L$ which lie inside H_L are all $T = 0$ planes (and such that the eight planes $P \supset L$ which are not in H_L are all $T = 1$ planes). Thus, for $L = \langle a, b \rangle \in \Phi_{630}^{(1)}$, the hyperplane H_L has equation $(f_L(x) =)$ $T(a, b, x) = 0$, that is H_L is the T-associate $L^{\#}$ of L.*

The results for the T-associate of a solid D in $\mathrm{PG}(5, 2)$ are as follows.

Theorem 20 *(i) If $D \in \Theta_{21}^{(3)}$ then $D^{\#} = D$.*
(ii) If $D \in \Theta_{630}^{(3)}$ then $D^{\#} = n_D$.

Proof. (i) Since, see Lemma 12, $T(a, b, c) = 0$ for all $a, b, c \in D$, it follows that $D^{\#} \supseteq D$. But for any plane $P \subset D$ we have $D^{\#} \subseteq P^{\#}$, where, see Theorem 17(i), $P^{\#} = D$. So $D^{\#} = D$.

(ii) Certainly, see (43), the T-nucleus n_D of D is in $D^{\#}$. But we have $D^{\#} \subseteq P^{\#}$, and hence, see Theorem 17(ii), $D^{\#} \subseteq P$, for each of the four non-singular $T = 0$ planes $P \subset D$. But these four planes meet in n_D, and so $D^{\#} = n_D$. ■

Remark 21 *The mapping $L \longmapsto L^{\#}$ is a 10:1 mapping of the 630 non-singular lines $\Theta_{630}^{(1)}$ onto the 63 hyperplanes, and the mapping $D \longmapsto D^{\#}$ is a 10:1 mapping of the 630 non-super-singular solids $\Theta_{630}^{(3)}$ onto the 63 points in* $\mathrm{PG}(5, 2)$.

Finally, concerning hyperplanes we quickly see that $H^{\#} = \emptyset$ for any hyperplane H in $\mathrm{PG}(5, 2)$. This follows since for each solid $D \subset H$ we have $H^{\#} \subseteq D^{\#}$, and so, from Theorem 20, $H^{\#} = \emptyset$.

6 Commentary

(i) There are $|\,\mathrm{GL}(6, 2)|/|\,\mathrm{SL}(3, 4).2| = 2^8.3.7.31 = 166656$ distinct elements $T \in \mathrm{Alt}(\times^3 V_6)$ such that $\mathcal{L}_T$ is a Desarguesian line-spread in $\mathrm{PG}(5, 2)$, and for each $T \in \mathrm{Alt}(\times^3 V_6)$ there are 2^{21} cubic hypersurfaces ψ in $\mathrm{PG}(5, 2)$ such that $T_Q = T$. Consequently there are $N = 2^{29}.3.7.31 = 349502963712$ different cubic hypersurfaces ψ such that $\mathcal{L}_{T_\psi}$ is a Desarguesian line-spread in $\mathrm{PG}(5, 2)$. As well as sharing this last property these N cubics share, despite their great variety, various consequent properties. In particular, for each of these N cubic hypersurfaces ψ any of the 630 solids D which contains just one line of the line-spread $\mathcal{L}_{T_\psi}$ possesses a distinguished point n_D with the property that all planes $P \subset D$ which contain n_D meet ψ in an odd number of points.

(ii) For a given $T \in \mathcal{T}$ the 2^{21} hypersurfaces ψ in $\mathrm{PG}(5, 2)$ having polynomial $Q \in F_3$ such that $T_Q = T$ fall into a large number of $\mathcal{G}_T$-orbits, and it will probably need computer assistance to carry out a complete classification. This complexity is in sharp contrast with the corresponding concerns involving an alternating bilinear form $B \in \mathrm{Alt}(\times^2 V_6)$. For consider the case of $b = b_3$ in (34) for which the alternating bilinear form B is non-degenerate, having as stabilizer

$\mathcal{G}_B \cong \mathrm{Sp}(6,2)$. Such a B arises from just $|F_1| = 64$ quadratic polynomials Q, and the classification of the latter into $\mathrm{Sp}(6,2)$-orbits is well-known: the 64 quadrics $Q(x) = 0$ comprise a family $\mathcal{H}$ of 36 hyperbolic quadrics and a family $\mathcal{E}$ of 28 elliptic quadrics; see for example [4, Lemma 20.7.2], [6, Section 2.2]. Returning to our present concerns, although many of the cubic hypersurfaces such that $T_Q \in \mathcal{T}$ may have small symmetry and be of little interest, it seems to the author that it would be worthwhile investigating those cubic hypersurfaces having a symmetry group $\mathcal{G}_\psi$ of reasonable size. One such cubic hypersurface is the Segre variety $\psi = \mathcal{S}_{1,2} \subset \mathrm{PG}(5,2)$, for which $\mathcal{G}_\psi \cong \mathrm{GL}(2,2) \times \mathrm{GL}(3,2)$ is of order 1008. This cubic hypersurface is investigated in detail in [8]. The present paper throws light upon many of the results found in [8].

(iii) The present paper has been dealing with just one of the $\mathrm{GL}(6,2)$-orbits of trivectors $t \in \wedge^3 V_6^*$. In fact, see [9], there are precisely five $\mathrm{GL}(6,2)$-orbits $\Omega_1,\ \ldots\ ,\Omega_5$ of nonzero trivectors $t \in \wedge^3 V_6^*$ as given by the following choices of orbit representatives $t_i \in \wedge_3 V_6^*,\ i = 1,2,3,4,5$:

$\Omega_1 \quad t_1 = f_{123}; \qquad |\mathcal{L}_{T_1}| = 203; \qquad \mathrm{rad}\, T_1 = \langle e_4, e_5, e_6 \rangle$
$\Omega_2 \quad t_2 = f_{123} + f_{145}; \qquad |\mathcal{L}_{T_2}| = 91; \qquad \mathrm{rad}\, T_2 = \langle e_6 \rangle$
$\Omega_3 \quad t_3 = f_{123} + f_{456}; \qquad |\mathcal{L}_{T_3}| = 49; \qquad \mathrm{rad}\, T_3 = \emptyset;$
$\Omega_4 \quad t_4 = f_{234} + f_{135} + f_{126}; \qquad |\mathcal{L}_{T_4}| = 35; \qquad \mathrm{rad}\, T_4 = \emptyset;$
$\Omega_5 \quad t_5 = f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345}; \quad |\mathcal{L}_{T_5}| = 21; \quad \mathrm{rad}\, T_5 = \emptyset$.

It follows that the $2^{21}(2^{20} - 1)$ cubic hypersurfaces $\psi : Q = 0$ in $\mathrm{PG}(5,2)$ can be classified into five large families $\mathcal{F}_1,\ \ldots\ ,\mathcal{F}_5$, with $\psi \in \mathcal{F}_i$ if $t_Q \in \Omega_i$. The alternating trilinear forms T for the orbits Ω_3 and Ω_4 are particularly worthy of further study since they share with the orbit Ω_5 of the present paper the non-existence of singular points. Presumably there will exist at least a few cubics in the families $\mathcal{F}_3$ and $\mathcal{F}_4$ which have sizeable symmetry groups making them also worthy of further study.

(iv) The $\mathrm{PG}(5,2)$ material in Section 2 surrounding the use of Equation 21 lends itself to generalization to $\mathrm{PG}(n,2)$ for certain values of n. For example, the 73 Points in $\mathrm{PG}(2,8)$ give rise to a Desarguesian 2-spread $\mathcal{P}_{73}$ of 73 planes in $\mathrm{PG}(8,2)$. If $T \in \mathrm{Alt}(\times^3 V_{9,2})$ is defined by $T(x,y,z) = \mathrm{Tr}(\tau(x,y,z))$ where $\tau(\neq 0) \in \mathrm{Alt}(\times^3 V_{3,8})$, then the T-singular lines $\mathcal{L}_T$ are precisely those lines in $\mathrm{PG}(8,2)$ which lie inside a plane $\in \mathcal{P}_{73}$. Further the planes $P \in \mathcal{P}_{73}$ are precisely those planes P whose T-associate $P^{\#}$ is the whole of $\mathrm{PG}(8,2)$.

As another example, the 85 Points in $\mathrm{PG}(3,4)$ give rise to a Desarguesian 1-spread $\mathcal{L}_{85}$ of 85 lines in $\mathrm{PG}(7,2)$. Let $T \in \mathrm{Alt}(\times^4 V_{4,4})$ be defined by $T(x,y,z,w) = \mathrm{Tr}(\tau(x,y,z,w))$ where $\tau(\neq 0) \in \mathrm{Alt}(\times^4 V_{4,4})$. Now for the alternating quadrilinear form T a line $L = \langle a,b \rangle$ in $\mathrm{PG}(7,2)$ is defined to be *T-singular* if

$$T(a,b,x,y) = 0 \qquad \text{for all } x,y \in V_{8,2}. \tag{44}$$

It then follows that the set $\mathcal{L}_T$ of T-singular lines in $\mathrm{PG}(7,2)$ is precisely $\mathcal{L}_{85}$.

References

[1] N.A. Gordon, T.M. Jarvis and R. Shaw, Aspects of the linear groups GL(n,2), *J. Combin. Math. Combin. Comput.* **53** (2005), 13-31.

[2] N.A. Gordon, T.M. Jarvis and R. Shaw, Some aspects of the linear groups GL(n,q), *Research Report,* (2003), see http://www.hull.ac.uk/php/masrs.

[3] R. Lidl & H. Niederreiter, *Introduction to Finite Fields and their Applications,* Cambridge University Press, Cambridge 1994.

[4] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions,* Clarendon, Oxford, 1985.

[5] R. Shaw, Finite geometries and Clifford algebras III, in A.Micali et al., eds., *Proceedings of the 2nd Workshop on Clifford Algebras and their Applications in Mathematical Physics, Montpellier, France,* Kluwer Acad. Pubs. (1992), 121-132.

[6] R. Shaw, Finite geometry, Dirac groups and the table of real Clifford algebras, in R. Ablamowicz and P. Lounesto, eds., *Clifford Algebras and Spinor Structures,* Kluwer Acad. Pubs., Dordrecht, (1995), 59-99.

[7] R. Shaw, The ψ-associate $X^{\#}$ of a flat X in PG$(n,2)$, *Des. Codes Cryptogr.,* **21** (2007), 229-246. [Correction: Eq. (2.13) should end with $(P_3P_5P_6)$, *not* $(P_6P_5P_3)$.]

[8] R. Shaw and N.A. Gordon, The cubic Segre variety in PG$(5,2)$, *Des. Codes Cryptogr.,* **51** (2009), 141-156.

[9] R. Shaw, Trivectors and cubics: PG(5,2) aspects, *Research Report,* (2008), see http://www.hull.ac.uk/php/masrs.

[10] R.A.Wilson *et al.*, *Atlas of Finite Group Representations,* http://brauer.maths.qmul.ac.uk/Atlas/ .

Prof. R. Shaw, Centre for Mathematics,
University of Hull, Hull HU6 7RX, UK