The associate $U^{\#}$ of a flat U with respect to a cubic hypersurface in $\mathrm{PG}(5,2)$

Ron Shaw

Abstract

Given a hypersurface ψ in $\mathrm{PG}(n,2)$ with cubic equation $Q(x)=0$ let $T(x,y,z)$ denotes the alternating trilinear form obtained by completely polarizing Q. The *ψ-associate* $U^{\#}$ of a flat U in $\mathrm{PG}(n,2)$ is defined by

$$U^{\#}=\{z\in\mathrm{PG}(n,2)\,|\,T(x,y,z)=0\text{ for all }x,y\in U\}.$$

Further a projective line L is said to be *singular* if $L^{\#}=\mathrm{PG}(n.2)$, and a point $p\in\mathrm{PG}(n,2)$ is said to be *singular* if every line L containing p is singular. Some general results are first expounded, but thereafter attention is confined to projective dimension $n=5$, and special consideration is given to the following three choices for $\psi\subset\mathrm{PG}(5,2)$:

(a) ψ is the underlying 21-set of a partial spread Σ_3 of three planes in $\mathrm{PG}(5,2)$;

(b) ψ is the underlying 35-set of a non-maximal partial spread Σ_5 of five planes in $\mathrm{PG}(5,2)$;

(c) ψ is the underlying 35-set of a maximal partial spread Ψ_5 of five planes in $\mathrm{PG}(5,2)$.

Results worthy of note include the following.

Choice (a). The singular lines form a complete spread of 21 lines. Further the 48 planes external to ψ fall into eight pairs of ordered triplets $\{(P_1,R_1,S_1),(P_2,R_2,S_2)\}$ such that $\psi^{c}=P_1\cup R_1\cup S_1\cup P_2\cup R_2\cup S_2$ and $P_i^{\#}=R_i,\quad R_i^{\#}=S_i,\quad S_i^{\#}=P_i,\quad i=1,2.$

Choice (b). One plane $W\in\Sigma_5$ is picked out by the property that every line $L\subset W$ is singular. Moreover W is the *privileged* member of Σ_5 as introduced in [R. Shaw, Double-fives and partial spreads in $\mathrm{PG}(5,2)$, in: *Proceedings of the First Pythagorean Conference, Spetses, Greece 1996*].

Choice (c). There is a unique singular point, namely the *nucleus* of Ψ_5 [see R. Shaw, Icosahedral sets in $\mathrm{PG}(5,2)$, *Europ. J. of Combinatorics* **18** (1997), 315-339].

Keywords hypersurfaces in $\mathrm{PG}(n,2)$, associate of a projective flat, partial spreads of planes in $\mathrm{PG}(5,2)$

MSC 2000: 51E20

1 Introduction

Throughout we work over GF(2), and so *we may identify the nonzero elements of a vector space* $V(n+1,2)=V_{n+1}$ *with the points* S_0 *of the associated projective space* $\mathbb{P}V_{n+1}=\mathrm{PG}(n,2)$. Consequently we identify $\mathrm{GL}(V_{n+1})=\mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. We use $\prec u,v,\ ...\ \succ$ for the vector subspace spanned by vectors $u,v,\ ...$, and $\langle u,v,\ ...\ \rangle$ for the flat (projective subspace) generated by projective points $u,v,\ ...$. The vector space $F(S_0)$ of all functions $S_0\to\mathrm{GF}(2)$ is of dimension $|S_0|=2^{n+1}-1$, one basis being $\{\chi_a\}_{a\in S_0}$, where χ_a is the characteristic function of the singleton subset $\{a\}\subset S_0$. Given a choice of coordinates $x_1,x_2,\ ...,x_{n+1}$ in V_{n+1}, there are $\binom{n+1}{r}$ monomials $\{x_{i_1}x_{i_2}...x_{i_r}\}_{1\le i_1<\ ...\ <i_r\le n+1}$ of (reduced) degree r. So altogether we have $\Sigma_{r=1}^{n+1}\binom{n+1}{r}=|S_0|$ linearly independent monomials, and these form another basis for $F(S_0)$. Given a point-set ψ of $\mathrm{PG}(n,2)$ it follows that it has equation $Q(x)=0$ for some *uniquely determined* polynomial Q of minimal degree on V_{n+1} and satisfying $Q(0)=0$. Briefly stated, *every point-set of* $\mathrm{PG}(n,2)$ *is a hypersurface.* The (reduced) degree $d=\deg Q$ of Q is the *polynomial degree* of the point-set ψ. Observe that $d\le n+1$. In fact, see [6, Section 1.2], the following is easily seen to hold.

Lemma 1.1 *If* $|\psi|$ *is even then the point-set* ψ *of* $\mathrm{PG}(n,2)$ *has polynomial degree* $d=n+1$, *while if* $|\psi|$ *is odd then* $d\le n$.

From now onwards we will deal with situations where some particular choice of a non-empty point-set ψ of $\mathrm{PG}(n,2)=PV_{n+1}$ *has been made.* Points, lines, ... , which lie inside ψ will be called *internal* points, lines, ... , and those which lie in ψ^c are termed *external.* Let $\mathcal{G}_\psi<\mathrm{GL}(n+1,2)$ be the stabilizer of the set ψ, and denote by $\mathcal{G}_X$ that subgroup of $\mathcal{G}_\psi$ which stabilizes a subset X of $\mathrm{PG}(n,2)$. Let us suppose that the subset ψ has equation $Q(x)=0$ where the polynomial $Q=Q_\psi$ has (reduced) degree d.

Definition 1.2 *Given the choice of subset* ψ *of* $\mathrm{PG}(n,2)$ *a flat* U *of* $\mathrm{PG}(n,2)$ *is termed* ψ*-odd whenever* $|U\cap\psi|$ *is odd and* ψ*-even whenever* $|U\cap\psi|$ *is even. Once a particular* ψ *has been agreed then we omit the "*ψ*-".*

The next theorem shows that the degree $d=\deg Q$ can be determined from the point-set ψ purely by incidence properties. The theorem deals with an odd point-set ψ, since if $|\psi|$ is even then $d=n+1$ by lemma 1.1.

Theorem 1.3 *(See [7, Theorem 1.1].) If* $|\psi|$ *is odd then* Q *has polynomial degree* d *if and only if* (i) *every* d*-flat is* ψ*-odd and* (ii) *there exists at least one* $(d-1)$*-flat which is* ψ*-even.*

1.1 Cubic hypersurfaces in $\mathrm{PG}(n,2)$

From now onwards we deal with a hypersurface ψ in $\mathrm{PG}(n,2)$ having equation $Q(x)=0$ where the (reduced) degree $d=\deg Q$ of Q is $d=3$. (See [8] for general degree d.) Let $T(v_1,v_2,v_3)$, $v_i\in V_{n+1}$, denote the trilinear alternating form obtained by completely polarizing the cubic polynomial Q :

$$T(v_1,v_2,v_3)=\textstyle\sum_{i=1}^{3}Q(v_i)+\sum_{1\le i<j\le 3}Q(v_i+v_j)+Q(v_1+v_2+v_3). \tag{1.1}$$

Consequently the following lemma holds.

Lemma 1.4 *(i) If the points v_1, v_2, v_3 are dependent then $T(v_1, v_2, v_3) = 0$.*

(ii) If the points v_1, v_2, v_3 are independent, generating a projective plane P, then

$$T(v_1, v_2, v_3) = \textstyle\sum_{v \in P} Q(v), \tag{1.2}$$

and so $T(v_1, v_2, v_3) = 0$ if and only if the plane P is ψ-odd.

Definition 1.5 *For a cubic hypersurface ψ in $\mathrm{PG}(n, 2)$ the ψ-*associate *$U^{\#}$ of an r-flat U is*

$$U^{\#} = \{v \in \mathrm{PG}(n, 2) \,|\, T(u_1, u_2, v) = 0 \textit{ for all } u_1, u_2 \in U\}. \tag{1.3}$$

Since T is trilinear, $U^{\#}$ is an s-flat for some s. Since $U^{\#}$ is uniquely determined by U it follows that $\mathcal{G}_U \leq \mathcal{G}_{U^{\#}}$, where $\mathcal{G}_U$ denotes that subgroup of $\mathcal{G}_\psi$ which stabilizes the flat U. Further, if W is also a flat, then from $W \subset U$ it follows that $U^{\#} \subseteq W^{\#}$.

Remark 1.6 *Definition 1.5 is the special case $d = 3$ of the general definition, see [8], of the ψ-associate of a flat in $\mathrm{PG}(n, 2)$ with respect to a given subset ψ of polynomial degree d. (Note that if $d = 2$ then $U^{\#}$ is the flat, usually denoted $U^{\perp}$, orthogonal to U.) The hope is that, for some subsets ψ of $\mathrm{PG}(n, 2)$, an investigation of the ψ-associates $X^{\#}$ of r-flats may illuminate in an interesting way some of the known geometry of ψ, or conceivably even bring to light some new geometrical aspects.*

Note that the associate $\langle u\rangle^{\ddagger}$ of a point $u \in \mathrm{PG}(n, 2)$ is the whole of $\mathrm{PG}(n, 2)$. Concerning the associate of a line L of $\mathrm{PG}(n, 2)$ we make the following definition.

Definition 1.7 *The* associated linear form *of a line $L = \langle a_1, a_2\rangle$ of $\mathrm{PG}(n, 2)$ is f_L, where*

$$f_L(x) = T(a_1, a_2, x), \quad x \in V_6. \tag{1.4}$$

This is a good definition: since T is alternating and trilinear, any choice of independent points $a_1, a_2 \in L$ yield the same linear form.

The associate $L^{\#}$ of a line L of $\mathrm{PG}(n, 2)$ is thus the flat H_L :

$$H_L := \{x \in \mathrm{PG}(n, 2) |\; f_L(x) = 0\}. \tag{1.5}$$

So $L^{\#}$ is a hyperplane whenever f_L is not the zero form, while $L^{\#} = \mathrm{PG}(n, 2)$ if f_L is the zero form.

Definition 1.8 *If f_L is the zero form the line L will be termed* singular*. Also, for $r \geq 1$, an r-flat will be termed* singular *if it contains one or more singular lines, and* non-singular *if it contains no singular lines. Further a point $p \in \mathrm{PG}(n, 2)$ is termed* singular *if every line L containing p is singular, that is if*

$$T(a_1, a_2, p) = 0, \qquad \textit{for all } a_1, a_2 \in \mathrm{PG}(n, 2). \tag{1.6}$$

Since T is trilinear the set of all singular points is a flat in $\mathrm{PG}(n, 2)$ which we will call the *radical* $\mathrm{rad}\,\psi$ of ψ (and of T).

Because T is trilinear and alternating the next theorem follows quickly— see [8, Lemma 1.11] — from lemma 1.4(ii). It is noteworthy, since it rephrases the definition of the associate $U^{\#}$ solely in terms of certain incidence properties. Because a coordinate form for T is usually quite complicated, use of theorem 1.9, and of its offshoot theorem 1.10, is usually the best way to determine $U^{\#}$.

Theorem 1.9 *Given a cubic hypersurface* ψ *in* $\mathrm{PG}(n,2)$ *let* U *be an* r*-flat of* $\mathrm{PG}(n,2)$. *Then*

(i) for $r \geq 1$ *a point* $y \in U^c$ *is in the* ψ*-associate* $U^{\#}$ *of* U *if and only if for each line* L *of* U *the plane* $\langle y, L\rangle$ *is* ψ*-odd;*

(ii) for $r \geq 2$ *a point* $y \in U$ *is in the associate* $U^{\#}$ *of* U *if and only if every plane* P *of* U *which contains* y *is* ψ*-odd.*

Theorem 1.10 *The following hold for any cubic hypersurface* ψ *in* $\mathrm{PG}(n,2)$.

(i) A point $y \in L^c$ *is in the associate* $L^{\#}$ *of a line* L *if and only if the plane* $\langle y, L\rangle$ *is* ψ*-odd.*

(ii) A line L *is singular if and only if every plane* $P \supset L$ *is* ψ*-odd.*

(iii) The associate of a plane $P = \langle a_1, a_2, a_3\rangle$ *is*

$$P^{\#} = H_{L_{12}} \cap H_{L_{13}} \cap H_{L_{23}},$$

where $L_{ij} = \langle a_i, a_j\rangle$ *and* $H_L = \{x \in \mathrm{PG}(n,2) |\ f_L(x) = 0\}$. *Further,* $P^{\#}$ *is an* s*-flat for some* $s \geq n-3$, *and a point* $y \in P^c$ *lies in* $P^{\#}$ *if and only if each of the three planes* $\langle y, L_{ij}\rangle$ *is odd.*

(iv) If P *is a* ψ*-even plane then* P *is non-singular and* $P^{\#}$ *is a disjoint* $(n-3)$*-flat.*

(v) If P *is a* ψ*-odd plane then* $P^{\#} \supseteq P$; *moreover* $P^{\#}$ *is* ψ*-odd.*

Proof. (i) This is an instance of theorem 1.9(i); (ii) then follows.

(iii) Since T is alternating and trilinear, if y satisfies the three equations

$$T(a_2, a_3, y) = 0, \quad T(a_1, a_3, y) = 0, \quad T(a_1, a_2, y) = 0, \tag{1.7}$$

then y satisfies $T(x_1, x_2, y) = 0$ for all $x_1, x_2 \in X$. These 3 equations being linear, it follows that $P^{\#}$ is an s-flat for some $s \geq n-3$.

(iv) If the plane P is even then by theorem 1.9(ii) no point y of P can lie in $P^{\#}$. Hence $P^{\#}$ is an s-flat for some $s \leq n-3$. Hence, by (iii), $s = n-3$.

(v) If the plane P is odd then theorem 1.9(ii) entails that $P^{\#} \supseteq P$, and so $P^{\#}$ is an s-flat for some $s \geq 2$. If $s > 2$ then $P^{\#}$ is odd since theorem 1.3 (with $d = 3$) entails that every s-flat is odd for $s > 2$. If $s = 2$, then $P^{\#} = P$ and so $P^{\#}$ odd. ∎

1.2 Cubic hypersurfaces in $\mathrm{PG}(5,2)$

From now onwards we confine our attention to $d = 3$ and $n = 5$. From theorem 1.10(iv) we have part (i) of the next theorem, with part (ii) then following from theorem 1.10(iii).

Theorem 1.11 *For any cubic* $\psi \subset \mathrm{PG}(5,2)$ *suppose that* P *is an even plane.*

(i) Then P *is non-singular; further* $P^{\#}$ *is a plane which is disjoint from* P.

(ii) If $P = \langle p_1, p_2, p_3\rangle$ *suppose that* $N = \langle n_1, n_2, n_3\rangle$ *is a plane disjoint from* P *such that each of the nine planes* $\langle p_i, p_j, n_k\rangle$, $1 \leq i < j \leq 3$, $k = 1, 2, 3$, *is odd. Then* $N = P^{\#}$.

The ψ*-even planes* P *in* $\mathrm{PG}(5,2)$ *are thus particularly enticing,* If $P^{\#}$ also turns out to be even, then $P^{\#\#}$ $(= (P^{\#})^{\#})$ is a further plane, which is moreover disjoint from $P^{\#}$ (but not necessarily from P).

Definition 1.12 *An even plane* P *of* $\mathrm{PG}(5,2)$ *is termed* faithfully-even *if each of the planes* $P_1 = P^{\#},\ P_2 = P^{\#\#},\ ...,\ P_{r+1} = (P_r)^{\#}$ *... is also even.*

For any faithfully-even plane P in $\mathrm{PG}(5,2)$ we can define an ordered pair (r,s) of integers r,s, with $0 \leq r < s$, such that the members of the finite "#-sequence" $P_0(= P), P_1, ... P_s$ are distinct and such that $(P_s)^{\#} = P_r$. For example, for a #-sequence of type $(1,2)$ the planes $P, P^{\#}, P^{\#\#}$ are distinct and $P^{\#\#\#}$ coincides with $P^{\#}$.

Concerning odd planes in $\mathrm{PG}(5,2)$, part (v) of theorem 1.10 can be strengthened as follows.

Theorem 1.13 *For any cubic* $\psi \subset \mathrm{PG}(5,2)$ *suppose that* P *is an odd plane. Then* $P^{\#} \supseteq P$ *and* $P^{\#}$ *is odd; further*
(i) if P *is non-singular then* P *is self-associate:* $P^{\#} = P$;
(ii) if P *contains just one singular line then* $P^{\#}$ *is a solid;*
(iii) if P *contains just one pencil of singular lines then* $P^{\#}$ *is a hyperplane;*
(iv) if every line $L \subset P$ *is singular then* $P^{\#} = \mathrm{PG}(5,2)$.

Proof. From theorem 1.10(iii), $x \in P^{\#}$ if and only if $f_L(x) = 0$ holds for three independent choices of the line $L \subset P$.
(i) If no line of P is singular we show now that the three linear forms f_L are linearly independent, and hence that $P^{\#}$ is a plane. For since the lines L are non-singular the forms f_L are not the zero form. Further, a linear relation such as $f_{\langle a_1,a_2\rangle} = f_{\langle a_1,a_3\rangle}$ between two of the f_L is ruled out, for it would entail that $f_{\langle a_1,a_2+a_3\rangle}$ is the zero form, that is that P possesses a singular line $\langle a_1, a_2+a_3\rangle$. Finally the linear relation $f_{\langle a_1,a_2\rangle} + f_{\langle a_1,a_3\rangle} = f_{\langle a_2,a_3\rangle}$ between all three f_L is ruled out for it entails that $f_{\langle a_1+a_2,a_2+a_3\rangle}$ is the zero form, that is that P possesses a singular line $\langle a_1+a_2, a_2+a_3\rangle$.
(ii) If P contains just one singular line then there exists just one linear relation between the three linear forms, whence $P^{\#}$ is a 3-flat .
(iii), (iv) These cover similarly the remaining possibilities, namely that (iii) there are two independent linear relations between the three linear forms, and (iv) all the f_L are zero. ■

2 Cubic hypersurfaces in $\mathrm{PG}(5,2)$ arising from partial spreads of planes

We now confine attention to certain hypersurfaces ψ which arise from partial spreads of planes in $\mathrm{PG}(5,2)$. Consequently *we will have frequent need to appeal to certain basic results concerning such partial spreads which can be found in [2], [3] and [4].* Often we will pay special attention to the associate of even planes P, knowing, from theorem 1.11, that $P^{\#}$ is then necessarily a disjoint plane.

2.1 Introduction

If we view V_6 as $V_3 \oplus V_3$ then a general point $z \in V_6$ is

$$z = (x,y) \in V_6 = V_3 \oplus V_3,\ x,y \in V_3. \tag{2.1}$$

Let A be an element of $\mathrm{GL}(V_3)$ of order 7 and satisfying $A^3 = I + A$. Since A is fixed-point-free on $\mathbf{P}V_3 = PG(2,2)$ it gives rise, *cf.* [3, Section 4.1], to the following plane-spread Σ_9 for $\mathbf{P}V_6 = \mathrm{PG}(5,2)$:

$$\Sigma_9 = \{X, Y, P_0, P_1, \ \dots \ , P_6\} \tag{2.2}$$

upon defining the nine planes $X, Y, P_0, P_1, \ \dots \ , P_6$ of $\mathrm{PG}(5,2)$ by:

$$\begin{aligned} X &= \{(x,0),\ x \in \mathbb{P}V_3\},\ Y = \{(0,y),\ y \in \mathbb{P}V_3\}, \\ P_r &= \{(x, A^r x),\ x \in \mathbb{P}V_3\}, \quad r = 0,1,2 \ \dots \ ,6. \end{aligned} \tag{2.3}$$

Upon choosing a point $a_1 \in \mathbb{P}V_3$ and setting $a_{r+1} = A^r a_1,\ r = 0,1,\ \dots \ ,6$, the 49 points underlying the seven planes $P_0, P_1, \ \dots \ , P_6$ are the points $(a_i, a_j),\ 1 \leq i,j \leq 7$. In detail

$$P_r = \{(a_i, a_{i+r}),\ i = 1,2,\ \dots \ ,7\}, \qquad r = 0,1,2,3,4,5,6, \tag{2.4}$$

$$X = \{(a_i,0),\ i = 1,2,\ \dots \ ,7\},\ Y = \{(0,a_i),\ i = 1,2,\ \dots \ ,7\}, \tag{2.5}$$

where in (2.4) the value of the index $i + r$ is taken $\bmod 7$.

Shorthand notation. It helps at times to adopt the shorthand notation ij for (a_i, a_j), $i0$ for $(a_i, 0)$ and $0j$ for $(0, a_j)$.

For future reference let us note here a particular subgroup of the symmetry group $\mathcal{G}(\Sigma_9)$ of the spread Σ_9. Recall that the normalizer in $\mathrm{GL}(V_3)$ of the Z_7 subgroup $\langle A \rangle$ is a subgroup $\mathcal{F}_{21} \cong Z_7 \rtimes Z_3$ of order 21 which is generated by A and C where $C \in \mathrm{GL}(V_3)$, of order 3, satisfies $CAC^{-1} = A^2$. Setting $\Phi_A = A \oplus A$ and $\Phi_C = C \oplus C$ observe that the group $\mathcal{F} = \langle \Phi_A, \Phi_C \rangle$ is a subgroup $\cong Z_7 \rtimes Z_3$ of $\mathcal{G}(\Sigma_9)$: for Φ_A stabilizes each plane of Σ_9, while Φ_C stabilizes each of X, Y, P_0 and effects the permutations $(P_1 P_2 P_4)$, $(P_6 P_5 P_3)$ of the other six planes of Σ_9.

It is known, see [4, Theorem 4.1], that in $\mathrm{PG}(5,2)$ all complete plane-spreads are projectively equivalent to the spread Σ_9; also if $\Sigma_r, r < 9$, is any r-subset of Σ_9 then all non-maximal partial plane-spreads of size r are projectively equivalent to Σ_r. Further, up to projective equivalence, there is just one kind of maximal partial spread: it is of size 5 and is represented by

$$\Psi_5 = \{X, Y, P_0, P_1, S\}, \tag{2.6}$$

where S is one of the seven planes external to $X \cup Y \cup P_0 \cup P_1$ other than the five planes $P_2, \ \dots \ , P_6$. (The plane S is in fact of the form $S = \{(x, Bx),\ x \in \mathbb{P}V_3\}$ where B is an element of $\mathrm{GL}(V_3)$ of order 7 such that $A^{-1}B$ is of order 7 and $AB \neq BA$. The other six planes arise by replacing B by $A^k B A^{-k}$, $1 \leq k \leq 6$.)

From now on we will confine our attention to three particular cubic hypersurfaces given by the point-sets underlying (a) a partial spread Σ_3 (b) a non-maximal partial spread Σ_5 and (c) a maximal partial spread Ψ_5. The following choices of point-set ψ will prove to be convenient:

$$\begin{aligned} &\text{(a) } \psi_a = X \cup Y \cup P_0 \\ &\text{(b) } \psi_b = Y \cup P_1 \cup P_2 \cup P_4 \cup P_0 = (X \cup P_6 \cup P_5 \cup P_3)^c \\ &\text{(c) } \psi_c = X \cup Y \cup P_0 \cup P_1 \cup S. \end{aligned} \tag{2.7}$$

That each of these sets ψ indeed has polynomial degree $d = 3$ follows quickly from theorem 1.3. For each ψ is the union of an odd number of pairwise disjoint planes, whence $|D \cap \psi|$ is odd for any solid D, since a solid in $\mathrm{PG}(5,2)$ meets every plane in an odd number of points. On the other hand there exist in each case ψ-even planes: in the first two cases there exist, see (2.2), external planes, and in the third case it is not difficult to find planes which meet ψ_c in 2 and in 4 points. (See [2, Section 5.2].)

Remark 2.1 *The 35-set ψ_c is called a* double-five *since, see [2], [3], [4], it admits two different decompositions into five disjoint planes.*

2.2 The case $\psi_a = X \cup Y \cup P_0 = \mathcal{S}_{1,2,2}$

It is worth noting that ψ_a may be viewed as the Segre variety

$$\mathcal{S}_{1,2,2} = \{v \otimes w : v \in \mathbb{P}V_2,\ w \in \mathbb{P}V_3\}, \tag{2.8}$$

which lies in $\mathrm{PG}(5,2) = \mathbb{P}V_6$, and which consists of the 21 decomposable elements of a tensor product space $V_6 = V_2 \otimes V_3$. For note that $\mathcal{S}_{1,2,2}$ contains three internal planes, namely $\mathbb{P}(v \otimes V_3)$, $v \in \mathbb{P}V_2$, which may be identified with the three internal planes X, Y, P_0 of ψ_a. The seven internal lines $\mathbb{P}(V_2 \otimes w)$, $w \in \mathbb{P}V_3$, of $\mathcal{S}_{1,2,2}$ are the transversals of the planes X, Y, P_0. The stabilizer $\mathcal{G}_{\psi_a}$ of $\psi_a = \mathcal{S}_{1,2,2}$ consists of elements of $\mathrm{GL}(V_2 \otimes V_3)$ of the form $C \otimes D$, $C \in \mathrm{GL}(V_2)$, $D \in \mathrm{GL}(V_3)$ and so $\mathcal{G}_{\psi_a}$ is isomorphic to $\mathrm{GL}(2,2) \times \mathrm{GL}(3,2)$, of order $6 \times 168 = 1008$.

Worthy of note are two particular elements $J, K \in \mathrm{GL}(V_3 \oplus V_3)$ defined by

$$J(x,y) = (y,x), \qquad K(x,y) = (y, x+y), \qquad x, y \in V_3, \tag{2.9}$$

and satisfying $J^2 = I$ and $K^3 = I$.

Lemma 2.2 *Both J and K are elements of $\mathcal{G}_{\psi_a}$, and also of $\mathcal{G}(\Sigma_9)$.*

Proof. This follows since J and K effect the following permutations

$$J : (XY)(P_0)(P_1P_6)(P_2P_5)(P_4P_3), \qquad K : (XYP_0)(P_1P_2P_4)(P_3P_5P_6) \tag{2.10}$$

of the members of Σ_9. For example, K maps the element $(x, Ax) \in P_1$ to $(Ax, (I+A)x) = (Ax, A^3x)$, which last is an element of P_2. ■

Notation 2.3 *We denote by $\Omega_N^{(r)}(n_1n_2n_3))$ a $\mathcal{G}_{\psi_a}$-orbit of r-flats of length $|\Omega_N^{(r)}(n_1n_2n_3)| = N$ whose 'intersection pattern' with the three internal planes of ψ_a is $n_1n_2n_3 := \{n_1, n_2, n_3\}$. For example the $\mathcal{G}_{\psi_a}$-orbit $\Omega_7^{(3)}(333)$ consists of those 7 solids which meet each of the internal planes X, Y, P_0 in a line.*

There are two $\mathcal{G}_{\psi_a}$-orbits of internal lines, namely the 7 transversals $\Omega_7^{(1)}(111)$ of the three planes X, Y, P_0, and the $(3 \times 7 =)21$ lines $\Omega_{21}^{(1)}(300)$ which lie inside one of the planes X, Y, P_0. Through a line $L \in \Omega_{21}^{(1)}(300)$ pass three lines $\in \Omega_7^{(1)}(111)$, and these three generate a solid of an orbit $\Omega_7^{(3)}(333)$. A solid $D \in \Omega_7^{(3)}(333)$ is of the form $D = \mathcal{H} \cup L' \cup L''$, where the 9-set $\mathcal{H} := D \cap \psi_a$ is a hyperbolic quadric in the 3-flat D and where L', L'' are the two lines of D which are external to $\mathcal{H}$. Such external lines L', L'' form a $\mathcal{G}_{\psi_a}$-orbit $\Omega^{(1)}_{14}(000)$. (There exist in fact a further 168 external lines and these form an orbit $\Omega^{(1)}_{168}(000)$.)

Of course it is fairly easy to determine all the $\mathcal{G}_{\psi_a}$-orbits of flats. However, rather than pursuing the full story of the ψ_a-associates of all flats, we content ourselves here with reporting just a couple of particularly interesting aspects, one concerning the singular lines and the other the even planes.

For most lines L it is easy to find a plane $P \supset L$ which is ψ_a-even, whence by theorem 1.10(ii) L is non-singular. But if L belongs to one of the orbits $\Omega^{(1)}_7(111), \Omega^{(1)}_{14}(000)$ one checks that all 15 planes $P \supset L$ are odd. Hence we have the next theorem, from which it follows that there are no singular points: $\operatorname{rad} \psi_a = \emptyset$. Another consequence is that a plane of $\mathrm{PG}(5,2)$ can contain at most one singular line. (Hence only parts (i) and (ii) of theorem 1.13 are relevant for $P^{\#}$ if P is an odd plane.)

Theorem 2.4 *There are precisely* 21 *ψ_a-singular lines in* $\mathrm{PG}(5,2)$, *namely the lines* $L \in \Omega^{(1)}_7(111) \cup \Omega^{(1)}_{14}(000)$. *Moreover these constitute a line-spread for* $\mathrm{PG}(5,2)$.

Amongst the ψ_a-even planes are the planes P which are external to $\psi_a = X \cup Y \cup P_0$. Without loss of generality, see [3, Section 4.1], [4, Section 4], we may assume that such a P is one of the planes $P_r = \{(x, A^r x),\ x \in \mathbb{P}V_3\}$, $r = 1, 2\ \ldots, 6$, in eq. (2.3), for some element $A \in \mathrm{GL}(3,2)$ of order 7 and satisfying $A^3 = I + A$. Since there exist precisely eight Z_7-subgroups of $\mathrm{GL}(3,2)$ there are $8 \times 6 = 48$ planes external to ψ_a, forming a $\mathcal{G}_{\psi_a}$-orbit $\Omega^{(2)}_{48}(000)$.

Theorem 2.5 *If ψ_a and P_r are as in (2.7) and (2.3) then the associates of the external planes P_r are as follows:*

$$(P_1)^{\#} = P_2, \qquad (P_2)^{\#} = P_4, \qquad (P_4)^{\#} = P_1, \tag{2.11}$$

$$(P_6)^{\#} = P_5, \qquad (P_5)^{\#} = P_3, \qquad (P_3)^{\#} = P_6. \tag{2.12}$$

So for any external plane P we have $P^{\#\#\#} = P$.

Proof. Knowing from theorem 1.11 that $(P_1)^{\#}$ is a plane, to show that $(P_1)^{\#} = P_2$ we need to show, see theorem 1.9(i), that the plane $\langle z, L\rangle$ is ψ-odd for each point $z \in P_2$ and each line $L \subset P_1$. Because the planes of the spread 2.2 are invariant under the Z_7 subgroup of $\mathcal{G}_{\psi_a}$ generated by $\Phi_A = A \oplus A$, it suffices to consider just one choice of line $L \subset P_1$, say $L_1 := \{(a_1, a_2), (a_2, a_3), (a_4, a_5)\}$ (in the notation of equation (2.4)). Then one sees that:
(a) if $z = (a_1, a_3) \in P_2$ then the plane $\langle z, L_1\rangle$ meets $\mathcal{S}_{1,2,2}$ in the three points $(0, a_5) \in Y$, $(a_4, 0) \in X$ and $(a_2, a_2) \in P_0$;
(b) for the other six points $z \in P_2$ the plane $\langle z, L_1\rangle$ meets just one of the planes X, Y, P_0.
So indeed $\langle z, L\rangle$ is ψ-odd for each point $z \in P_2$ and each line $L \subset P_1$, whence $P_2 = (P_1)^{\#}$. On applying to this last relation the element $K \in \mathcal{G}_{\psi_a}$ of lemma 2.2 we obtain the other two relations in (2.11). Further the relations (2.12) follow from the relations (2.11) upon applying the involution $J \in \mathcal{G}_{\psi_a}$, see (2.10). ∎

Corollary 2.6 *The* 48 *planes external to the Segre variety* $\psi_a = \mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$ *fall into eight pairs of ordered triplets* $\{(R_1, S_1, U_1), (R_2, S_2, U_2)\}$ *satisfying* $\psi^c = R_1 \cup S_1 \cup U_1 \cup R_2 \cup S_2 \cup U_2$ *and*

$$R_i^{\#} = S_i, \quad S_i^{\#} = U_i, \quad U_i^{\#} = R_i, \quad i = 1, 2. \tag{2.13}$$

Besides the external planes $\Omega^{(2)}_{48}(000)$ one finds that there are just two further orbits of even planes, namely $\Omega^{(2)}_{168}(310$ and $\Omega^{(2)}_{504}(110)$. In order to deal with these planes of patterns 310 and 110 it will prove convenient to adopt the temporary notation W_1, W_2, W_3 for the three internal planes X, Y, P_0 of ψ_a. Further we will make a choice of a solid $\sigma \in \Omega^{(3)}_7(333)$, and also choose a line $M \in \Omega^{(1)}_7(111)$ which is disjoint from σ. So $M = \{m_1, m_2, m_3\}$ where $\{m_i\} = M \cap W_i$.

Theorem 2.7 *In the foregoing notation, denote by* $L_i = \sigma \cap W_i,\ i = 1, 2, 3,$ *the three lines of* σ *which belong to the orbit* $\Omega^{(1)}_{21}(300)$*. Define planes* N_1, N_2, N_3 *and* N'_1, N'_2, N'_3 *as follows:*

$$N_1 = \langle m_2, L_3\rangle, \quad N_2 = \langle m_3, L_1\rangle, \quad N_3 = \langle m_1, L_2\rangle, \tag{2.14}$$

$$N'_1 = \langle m_3, L_2\rangle, \quad N'_2 = \langle m_1, L_3\rangle, \quad N'_3 = \langle m_2, L_1\rangle. \tag{2.15}$$

Then N_1, N_2, N_3 *are pairwise disjoint planes of the orbit* $\Omega^{(2)}_{168}(310)$, *and so are* N'_1, N'_2, N'_3*. Moreover*

$$(N_1)^{\#} = N_2, \quad (N_2)^{\#} = N_3, \quad (N_3)^{\#} = N_1, \tag{2.16}$$

$$(N'_1)^{\#} = N'_3, \quad (N'_2)^{\#} = N'_1, \quad (N'_3)^{\#} = N'_2. \tag{2.17}$$

Proof. Since N_3 meets ψ in the four points $\{m_1\} \cup L_2$ of pattern 130, then $N_3 \in \Omega^{(2)}_{168}(310)$; similarly $N_1, N_2, N'_1, N'_2, N'_3 \in \Omega^{(2)}_{168}(310)$. In order to prove the results (2.16), (2.17) we may without loss of generality suppose that, in shorthand notation, $L_1 = \{10, 20, 40\}$, $L_2 = \{01, 02, 04\}$, $L_3 = \{11, 22, 44\}$ and $M = \{30, 03, 33\}$. Then a straightforward application of theorem 1.11(ii) quickly confirms the result $(N_1)^{\#} = N_2$. From this last result the remaining five results (2.16), (2.17) follow upon applying the symmetries J and K of lemma 2.2. For K effects the permutations $(L_1L_2L_3)$ and $(m_1m_2m_3)$, and hence effects $(N_1N_2N_3)$ and $(N'_1N'_2N'_3)$, while the involution J effects the permutations $(L_1L_2)(L_3)$ and $(m_1m_2)(m_3)$, and hence effects $(N_1N'_2)(N_2N'_1)(N_3N'_3)$. ∎

Theorem 2.8 *In the foregoing notation, let* $L = \{p_1, p_2, p_3\}$*, where* $\{p_i\} = L \cap W_i$*, be one of the three lines of* σ *which belong to the orbit* $\Omega^{(1)}_7(111)$*. Denote by* $T_i,\ i = 1, 2, 3,$ *that tangent to* ψ_a *which contains* p_i *and which lies in* σ*. Define planes* Q_1, Q_2, Q_3 *and* Q'_1, Q'_2, Q'_3 *as follows:*

$$Q_1 = \langle m_2, T_3\rangle, \quad Q_2 = \langle m_3, T_1\rangle, \quad Q_3 = \langle m_1, T_2\rangle, \tag{2.18}$$

$$Q'_1 = \langle m_3, T_2\rangle, \quad Q'_2 = \langle m_1, T_3\rangle, \quad Q'_3 = \langle m_2, T_1\rangle. \tag{2.19}$$

Then Q_1, Q_2, Q_3 *are pairwise disjoint planes of the orbit* $\Omega^{(2)}_{504}(110)$, *and so are* Q'_1, Q'_2, Q'_3*. Moreover*

$$(Q_1)^{\#} = Q_2, \quad (Q_2)^{\#} = Q_3, \quad (Q_3)^{\#} = Q_1, \tag{2.20}$$

$$(Q'_1)^{\#} = Q'_3, \quad (Q'_2)^{\#} = Q'_1, \quad (Q'_3)^{\#} = Q'_2, . \tag{2.21}$$

Proof. Since Q_3 meets ψ in the two points $\{m_1, p_2\}$ of pattern 110, then $Q_3 \in \Omega^{(2)}_{504}(110)$; and similarly $Q_1, Q_2, Q'_1, Q'_2, Q'_3 \in \Omega^{(2)}_{504}(110)$. In order to prove the results (2.20), 2.21 we may, without loss of generality, suppose that, in shorthand notation, $\sigma = \langle 20, 30, 02, 03\rangle$, $M = \{10, 01, 11\}$ and $L = \{20, 02, 22\}$, whence $T_1 = \{20, 32, 52\}$, $T_2 = \{02, 23, 25\}$, $T_3 = \{22, 35, 53\}$. Then a straightforward application of theorem 1.11(ii) quickly confirms the result $(Q_{\ 1})^{\#} = Q_2$. The remaining five results then follow upon applying, *cf.* the preceding proof, the symmetries J and K of lemma 2.2. ■

The next theorem summarizes results in the preceding three theorems.

Theorem 2.9 *There are three $\mathcal{G}_{\psi_a}$-orbits of even planes in* $\mathrm{PG}(5, 2)$*:*

$$\Omega^{(2)}_{168}(310, \quad \Omega^{(2)}_{504}(110), \quad \Omega^{(2)}_{48}(000). \tag{2.22}$$

If P is any even plane then $P, P^{\#}, P^{\#\#}$ are pairwise disjoint planes belonging to the same $\mathcal{G}_{\psi_a}$-orbit; further $P^{\#\#\#} = P$. Thus every even plane is faithfully-even, with $\#$-sequence $(0, 2)$.

Remark 2.10 *One should not attribute this property $P^{\#\#\#} = P$ of even planes solely to the fact that ψ_a has degree $d = 3$. For consider the case of the Grassmannian variety $\psi = \mathcal{G}_{1,4,2}$ in* $\mathrm{PG}(9, 2)$*, where $d = 5$ and so where, see [5, Corollary 3.2] the analogues of the ψ_a-even planes are the $\mathcal{G}_{1,4,2}$-even* 4*-flats in* $\mathrm{PG}(9, 2)$*. Now it was found in [5, Section 3] (using the definition of $X^{\#}$ appropriate to $d = 5$) that there exist $\mathcal{G}_{1,4,2}$-even 4-flats X such that $X, X^{\#}, X^{\#\#}$ are pairwise disjoint even 4-flats satisfying $X^{\#\#\#} = X$.*

2.3 The case $\psi_b = Y \cup P_1 \cup P_2 \cup P_4 \cup P_0$

We now consider the 35-set ψ_b underlying the non-maximal partial spread $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$.

Lemma 2.11 *(i) There exists a subgroup $\mathcal{A} \cong$* $\mathrm{Alt}(4)$ *of $\mathcal{G}_{\psi_b}$ which fixes the plane P_0 and effects all even permutations of the four planes $\{Y, P_1, P_2, P_4\}$. The normal subgroup $\mathcal{N} \cong (Z_2)^2$ of $\mathcal{A}$ is $\mathcal{N} = \{I, J', J'', J'''\}$ where*

$$\begin{aligned} J'(x, y) &= (A^3x + Ay, Ax + A^3y), \\ J''(x, y) &= (A^6x + A^2y, A^2x + A^6y), \\ J'''(x, y) &= (A^5x + A^4y, A^4x + A^5y). \end{aligned} \tag{2.23}$$

(ii) The subgroup $\mathcal{B} := \langle \Phi_A, \mathcal{N} \rangle$ of $\mathcal{G}_{\psi_b}$, of order 28*, has the structure $Z_7 \times (Z_2)^2$. It acts transitively upon the $7 \times 4 = 28$ internal lines, and also upon the* 28 *internal points, of the four planes Y, P_1, P_2, P_4.*

Proof. (i) Recall from section 2.1 that the element $\Phi_C \in \mathcal{G}(\Sigma_9)$, of order 3, stabilizes both Y and P_0 and effects the permutation $(P_1P_2P_4)$; so $\Phi_C \in \mathcal{G}_{\psi_b}$. Defining $J' \in \mathrm{GL}(V_3 \oplus V_3)$ by $J'(x, y) = (A^3x + Ay, Ax + A^3y)$ one sees that J' is an involution which effects the permutation $(YP_2)(P_1P_4)(P_0)$ of the members of Σ_5; so $J' \in \mathcal{G}_{\psi_b}$. The subgroup $\mathcal{A} := \langle \Phi_C, J' \rangle$ of $\mathcal{G}_{\psi_b}$ thus has the stated permutation property. The element $\Phi_C J'$ of $\mathcal{A}$ is seen to have order 3, and so it follows, see [1, Table 1], that the group $\mathcal{A}$ generated by Φ_C and J' is isomorphic to $\mathrm{Alt}(4)$.

Besides J' the other two involutions inside $\mathcal{A} \cong \mathrm{Alt}(4)$ are $J'' := \Phi_C J' \Phi_{C^{-1}}$ and $J''' := \Phi_{C^{-1}} J' \Phi_C$, and so are as stated in (2.23).

(ii) Since Φ_A commutes with J', J'' and J''' in (2.23) the subgroup $\mathcal{B}$ has the stated direct product structure. The rest follows since by part (i) $\mathcal{N}$ is transitive upon the four planes Y, P_1, P_2, P_4 while $\langle \Phi_A \rangle \cong Z_7$ is transitive upon the lines, and also the points, of each of these four planes. ■

Remark 2.12 *Setting* $y = x$ *in (2.23) we see, since* $A^3 + A = I$, *that all three involutions in* $\mathcal{N}$ *fix the plane* P_0 *pointwise. Moreover observe that the involution* J, *see (2.9), not only fixes the plane* P_0 *pointwise but also centralizes* $\mathcal{N}$. *In fact the seven involutions of the subgroup* $\langle J, \mathcal{N}\rangle \cong (Z_2)^3$ *of* $\mathcal{G}(\Sigma_9)$ *are precisely those involutions of* $\mathcal{G}(\Sigma_9)$ *which stabilize the plane* P_0.

Theorem 2.13 *If* ψ_b *is the* 35*-set underlying the non-maximal partial spread* $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$ *then*
(i) every line L *of the plane* P_0 *is* ψ_b*-singular;*
(ii) the lines of the other four planes of Σ_5 *are all non-singular.*

Proof. (i) In shorthand notation let D be the solid $\langle 01, 02, 12, 23\rangle$ and observe that D is disjoint from the line $L = \{22, 33, 55\} \subset P_0$. Hence the fifteen planes P which contain L are the fifteen planes $\langle p, L\rangle, p \in D$. By a straightforward check we find that all fifteen planes are ψ_b-odd, with twelve meeting ψ_b in 5 points, one in 3 points and two (P_0 and one other) being internal to ψ_b. Hence, by theorem 1.10(ii), L is singular. The same is true for all 7 lines of P_0, since these form a single orbit under the action of the subgroup $\langle \Phi_A\rangle \cong Z_7$ of $\mathcal{G}_{\psi_b}$.

(ii) By lemma 2.11(ii) the 28 internal lines of the four planes Y, P_1, P_2, P_4 lie on the same $\mathcal{G}_{\psi_b}$-orbit, so it suffices to check that just one of them is non-singular. In shorthand notation consider the line $L = \{01, 02, 04\} \subset Y$ and observe that the plane $\langle 30, L\rangle$ is even, meeting ψ_b in the four points $\{34\} \cup L$. ■

The author was on the verge of being surprised by the fact that for any non-maximal partial plane-spread Σ_5 in PG(5, 2) one plane of Σ_5 is thus singled out as being that unique plane of Σ_5 which is singular. But then he recalled that eleven years ago he had, in talks delivered at the Pythagorean and Italian Combinatorics conferences in 1996 (see [3, Theorem 4.3] and [4, Theorem 4.2]), uncovered the existence of a *privileged plane*, as described in part (i) of the next theorem. It was then hardly a surprise to find that this privileged plane of Σ_5 coincided with the singular plane of Σ_5.

Theorem 2.14 *(i) Any non-maximal partial spread* Σ_5 *of* 5 *planes in* PG(5, 2) *possesses a privileged member, say* $W_* \in \Sigma_5$, *such that each of the* 7 *further internal planes* S_i *of* ψ_b *(see after (2.6)) meets* W_* *in a line and meets each of the four other planes* $W \in \Sigma_5$ *in a point. The* 7 *lines* $S_i \cap W_*$ *are distinct, as are the* 28 *points* $S_i \cap W$, $i = 0, 1, \ldots 6$, $W \in \Sigma_5 \setminus \{W_*\}$.

(ii) If ψ *is the underlying* 35*-set of* Σ_5 *then the privileged plane of* Σ_5 *is that plane of* Σ_5 *which is* ψ*-singular.*

Proof. (i) See [4, Theorem 4.2].

(ii) If ψ_b and Σ_5 is as in theorem 2.13 then P_0 must be the privileged plane of Σ_5 since it is the only plane of Σ_5 fixed by the subgroup $\mathcal{N}$ of $\mathcal{G}_{\psi_b}$. (*Aliter:* see [4, after eq. (4.23)].) ■

Remark 2.15 *Consider the following* $4 + 1 + 4$ *partition of the spread* Σ_9*:*

$$\Sigma_9 = \Sigma_4 \cup \{P_0\} \cup \Sigma'_4, \text{ where } \Sigma_4 = \{Y, P_1, P_2, P_4\},\ \Sigma'_4 = \{X, P_6, P_5, P_3\}. \quad (2.24)$$

It is known, see [3, Section 4], that P_0 *is the privileged plane of both* $\Sigma_5 = \Sigma_4 \cup \{P_0\}$ *and* $\Sigma'_5 = \Sigma'_4 \cup \{P_0\}$, *and that the symmetry groups* $\mathcal{G}(\Sigma_4), \mathcal{G}(\Sigma_5), \mathcal{G}(\Sigma'_4)$

and $\mathcal{G}(\Sigma'_5)$ *are all equal to* $\mathcal{G}_{\psi_b}$. *Further the full stabilizer of* ψ_b *is*

$$\mathcal{G}_{\psi_b} = \langle \Phi_A, \mathcal{A} \rangle \cong (Z_7 \times (Z_2)^2) \rtimes Z_3; \tag{2.25}$$

moreover the subgroup $\mathcal{G}(\Sigma_4, \Sigma'_4)$ *of* $\mathcal{G}(\Sigma_9)$ *consisting of all those elements which preserve the* $4+1+4$ *partition (2.24) is the following direct product:*

$$\mathcal{G}(\Sigma_4, \Sigma'_4) = \mathcal{G}_{\psi_b} \times \langle J \rangle. \tag{2.26}$$

Here the involution J *of (2.9) effects the interchanges* $\Sigma_4 \rightleftarrows \Sigma'_4$ *and* $\Sigma_5 \rightleftarrows \Sigma'_5$ *via its effect* $(XY)(P_0)(P_1P_6)(P_2P_5)(P_4P_3)$, *see (2.10), on the planes of* Σ_9.

Theorem 2.16 *As previously let* ψ_b *be the* 35*-set underlying the non-maximal partial spread* $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$ *whose (unique) extension to a complete spread* Σ_9 *is as in (2.24). Then*
(i) $(P_0)^\# = \mathrm{PG}(5,2)$;
(ii) if $P \in \Sigma_4 := \Sigma_5 \setminus \{P_0\}$ *then* P *is self-associate:* $P^\# = P$;
(iii) if $P \in \Sigma'_4 := \Sigma_9 \setminus \Sigma_5$ *then* $P^\# = J(P)$, *where* J *is as in (2.26).*

Proof. (i) This follows immediately from theorems 2.13(i) and 1.13(iv).
(ii) This follows immediately from theorems 2.13(ii) and 1.13(i).
(iii) First let us check that $X^\# = Y$. Since X is an even plane, we know from theorem 1.11 that $X^\#$ is a plane disjoint from X. Now X is stabilized by the element $\Phi_C \in \mathcal{G}_{\psi_b}$; moreover the only other two planes in $\mathrm{PG}(5,2)$ which are stabilized by Φ_C are P_0 and Y. Hence either $X^\# = Y$ or $X^\# = P_0$. The last possibility is easily ruled out: for example, in shorthand notation, the plane $R = \langle p, L \rangle$ joining the point $p = 77$ of P_0 to the line $L = \langle 10, 20 \rangle$ of X is even, meeting ψ_b in the four points $\{77, 37, 57.67\}$. Consequently $P^\# = J(P)$ holds in the case of $X \in \Sigma_4$. Since J centralizes the subgroup $\mathcal{N}$ of $\mathcal{G}_{\psi_b}$, and since $\mathcal{N}$ is transitive on the four planes Σ_4, result (iii) follows. ∎

Note that in the present $\psi = \psi_b$ case the even planes $P \in \Sigma'_4$ are thus very far from being faithfully even, since their first associate $P^\#$ is odd.

The next theorem relates the privileged plane of a non-maximal partial spread Σ_5 to the ψ_a-associate of section 2.2.

Theorem 2.17 *Suppose that* Σ_4 *is any partial spread of four planes in* $\mathrm{PG}(5,2)$. *Let* $\Sigma_9 = \Sigma_4 \cup \Sigma_5$ *be its unique extension to a complete spread. Choosing* $P \in \Sigma_4$, *let* ψ_a *denote the underlying* 21*-set of the three planes* $\Sigma_4 \setminus \{P\}$ *and use* ψ_a *to define the associate* $X^\#$ *of a flat* X *of* $\mathrm{PG}(5,2)$. *Then the privileged plane of* Σ_5 *is* $P^{\#\#}$.

Proof. If $\Sigma_4 = \{X, Y, P_0, P_1\}$ then, see [4, Theorem 4.3, Proof], the privileged plane of Σ_5 is P_4. But, see theorem 2.5, for $\psi_a = X \cup Y \cup P_0$ we have $P_4 = P_1^{\#\#}$. ∎

2.4 The case ψ_c of a double-five of planes

In [2] the 35-set $\psi_c \subset \mathrm{PG}(5,2)$ was constructed starting out from a regular icosahedron whose faces were coloured in two enantiomorphic ways — the two colourings being related to *two decompositions* $\psi_c = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5$ *and* $\psi_c = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$ *of* ψ_c *into mutually disjoint planes, each set of five* *planes being a maximal partial spread.* The symmetry group of ψ_c is icosahedral: $\mathcal{G}_{\psi_c} \cong \mathrm{Alt}(5) \times Z_2$. As noted after eq. (2.7) it has a cubic equation $Q(x) = 0$. In [3, Section 2] can be found the explicit cubic polynomial Q, and its associated alternating trilinear form T, using an appropriate choice of coordinates. Also in [3, Section 2] it is shown that there exists a $\mathcal{G}_{\psi_c}$-invariant non-degenerate symplectic form $x.y$ on V_6, with respect to which each plane α_r, and each plane β_r, is self-polar: $(\alpha_r)^\perp = \alpha_r$, $(\beta_r)^\perp = \beta_r$.

As discussed in [8, Section 2.6] the intricate geometric structure of the set ψ_c seems to hold out considerable promise for interesting results concerning the ψ_c-associate of flats. However, as partly spelled out in the next theorem, and ensuing remarks, these initial hopes appear not to be borne out, due to the existence of a (unique) fixed point u of $\mathcal{G}_{\psi_c}$, termed the *nucleus* of ψ_c. (In fact the hyperplane $u.x = 0$ meets ψ_c in a parabolic quadric $\mathcal{P}_4$ whose nucleus is u.)

Theorem 2.18 *(i) The nucleus u of a double-five ψ_c is a singular point.*

(ii) If U is any flat in $\mathrm{PG}(5,2)$ *then $U^\#$ passes through u.*

(iii) If P is any ψ_c-even plane in $\mathrm{PG}(5,2)$ *then $P^\#$ is an odd plane. (So no plane in* $\mathrm{PG}(5,2)$ *is faithfully even.)*

(iv) If P is any ψ_c-odd plane then P is singular.

Proof. (i) As noted in [3, Section 2.2, Remark (i)], $T(x, y, u) = 0$ for all $x, y \in V_6$.

(ii) This follows immediately from the definition of $U^\#$ in (1.3).

(iii) From (ii), and from theorem 1.11(i), we know that $P^\#$ is a plane containing u. From (1.2) it follows that $\sum_{v \in P^\#} Q(v) = 0$, and hence that $P^\#$ is odd.

(iv) If $u \in P$ then of course P contains the pencil of singular lines through u. Suppose $u \notin P$. Then by (ii) $u \in P^\#$; but by theorem 1.13 $P^\# \supseteq P$ and hence $P^\# \neq P$. So, by theorem 1.13, P can not be non-singular. ■

Remark 2.19 *If P is an odd plane and $u \notin P$ then by part (iv) P must contain a singular line L with $u \notin L$. In the case where $P = \alpha_r$ or $P = \beta_r$ then $L = \lambda_r := \alpha_r \cap \beta_r$ is the only singular line, and so, by theorem 1.13(ii) and theorem 2.18(ii), it follows that $P^\# = \langle u, P \rangle$. (*Cf. $P^\perp = P$.*)*

Observe also that, since every line of the plane $Q := \langle u, \lambda_r \rangle$ is thus singular, it follows, see theorem 2.18(iv), that $Q^\# =$ $\mathrm{PG}(5,2)$.

Remark 2.20 *If P is any ψ_c-even plane in* $\mathrm{PG}(5,2)$, *and so, by the theorem, $P^\#$ is an odd plane containing u, then $P^\#$ contains a pencil of singular lines through u. Hence, from theorem 1.13, it follows that $P^{\#\#}$ is a hyperplane, or possibly the whole of* $\mathrm{PG}(5,2)$.

References

[1] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, fourth ed., Springer, Berlin, 1980.

[2] R. Shaw, Icosahedral sets in PG(5, 2), *Europ. J. of Combinatorics* **18** (1997), 315-339.

[3] R. Shaw, Double-fives and partial spreads in $\mathrm{PG}(5,2)$, in: J.W.P. Hirschfeld, S.S. Magliveras & M.J. de Resmini, eds., *Geometry, Combinatorial Designs and Related Structures: Proceedings of the First Pythagorean Conference, Spetses, Greece 1996,* Cambridge University Press, (1997), 201-216.

[4] R. Shaw, Configurations of planes in $\mathrm{PG}(5,2)$, *Discrete Math.* **208/209** (1999), 529-546.

[5] R. Shaw and N. A. Gordon, The quintic Grassmannian $\mathcal{G}_{1,4,2}$ in $\mathrm{PG}(9,2)$, *Des. Codes Cryptogr.* **32** (2004) 381-396.

[6] R. Shaw and N.A. Gordon, The polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$, *Des. Codes Cryptogr.,* 39 (2006) 289-306.

[7] R.Shaw, The polynomial degree of Grassmann and Segre varieties over GF(2). (Submitted to Discrete Mathematics, apropos talk at BCC20, Durham, July 2005. Accessible from: http://www.hull.ac.uk/php/masrs/)

[8] R. Shaw, The ψ-associate $X^{\#}$ of a flat X in $\mathrm{PG}(n,2)$, (Sept 2006). (Accessible from: http://www.hull.ac.uk/php/masrs/)

Ron Shaw
CENTRE FOR MATHEMATICS, UNIVERSITY OF HULL, HULL HU6 7RX
email: r.shaw@hull.ac.uk
website: http://www.hull.ac.uk/php/masrs/