# The $\psi$-associate $X^{\#}$ of a flat $X$ in $\mathrm{PG}(n,2)$

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**Abstract**

For a given hypersurface $\psi$ in $\mathrm{PG}(n,2)$, with equation $Q(x)=0$, where $Q$ is a polynomial of (reduced) degree $d>1$, a definition is given of the $\psi$-associate $X^{\#}$ of a flat $X$ in $\mathrm{PG}(n,2)$. The definition involves the fully polarized form of the polynomial $Q$; in the cubic case $d=3$ it reads

$$X^{\#}=\{z\in\mathrm{PG}(n,2)\,|\,T(x,y,z)=0 \text{ for all } x,y\in X\},$$

where $T(x,y,z)$ denotes the alternating trilinear form obtained by completely polarizing $Q$. Some general results, valid for any degree $d$ and projective dimension $n$, are first expounded. Thereafter several choices of $\psi$ are visited, but for each choice just a few aspects are highlighted. Despite the partial nature of the survey quite a variety of behaviours of the $\psi$-associate are uncovered.

Many of the choices of $\psi$ which are considered are of cubic hypersurfaces in $\mathrm{PG}(5,2)$. If $\psi$ is the Segre variety $\mathcal{S}_{1,2,2}\subset\mathrm{PG}(5,2)$ it is shown that the 48 planes external to $\mathcal{S}_{1,2,2}$ fall into eight pairs of ordered triplets $\{(P_1,R_1,S_1),(P_2,R_2,S_2)\}$ such that $\psi^{\mathrm{c}}=P_1\cup R_1\cup S_1\cup P_2\cup R_2\cup S_2$ and

$$P_i^{\#}=R_i,\quad R_i^{\#}=S_i,\quad S_i^{\#}=P_i,\quad i=1,2.$$

Further those lines $L$ of $\mathrm{PG}(5,2)$ which are *singular,* satisfying that is $L^{\#}=\mathrm{PG}(5.2)$, are in this case shown to form a complete spread of 21 lines. Another result of note arises in the case where $\psi$ is the underlying 35-set of a non-maximal partial spread $\Sigma_5$ of five planes in $\mathrm{PG}(5,2)$, where it is shown that one plane $W\in\Sigma_5$ is singled out by the property that every line $L\subset W$ is singular.

**Keywords** hypersurfaces in $\mathrm{PG}(n,2)$, associate of a projective flat, partial spreads of planes in $\mathrm{PG}(5,2)$

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## 1 Introduction

Throughout we work over $\mathrm{GF}(2)$, and so *we may identify the nonzero elements of a vector space $V(n+1,2)=V_{n+1}$ with the points $S_0$ of the associated projective space* $\mathbb{P}V_{n+1}=\mathrm{PG}(n,2)$. Consequently we identify $\mathrm{GL}(V_{n+1})=\mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. Similarly nonzero elements of the dual vector space $V^*_{n+1}$ will be identified with the points of the dual projective space $\mathbb{P}V^*_{n+1}=\mathrm{PG}(n,2)^*$. We use $\prec u,v,\ \dots\succ$ for the vector subspace spanned by vectors $u, v, \ldots$ , and $\langle u, v, \ldots \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$ . The vector space $F(S_0)$ of all functions $S_0 \to \mathrm{GF}(2)$ is of dimension $|S_0| = 2^{n+1} - 1$, one basis being $\{\chi_a\}_{a \in S_0}$, where $\chi_a$ is the characteristic function of the singleton subset $\{a\} \subset S_0$. Given a choice of coordinates $x_1, x_2, \ldots, x_{n+1}$ in $V_{n+1}$, there are $\binom{n+1}{r}$ monomials $\{x_{i_1} x_{i_2} \ldots x_{i_r}\}_{1 \le i_1 < \ldots < i_r \le n+1}$ of (reduced) degree $r$. So altogether we have $\Sigma_{r=1}^{n+1} \binom{n+1}{r} = |S_0|$ linearly independent monomials, and these form another basis for $F(S_0)$. Given a point-set $\psi \subset \mathrm{PG}(n, 2)$ it follows that it has equation $Q(x) = 0$ for some *uniquely determined* polynomial $Q$ of minimal degree on $V_{n+1}$ and satisfying $Q(0) = 0$. Briefly stated, *every point-set of* $\mathrm{PG}(n, 2)$ *is a hypersurface.* The (reduced) degree $d = \deg Q$ of $Q$ is the *polynomial degree* of the point-set $\psi$. Observe that $d \le n + 1$. In fact, see [14, Section 1.2], the following is easily seen to hold.

**Lemma 1.1** *If* $|\psi|$ *is even then the point-set* $\psi \subset \mathrm{PG}(n, 2)$ *has polynomial degree* $d = n + 1$, *while if* $|\psi|$ *is odd then* $d \le n$.

Note that if $F_r = F_r(S_0)$, $r > 0$, denotes the subspace of $F(S_0)$ which consists of functions $f$ expressible as a polynomial function $f(x_1, x_2, \ldots, x_{n+1})$ with $\deg f \le r$ and $f(0) = 0$, then the subspaces $F_r$ are nested:

$$F_1 \subset F_2 \subset \ \ldots \ \subset F_n \subset F_{n+1} = F(S_0). \tag{1.1}$$

**Remark 1.2** *If instead we consider subsets* $\psi$ *of the points* $S_0$ *of* $\mathrm{PG}(n, q)$ *for* $q > 2$ *then the vector space* $F(S_0)$ *of all functions* $S_0 \to \mathrm{GF}(q)$ *is of dimension* $|S_0| = (q^{n+1} - 1)/(q - 1)$, *and it is still true, see for example [4, Lemma 3.5(a)], that any subset* $\psi \subset S_0$ *can be described to consist of those points* $\prec x \succ \in S_0$ *such that* $x \in V(n + 1, q)$ *satisfies a single polynomial equation* $Q(x) = 0$. *However, for* $q > 2$, *the polynomial* $Q$ *is far from unique;* cf. *[5]. So in such cases the polynomial degree of the point-set* $\psi$ *is defined to be the (reduced) degree of the characteristic function* $\chi$ *of* $\psi^c$, *where* $\chi(x) = 0$ *for* $\prec x \succ \in \psi$ *and* $\chi(x) = 1$ *for* $\prec x \succ \in \psi^c$.

*From now onwards we will deal with situations where some particular choice of a non-empty point-set* $\psi \subset \mathrm{PG}(n, 2) = PV_{n+1}$ *has been made.* Points, lines, ... , which lie inside $\psi$ will be called *internal* points, lines, ... , and those which lie in $\psi^c$ are termed *external.* Let $\mathcal{G}_\psi < \mathrm{GL}(n + 1, 2)$ be the stabilizer of the set $\psi$, and denote by $\mathcal{G}_X$ that subgroup of $\mathcal{G}_\psi$ which stabilizes a subset $X \subset \mathrm{PG}(n, 2)$. Let us suppose that the subset $\psi$ has equation $Q(x) = 0$ where the polynomial $Q = Q_\psi$ has (reduced) degree $d \ge 2$.

**Definition 1.3** *Given the choice of subset* $\psi \subset \mathrm{PG}(n, 2)$ *a flat* $X$ *of* $\mathrm{PG}(n, 2)$ *is termed* $\psi$*-odd whenever* $|X \cap \psi|$ *is odd and* $\psi$*-even whenever* $|X \cap \psi|$ *is even. Once a particular* $\psi$ *has been agreed then we omit the "*$\psi$*-".*

Note that a flat $X$ is odd whenever $X$ contains an even number of external points. The next theorem shows that the degree $d = \deg Q$ can be determined from the point-set $\psi$ purely by incidence properties. The theorem deals with an odd point-set $\psi$, since if $|\psi|$ is even then $d = n + 1$ by Lemma 1.1.

**Theorem 1.4** *(See [15, Theorem 1.1].) If* $|\psi|$ *is odd then* $Q$ *has polynomial degree* $d$ *if and only if* (i) *every* $d$*-flat is* $\psi$*-odd and* (ii) *there exists at least one* $(d - 1)$*-flat which is* $\psi$*-even.*

### 1.1 The alternating multilinear form $T_Q$

Let $T_Q$ denote the completely polarized form of $Q$. So, granted that the (reduced) degree $\deg Q$ of $Q$ is $d(\geq 2)$, for $v_i \in V_{n+1}$ we have

$$T_Q(v_1, v_2, \dots, v_d) = \sum_{i=1}^{d} Q_i + \sum_{1\leq i<j\leq d} Q_{ij} + \sum_{1\leq i<j<k\leq d} Q_{ijk} + \dots + Q_{12\dots d}\ , \quad (1.2)$$

where we have abbreviated $Q(v_i), Q(v_i+v_j), Q(v_i+v_j+v_k), \dots$ as $Q_i, Q_{ij}, Q_{ijk}, \dots$ etc. Then $T_Q$ is an alternating multilinear form on $V_{n+1}$, in $d$ vector variables $v_1, \dots, v_d$, such that (i) and (ii) in the following lemma hold:

**Lemma 1.5** *(i) If the points $v_1, \dots, v_d$ are dependent then $T_Q(v_1, \dots, v_d) = 0$.*
*(ii) If $\langle v_1, \dots, v_d\rangle$ is a $(d-1)$-flat $U$, then*

$$T_Q(v_1, \dots, v_d) = \sum_{u\in U} Q(u), \quad (1.3)$$

*and so $T_Q(v_1, \dots, v_d) = 0$ if and only if the flat $U$ is $\psi$-odd.*

It is worth stressing that many different polynomials $Q$ give rise to the same alternating form $T = T_Q$. Relative to a choice of coordinates $x_1, x_2, \dots, x_{n+1}$ in $V_{n+1}$ we have a direct sum decomposition $F_d = F_{d-1} \oplus Y_d$ where $Y_d$ is the subspace spanned by the $\binom{n+1}{d}$ monomials $\{x_{i_1}x_{i_2}\dots x_{i_d}\}_{1\leq i_1<\dots<i_d\leq n+1}$. Then for $0 \neq Q \in Y_d$ we have $T_{Q+Q'} = T_Q$ for any element $Q' \in F_{d-1}$. So $|F_{d-1}|$ different polynomials $Q$ share the same alternating form $T$. Of course this aspect is familiar in the $d = 2$ cases. For example, in $\mathrm{PG}(5,2)$ each of the 28 non-degenerate symplectic forms $T$ arises from $(|F_1| =)$ 64 quadrics; further, as in [9, Lemma 2.1], of these 64 quadrics 36 are hyperbolic quadrics $\mathcal{H}_5$ and 28 are elliptic quadrics $\mathcal{E}_5$.

In Section 1.2 we make use of $T_Q$ to define the $\psi$-associate $X^{\#}$ of a flat $X$ in $\mathrm{PG}(n,2)$, and the rest of the present paper is taken up with a study of the $\psi$-associate. For this study we basically only need to know that $T_Q$ is an alternating multilinear form satisfying Lemma (1.5). Nevertheless let us in the next subsection spell out a little more about $T_Q$.

#### 1.1.1 The linear isomorphisms $F_d/F_{d-1} \cong \wedge^d V_{n+1}^* \cong \mathrm{Alt}(\times^d V_{n+1}, \mathbb{F}_2)$

In the direct sum decomposition $F_d = F_{d-1} \oplus Y_d$ of the preceding section it should be stressed that $Y_d$ depends on the choice of coordinates: there is (if $d \leq n$) no basis-independent choice of a complementary subspace to the subspace $F_{d-1}$ of $F_d$. In contrast, we now deal with some natural linear isomorphisms surrounding $T_Q$ and the property $T_{Q+Q'} = T_Q$, for $Q' \in F_{d-1}$. In the next theorem $\mathrm{Alt}(\times^d V_{n+1}, \mathbb{F}_2)$ denotes the vector space of all alternating multilinear forms $\times^d V_{n+1} \to \mathrm{GF}(2)$. Also, for $Q$ of degree $d$, $Q + F_{d-1}$ is a general element of the quotient space $F_d/F_{d-1}$ and $T_Q$ is as previously. Further $t_Q$ is defined to be that element of $\wedge^d V_{n+1}^*$ such that if $Q$ is the product $f_1 f_2 \dots f_d$ of $d$ linear forms $f_i \in V_{n+1}^*$ then $t_Q = f_1 \wedge f_2 \wedge \dots \wedge f_d$.

**Theorem 1.6** *There exist natural linear isomorphisms $F_d/F_{d-1} \cong \wedge^d V_{n+1}^* \cong \mathrm{Alt}(\times^d V_{n+1}, \mathbb{F}_2)$, giving rise, for $Q \in F_d$, to the bijective correspondences*

$$Q + F_{d-1} \longleftrightarrow t_Q \longleftrightarrow T_Q.$$

**Proof.** (*cf.* [8, Theorem B].) Concerning the first isomorphism, consider the multilinear map $M : \times^d V_{n+1}^* \to F_d/F_{d-1}$ defined by

$$M(f_1, f_2, \ \dots \ , f_d) = f_1 f_2 \dots f_d + F_{d-1}$$

and observe that $M$ is alternating, since if $f_i = f_j$ for some $i \neq j$, then $f_1 f_2 \dots f_d \in F_{d-1}$. Hence there exists a unique linear map $\rho : \wedge^d V_{n+1}^* \to F_d/F_{d-1}$ such that

$$\rho(f_1 \wedge f_2 \wedge \ \dots \ \wedge f_d) = f_1 f_2 \dots f_d + F_{d-1}.$$

Now $\rho$ is surjective, and moreover the dimension $\binom{n+1}{d}$ of $\wedge^d V_{n+1}^*$ tallies with that of $F_d/F_{d-1}$, since there exist $\binom{n+1}{d}$ monomials $\{x_{i_1} x_{i_2} \dots x_{i_d}\}_{1 \leq i_1 < \dots < i_d \leq n+1}$. Hence $\rho$ is a linear isomorphism.

The second isomorphism, that of $\wedge^d V_{n+1}^*$ with $\mathrm{Alt}(\times^d V_{n+1}, \mathbb{F}_2)$, is a standard one, each $t \in \wedge^d V_{n+1}^*$ giving rise to an element $T \in \mathrm{Alt}(\times^d V_{n+1}, \mathbb{F}_2)$ by way of

$$T(v_1, \dots, v_d) = < t | v_1 \wedge v_2 \wedge \ \dots \wedge v_d > .$$

Here $< \cdot | \cdot >$ is given on decomposable $d$-vectors by

$$< f_1 \wedge f_2 \wedge \ \dots \ \wedge f_d | v_1 \wedge v_2 \wedge \ \dots \wedge v_d > = \det[f_i(v_j)],$$

and is the standard determinantal pairing of $\wedge^d V_{n+1}^*$ with $\wedge^d V_{n+1}$, ∎

**Remark 1.7** *From the foregoing, for any $Q$ the explicit coordinate expression of the alternating form $T_Q$ is easily given as a sum of $d \times d$ determinants. For example, in the cubic case $d = 3$, if $Q \in F_3$ has the coordinate form $Q(x) = \Sigma_{i<j<k} c_{ijk} x_i x_j x_k + Q'$, $Q' \in F_2$, then*

$$T_Q(x, y, z) = \sum\nolimits_{i<j<k} c_{ijk} \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}. \qquad (1.4)$$

## 1.2 The $\psi$-associate $X^\#$ of a flat $X \subset \mathrm{PG}(n, 2)$

**Definition 1.8** *If a subset $\psi \subset \mathrm{PG}(n, 2)$ has equation $Q(x) = 0$, where $\deg Q = d \geq 2$, then the $\psi$-associate $X^\#$ of an $r$-flat $X$ of $\mathrm{PG}(n, 2)$ is the following subset of $\mathrm{PG}(n, 2)$:*

$$X^\# = \{y \in \mathrm{PG}(n, 2) \,|\, T_Q(x_1, ..., x_{d-1}, y) = 0 \text{ for all } x_1, ..., x_{d-1} \in X\}. \qquad (1.5)$$

*For an agreed $\psi$ we simply refer to $X^\#$ as the associate of $X$.*

**Lemma 1.9** *Let $X$ be an $r$-flat in $\mathrm{PG}(n, 2)$. Then*
*(i) $X^\#$ is always a flat;*
*(ii) if $r$ is $< d - 2$ then $X^\# = \mathrm{PG}(n, 2)$;*
*(iii) if $r = d - 2$ then $X \subseteq X^\#$;*
*(iv) if $W$ is a flat then $W \subset X \Longrightarrow X^\# \subseteq W^\#$;*
*(v) $\mathcal{G}_X \leq \mathcal{G}_{X^\#}$.*

**Proof.** (i) $T(= T_Q)$ is multilinear in its $d$ arguments.
(ii) If $r < d - 2$ then any $d - 1$ points $x_1, ..., x_{d-1}$ of $X$ are dependent.
(iii) If $r = d - 2$ and $y \in X$ then $x_1, ..., x_{d-1}, y$ in (1.5) are dependent.
(iv) If $T(x_1, ..., x_{d-1}, y) = 0$ holds for all $x_1, ..., x_{d-1} \in X$ then it holds for all $x_1, ..., x_{d-1}$ in the subspace $W$ of $X$.
(v) $X^\#$ is uniquely determined by $X$. ∎

**Remark 1.10** *In most cases of interest the polynomial $Q$, and the multilinear form $T_Q$, will be quite complicated. Consequently, leaving aside the possible use of the computer, the determination of $X^\#$ directly from definition 1.8 will usually be an extremely daunting task. Fortunately, as in the next lemma, the definition of $X^\#$ may be rephrased in terms of certain incidence properties. So, in most cases, it would appear sensible to attempt to determine $X^\#$ by appeal to Lemma 1.11 and its off-shoots.*

**Lemma 1.11** *Let $X \subset \mathrm{PG}(n,2)$ be an $r$-flat. Then*
*(i) if $r \geq d-2$ a point $y \in X^c$ is in the associate $X^\#$ of $X$ if and only if for each $(d-2)$-flat $W$ of $X$ the join $\langle y, W\rangle$ is odd;*
*(ii) if $r \geq d-1$ a point $y \in X$ is in the associate $X^\#$ of $X$ if and only if the following holds: if $K$ is any $(d-1)$-flat of $X$ which contains $y$ then $K$ is odd.*

**Proof.** Since $y \in X^\#$ if and only if $T_Q(x_1, ..., x_{d-1}, y) = 0$ for all $x_1, ..., x_{d-1} \in X$, the present lemma follows immediately from Lemma 1.5. ■

**Corollary 1.12** *(i) If $X$ is a $(d-2)$-flat then a point $y \in X^c$ is in the associate $X^\#$ of $X$ if and only if $\langle y, X\rangle$ is odd.*
*(ii) If $X$ is an even $(d-1)$-flat then $X^\#$ is disjoint from $X$.*
*(iii) If $X$ is an odd $(d-1)$-flat then $X^\# \supseteq X$; moreover $X^\#$ is odd.*

**Proof.** Parts (i) and (ii) follow immediately from parts (i) and (ii) of the last lemma. Concerning part (iii), the relation $X^\# \supseteq X$ also follows from part (ii) of the last lemma; hence either $X^\# = X$, and so $X^\#$ is odd, or else $X^\#$ is a $r$-flat for some $r \geq d$, whence, by Theorem 1.4, $X^\#$ is odd. ■

### 1.2.1 The associate $X^\#$ of a $(d-2)$-flat $W$

If $W = \langle a_1, \ \dots \ , a_{d-1}\rangle$ is a $(d-2)$-flat of $\mathrm{PG}(n,2) = \mathbb{P}V_{n+1}$ then, with respect to the agreed $\psi$, its *associated linear form* $f_W \in (V_{n+1})^*$ is defined by

$$f_W(x) = T(a_1, \ \dots \ , a_{d-1}, x), \quad x \in V_{n+1}. \tag{1.6}$$

This is a good definition: $T$ is alternating and multilinear, and so any choice of independent points $a_1, \ \dots \ , a_{d-1} \in W$ yield the same linear form. The $\psi$-associate of $W$ is thus the flat

$$W^\# = H_W := \{x \in \mathrm{PG}(n,2) | \ f_W(x) = 0\}. \tag{1.7}$$

So $W^\#$ is a hyperplane whenever $f_W$ is not the zero form, while $W^\# = \mathrm{PG}(n,2)$ if $f_W$ is the zero form

### 1.2.2 Singular flats and singular points

A $(d-2)$-flat $W$ in $\mathrm{PG}(n,2)$ will be termed *singular* if $f_W$ is the zero form, that is if $W^\# = \mathrm{PG}(n,2)$. For $r \geq d-2$ an $r$-flat in $\mathrm{PG}(n,2)$ will be termed *singular* if it contains one or more singular $(d-2)$-flats, and *non-singular* if it contains no singular $(d-2)$-flats.

**Lemma 1.13** *A $(d-2)$-flat $W$ in $\mathrm{PG}(n,2)$ is singular if and only if every $(d-1)$-flat $U$ which contains $W$ is $\psi$-odd.*

**Proof.** By Corollary 1.12(i) $W^{\#} = \mathrm{PG}(n,2)$ holds if and only if every $(d-1)$-flat $U$ which contains $W$ is $\psi$-odd. ∎

For certain choices of $\psi$ there may exist points $u$ of $\mathrm{PG}(n,2)$ such that

$$T(a_1, \ldots, a_{d-1}, u) = 0, \quad \text{for all } a_1, \ldots, a_{d-1} \in \mathrm{PG}(n,2). \tag{1.8}$$

Such points will be termed *singular points.* Since $T$ is multilinear the set of all singular points is a flat in $\mathrm{PG}(n,2)$ which we will call the *radical* $\mathrm{rad}\,\psi$ of $\psi$ (and of $T$). The following lemma is immediate. (For (i)(c), recall Lemma (1.5)(ii).)

**Lemma 1.14** *(i) $u \in \mathrm{rad}\,\psi$ if and only if any one of the following holds:*
(a) *$u \in W^{\#}$ for each $(d-2)$-flat $W$;*
(b) *every $(d-2)$-flat containing $u$ is singular;*
(c) *every $(d-1)$-flat containing $u$ is odd.*
*(ii)* $\mathrm{rad}\,\psi = \mathrm{PG}(n,2)^{\#}$.
*(iii)* $\mathrm{rad}\,\psi$ *is stabilized by* $\mathcal{G}_\psi$.
*(iv)* $\mathrm{rad}\,\psi \subseteq X^{\#}$ *for all flats $X$ of* $\mathrm{PG}(n,2)$.

Presumably point-sets $\psi$ having $\mathrm{rad}\,\psi = \emptyset$ are the most promising ones to investigate. Note however that $\mathrm{rad}\,\psi = \emptyset$ does not at all preclude the existence of singular $(d-2)$-flats; for example see $\psi = \psi_4$ in Section 2.5, where $\mathrm{rad}\,\psi_4 = \emptyset$ but where, see Theorem 2.9, all lines of a certain plane are singular.

### 1.2.3 The associate $X^{\#}$ of a $(d-1)$-flat $X$

Observe that in the foregoing notation Lemma 1.11(i) asserts that a point $y \in X^c$ is in the associate $X^{\#}$ of the $r$-flat $X, r \geq d-2$, if and only if $f_W(y) = 0$ for each $(d-2)$-flat $W$ of $X$. The case $r = d-1$ is of particular interest, as in the next theorem.

**Theorem 1.15** *Given a $(d-1)$-flat $X \subset \mathrm{PG}(n,2)$ let $\mathcal{B} = \{a_1, \ldots, a_d\}$ be* any *choice of points which generate $X$, and consider the $d$ independent $(d-2)$-flats $W_i = \langle \mathcal{B}_i \rangle \subset X$, where $\mathcal{B}_i := \mathcal{B} \setminus \{a_i\}$, $i = 1, \ldots, d$. Then*

$$X^{\#} = H_{W_1} \cap \ \ldots \ H_{W_d}. \tag{1.9}$$

**Proof.** Since $T$ is alternating and multilinear, if $y$ satisfies the $d$ equations $T(a_2, a_3, ..., a_d, y) = 0$, $T(a_1, a_3, ..., a_d, y) = 0$, ... , $T(a_1, ..., a_{d-1}, y) = 0$, then $y$ satisfies $T(x_1, ..., x_{d-1}, y) = 0$ for all $x_1, ..., x_{d-1} \in X$. ∎

**Corollary 1.16** *If $X$ is a $\psi$-even $(d-1)$-flat then $X$ is non-singular and $X^{\#}$ is a disjoint $(n-d)$-flat.*

**Proof.** By the theorem, $X^{\#}$ is the intersection of $\leq d$ independent hyperplanes; hence $X^{\#}$ is an $s$-flat for some $s \geq n-d$. But if $X$ is even then, see Corollary 1.12(ii), $X \cap X^{\#} = \emptyset$, and so $s \leq n-d$. Hence $s = n-d$. Further $X$ can not be singular, for if $W_1$, say, were singular then $X^{\#}$ would be the intersection of at most $d-1$ independent hyperplanes, violating $X \cap X^{\#} = \emptyset$. ∎

## 1.3 The associate $X^{\#}$ in the cubic case $d = 3$

From now onwards we deal with degree $d > 2$. Note therefore from Lemma 1.9(ii) that the associate $\langle a\rangle^{\#}$ of a point $a$ is the whole of $\mathrm{PG}(n, 2)$. *In fact throughout Section 2 we will deal with* $d = 3$. In this $d = 3$ case the associate of a flat $X$ is accordingly

$$X^{\#} = \{y \in \mathrm{PG}(n, 2) \mid T(a_1, a_2, y) = 0 \text{ for all } a_1, a_2 \in X\}, \tag{1.10}$$

and the associated linear form $f_L$ of a line $L = \langle a_1, a_2\rangle$ is

$$f_L(x) = T(a_1, a_2, x), \quad x \in V_{n+1}. \tag{1.11}$$

A line $L$ in $\mathrm{PG}(n, 2)$ is singular if $f_L$ is the zero form. A flat in $\mathrm{PG}(n, 2)$ is singular if it contains one or more singular lines, and non-singular if it contains no singular lines. For future convenience we spell out in the next theorem the $d = 3$ instances of results in Section 1.2.

**Theorem 1.17** *For a cubic hypersurface $\psi$ in $\mathrm{PG}(n, 2)$ the following hold.*
*(i) Let $X \subset \mathrm{PG}(n, 2)$ be an $r$-flat. Then*
*(a) for $r \geq 1$ a point $y \in X^c$ is in the associate $X^{\#}$ of $X$ if and only if for each line $L$ of $X$ the plane $\langle y, L\rangle$ is $\psi$-odd;*
*(b) for $r \geq 2$ a point $y \in X$ is in the associate $X^{\#}$ of $X$ if and only if every plane $P$ of $X$ which contains $y$ is $\psi$-odd.*
*(ii) The associate of a plane $P = \langle a_1, a_2, a_3\rangle$ is*

$$P^{\#} = H_{L_{12}} \cap H_{L_{13}} \cap H_{L_{23}},$$

*where $L_{ij} = \langle a_i, a_j\rangle$ and $H_L = \{x \in \mathrm{PG}(n, 2) |\ f_L(x) = 0\}$. Further, $P^{\#}$ is an $s$-flat for some $s \geq n - 3$, and a point $y \in P^c$ lies in $P^{\#}$ if and only if each of the three planes $\langle y, L_{ij}\rangle$ is odd.*
*(iii) If $P$ is a $\psi$-even plane then $P$ is non-singular and $P^{\#}$ is a disjoint $(n-3)$-flat.*
*(iv) If $P$ is a $\psi$-odd plane then $P^{\#} \supseteq P$; moreover $P^{\#}$ is $\psi$-odd.*
*(v) A line $L$ in $\mathrm{PG}(n, 2)$ is singular if and only if every plane $P$ which contains $L$ is $\psi$-odd.*

In fact part (iv) of the theorem can be strengthened as follows.

**Theorem 1.18** *For a cubic $\psi \subset \mathrm{PG}(n, 2)$ suppose that $P$ is a $\psi$-odd plane. Then $P^{\#} \supseteq P$ and $P^{\#}$ is $\psi$-odd; further*
*(i) if $P$ is non-singular then $P^{\#}$ is a $(n - 3)$-flat;*
*(ii) if $P$ contains just one singular line then $P^{\#}$ is a $(n - 2)$-flat;*
*(iii) if $P$ contains just one pencil of singular lines then $P^{\#}$ is a hyperplane;*
*(iv) if every line $L \subset P$ is singular then $P^{\#} = \mathrm{PG}(n, 2)$.*

**Proof.** From Theorem 1.17(ii), $x \in P^{\#}$ if and only if $f_L(x) = 0$ holds for three independent choices of the line $L \subset P$.
(i) If no line of $P$ is singular we show now that the three linear forms $f_L$ are linearly independent, and hence that $P^{\#}$ is a $(n-3)$-flat . For since the lines $L$ are non-singular the forms $f_L$ are not the zero form. Further, a linear relation such as $f_{\langle a_1,a_2\rangle} = f_{\langle a_1,a_3\rangle}$ between two of the $f_L$ is ruled out, for it would

entail that $f_{\langle a_1,a_2+a_3\rangle}$ is the zero form, that is that $P$ possesses a singular line $\langle a_1, a_2 + a_3\rangle$. Finally the linear relation $f_{\langle a_1,a_2\rangle} + f_{\langle a_1,a_3\rangle} = f_{\langle a_2,a_3\rangle}$ between all three $f_L$ is ruled out for it entails that $f_{\langle a_1+a_2,a_2+a_3\rangle}$ is the zero form, that is that $P$ possesses a singular line $\langle a_1 + a_2, a_2 + a_3\rangle$.
(ii) If $P$ contains just one singular line then there exists just one linear relation between the three linear forms, whence $P^\#$ is a $(n - 2)$-flat .
(iii), (iv) These cover similarly the remaining possibilities, namely that (iii) there are two independent linear relations between the three linear forms, and (iv) all the $f_L$ are zero. ■

## 1.4 General plan of campaign

Further to the properties of the associate $X^\#$ outlined in Section 1.2 one can hardly expect to find many further which hold for a general choice of point-set $\psi$. For recall that even in the familiar case when $Q$ has degree $d = 2$ results such as $\dim U + \dim U^\perp = n+1$, where $U$ is a vector subspace of an $(n+1)$-dimensional vector space $V(n + 1, \mathbb{F})$, are valid only if the associated bilinear form is non-degenerate. In particular $\dim U + \dim U^\perp = n+1$ never holds if $\mathbb{F} = \mathrm{GF}(2)$ and $n$ is even. So in further investigations it seems sensible to investigate properties of the associate $X^\#$ for some particular choices of point-set $\psi$.

Ideally a plan of campaign should involve (at least) the following.

1. Choose a point-set $\psi \subset \mathrm{PG}(n, 2)$ — hopefully one which has some interesting geometry — and determine the (reduced) degree $d = \deg Q$.
2. Determine the stabilizer $\mathcal{G}_\psi < \mathrm{GL}(n + 1, 2)$.
3. For each $r > 0$ determine the $\mathcal{G}_\psi$-orbits of $r$-flats of $\mathrm{PG}(n, 2)$.
4. For a representative $X$ of each $\mathcal{G}_\psi$-orbit of $r$-flats determine its $\psi$-associate $X^\#$, listing the orbit to which $X^\#$ belongs.

The hope is that, for some sets $\psi \subset \mathrm{PG}(n, 2)$, an investigation of the $\psi$-associates $X^\#$ of $r$-flats may illuminate in an interesting way some of the known geometry of $\psi$, or conceivably even bring to light some new geometrical aspects.

Certainly in simple cases one expects to see clearly how $X^\#$ is related geometrically to $\psi$ and $X$, To take a couple of baby examples, consider first the case when $\psi$ is a 3-arc $\{a, b, c\}$ in $\mathrm{PG}(3, 2)$ and $P$ is one of the two planes in $\mathrm{PG}(3, 2)$ which are external to $\psi$. Here $d = 3$. (Indeed any odd subset of points in $\mathrm{PG}(3, 2)$ has $d \le 3$.) It follows from Theorem 1.17(iii) that $P^\#$ is a point $\langle p\rangle$ disjoint from $P$. Noting that the planes $\langle a + b + c, L\rangle$ are $\psi$-odd for each line $L \subset P$, it follows from Theorem 1.17(i)(a) that $P^\# = \langle a + b + c\rangle$. Similarly if $\psi = \mathrm{PG}(3, 2) \setminus (P \cup \{p\})$, where $p$ is a point disjoint from the plane $P$, then one finds that $P^\# = \langle p\rangle$, since, for each line $L \subset P$, the plane $\langle p, L\rangle$ meets $\psi$ in 3 points.

However it has to be admitted that for many choices of $\psi$ one encounters formidable difficulties in attempting to carry out in full the above plan, even in cases where $d$ and $n$ are of moderate size. In such cases perhaps one should concentrate one's energies upon finding the associate of the $\psi$-even $(d - 1)$-flats $X$, fortified by the knowledge, see Corollary 1.16, that $X^\#$ is necessarily a disjoint $(n - d)$-flat. The even $(d - 1)$-flats $X$ are particularly enticing in cases where $\psi$ is a subset of $\mathrm{PG}(2d-1,2)$ — for $X^{\#}$ is then also a $(d-1)$-flat. If $X^{\#}$ also turns out to be even, then one knows that $X^{\#\#}$ $(=(X^{\#})^{\#})$ is a further $(d-1)$-flat, which is moreover disjoint from $X^{\#}$ (but not necessarily from $X$).

**Definition 1.19** *In cases where $\psi$, of degree $d$, is a subset of* $\mathrm{PG}(2d-1,2)$, *a $\psi$-even $(d-1)$-flat $X$ is termed* faithfully $\psi$-even *if each of $X_1 = X^{\#}$, $X_2 = X^{\#\#}$, ..., $X_{r+1} = (X_r)^{\#}$ ... is also $\psi$-even.*

**Lemma 1.20** *If $\psi \subset \mathrm{PG}(2d-1,2)$ is of degree $d$, then a necessary condition for the existence of faithfully $\psi$-even $(d-1)$-flats is that* $\mathrm{rad}\,\psi = \emptyset$.

**Proof.** Let $X$ be any $\psi$-even $(d-1)$-flat and suppose that a $\psi$-singular point $u$ exists. Then, by parts (iv) and (i)(c) of Lemma 1.14, $u \in X^{\#}$ and so the $(d-1)$-flat $X^{\#}$ is odd. ■

Clearly faithfully-even $(d-1)$-flats in $\mathrm{PG}(2d-1,2)$ are worthy of special attention. For any faithfully-even $(d-1)$-flat $X$ we can define an ordered pair $(r,s)$ of integers $r,s$, with $0 \le r < s$, such that the members of the finite "#-sequence" $X_0(=X), X_1, \ldots X_s$, where $X_{r+1} = (X_r)^{\#}$, are distinct and such that $(X_s)^{\#} = X_r$. Thus for a #-sequence of type $(1,2)$ the $(d-1)$-flats $X, X^{\#}, X^{\#\#}$ are distinct and $X^{\#\#\#}$ coincides with $X^{\#}$.

### 1.4.1 Illustration: the Grassmannian $G_{1,4,2}$ in $\mathrm{PG}(9,2)$

As an illustration of the foregoing, consider the $(d,n) = (5,9)$ case where $\psi$ is the Grassmannian variety $\psi = G_{1,4,2}$ in $\mathrm{PG}(9,2)$, arising from the 155 lines of $\mathrm{PG}(4,2)$. That $d=5$ in this case was proved in [6]. (Recently it has been proved that the Grassmannian variety $\psi = G_{1,m,2}$ in $\mathrm{PG}(\binom{m+1}{2}-1,2)$ has $d = \binom{m}{2}-1$: see [2], [15].) An investigation of the $\psi$- associate in the case of $G_{1,4,2}$ was embarked upon in [13]. However, despite the large stabilizer $\mathcal{G}_\psi \cong \mathrm{GL}(5,2)$, step 3 of the above plan seems very hard to achieve. Even if one restricts attention to external flats the determination of all their $\mathcal{G}_\psi$-orbits is a far from trivial task, as witness the work [7] in which all ten orbits of external flats were determined. So in [13] the associate $X^{\#}$ was only determined for certain selected flats $X$, including (a) external 4-flats, and (b) certain flats obtained from partial spreads of lines in $\mathrm{PG}(5,2)$.
Concerning (a), there are just two $\mathcal{G}_\psi$-orbits of external 4-flats $X \subset \mathrm{PG}(9,2)$ and in [13, Theorem 3.4] it was shown that the associate $X^{\#}$ of an external 4-flat is also an external 4-flat, with $X$ and $X^{\#}$ belonging to different $\mathrm{GL}(5,2)$-orbits. Moreover $(X^{\#})^{\#} = X$. Thus, in the case $\psi = G_{1,4,2}$, external 4-flats are faithfully-even and of type $(0,1)$.
Concerning (b), several other even 4-flats were investigated in [13]. The examples considered all turned out to be faithfully-even, and to have sequences either of type $(0,2)$ or type $(1,2)$.

# 2 Cubic hypersurfaces in $\mathrm{PG}(5,2)$ arising from partial spreads of planes

Bearing in mind the difficulties mentioned in Section 1.4 "even in cases where $d$ and $n$ are of moderate size", let us in this section confine attention to certain $d=3$, $n=5$ cases, where $\psi$ is a cubic hypersurface in $\mathrm{PG}(5,2)$. Moreover, in order to ensure that the geometry is simple, we will deal solely with certain hypersurfaces $\psi$ which arise from partial spreads of planes in PG$(5,2)$. Consequently, in Sections 2.4 - 2.6, *we will have frequent need to appeal to certain basic results concerning such partial spreads which can be found in [10], [11] and [12].* Often we will pay special attention to the associate of even planes $P$, knowing, from Theorem 1.17(iii), that $P^{\#}$ is then necessarily a disjoint plane. Despite this lack of ambition we claim that nevertheless we gain some useful insights into the variety of behaviours to be expected of the $\psi$-associate in general, and, for some choices of $\psi$, throw light upon some little-known geometrical aspects.

## 2.1 Introduction

If we view $V_6$ as $V_3 \oplus V_3$ then a general point $z \in V_6$ is

$$z = (x, y) \in V_6 = V_3 \oplus V_3,\ x, y \in V_3. \tag{2.1}$$

Let $A$ be an element of GL$(V_3)$ of order 7 and satisfying $A^3 = I + A$. Since $A$ is fixed-point-free on $\mathbf{P}V_3 = PG(2,2)$ it gives rise, *cf.* [11, Section 4.1], to the following plane-spread $\Sigma_9$ for $\mathbf{P}V_6 = \mathrm{PG}(5,2)$ :

$$\Sigma_9 = \{X, Y, P_0, P_1, \ \dots\ , P_6\} \tag{2.2}$$

upon defining the nine planes $X, Y, P_0, P_1, \ \dots\ , P_6$ of PG$(5,2)$ by:

$$\begin{aligned} X &= \{(x,0),\ x \in \mathbb{P}V_3\},\ Y = \{(0,y),\ y \in \mathbb{P}V_3\}, \\ P_r &= \{(x, A^r x),\ x \in \mathbb{P}V_3\}, \quad r = 0,1,2 \ \dots\ ,6. \end{aligned} \tag{2.3}$$

Upon choosing a point $a_1 \in \mathbb{P}V_3$ and setting $a_{r+1} = A^r a_1,\ r = 0, 1,\ \dots\ , 6$, the 49 points underlying the seven planes $P_0, P_1,\ \dots\ , P_6$ are the points $(a_i, a_j),\ 1 \le i, j \le 7$. In detail

$$P_r = \{(a_i, a_{i+r}),\ i = 1, 2,\ \dots\ , 7\}, \qquad r = 0,1,2,3,4,5,6, \tag{2.4}$$

$$X = \{(a_i, 0),\ i = 1, 2,\ \dots\ , 7\},\ Y = \{(0, a_i),\ i = 1, 2,\ \dots\ , 7\}, \tag{2.5}$$

where in (2.4) the value of the index $i + r$ is taken mod 7.

For future reference let us note here a particular subgroup of the symmetry group $\mathcal{G}(\Sigma_9)$ of the spread $\Sigma_9$. Recall that the normalizer in GL$(V_3)$ of the $Z_7$ subgroup $\langle A \rangle$ is a subgroup $\mathcal{F}_{21} \cong Z_7 \rtimes Z_3$ of order 21 which is generated by $A$ and $C$ where $C \in$ GL$(V_3)$, of order 3, satisfies $CAC^{-1} = A^2$. Setting $\Phi_A = A \oplus A$ and $\Phi_C = C \oplus C$ observe that the group $\mathcal{F} = \langle \Phi_A, \Phi_C \rangle$ is a subgroup $\cong Z_7 \rtimes Z_3$ of $\mathcal{G}(\Sigma_9)$: for $\Phi_A$ stabilizes each plane of $\Sigma_9$, while $\Phi_C$ stabilizes each of $X, Y, P_0$ and effects the permutations $(P_1P_2P_4)$, $(P_6P_5P_3)$ of the other six planes of $\Sigma_9$.

It is known, see [12, Theorem 4.1], that in PG$(5,2)$ all complete plane-spreads are projectively equivalent to the spread $\Sigma_9$; also if $\Sigma_r, r < 9$, is any $r$-subset of $\Sigma_9$ then all non-maximal partial plane-spreads of size $r$ are projectively equivalent to $\Sigma_r$. Further, up to projective equivalence, there is just one kind of maximal partial spread: it is of size 5 and is represented by

$$\Psi_5 = \{X, Y, P_0, P_1, S\}, \tag{2.6}$$

where $S$ is one of the seven planes external to $X \cup Y \cup P_0 \cup P_1$ other than the five planes $P_2,\ \dots\ , P_6$. (The plane $S$ is in fact of the form $S = \{(x, Bx),\ x \in \mathbb{P}V_3\}$

where $B$ is an element of $\mathrm{GL}(V_3)$ of order 7 such that $A^{-1}B$ is of order 7 and $AB \neq BA$.)

We intend to consider the following cubic hypersurfaces (i)-(v):

$$\begin{gathered}\text{(i) } \psi_1 = X \quad \text{(ii) } \psi_2 = (X \cup Y)^c \quad \text{(iii) } \psi_3 = X \cup Y \cup P_0 \\ \text{(iv) } \psi_4 = Y \cup P_1 \cup P_2 \cup P_4 \cup P_0 = (X \cup P_6 \cup P_5 \cup P_3)^c \\ \text{(v) } \psi_5 = X \cup Y \cup P_0 \cup P_1 \cup S.\end{gathered} \tag{2.7}$$

That each of these sets $\psi_i$ indeed has polynomial degree $d = 3$ follows quickly from Theorem 1.4. For each $\psi_i$ is the union of an odd number of pairwise disjoint planes, whence $|D \cap \psi|$ is odd for any solid $D$, since a solid in $\mathrm{PG}(5,2)$ meets every plane in an odd number of points. On the other hand there exist in each case $\psi_i$-even planes: in the first four cases there exist, see (2.2), external planes, and in the fifth case it is not difficult to find planes which meet $\psi_5$ in 2 and in 4 points. (See [10, Section 5.2].)

**Remark 2.1** *The 35-set $\psi_5$ is called a double-five since, see [10], [11], [12], it admits two different decompositions into five disjoint planes.*

**Remark 2.2** *Recall after (1.3): since $\dim F_2 = 21$, there are $2^{21}$ different cubics $Q$ in $\mathrm{PG}(5,2)$ which share the same trilinear alternating form $T = T_Q$.*

**Notation 2.3** *We will denote by $\Omega_N^{(r)}(n_1n_2...)$ a $\mathcal{G}_\psi$-orbit of $r$-flats of length $|\Omega_N^{(r)}(n_1n_2...)| = N$ whose 'intersection pattern' is $n_1n_2... := \{n_1, n_2, ...\}$. Thus in the case of $\psi_2 = (X \cup Y)^c$ the $\mathcal{G}_{\psi_2}$-orbit $\Omega_{98}^{(2)}(30)$ consists of those $(49 \times 2 =)$ 98 planes in $\mathrm{PG}(5,2)$ which meet one of the planes $X, Y$ in 3 points (a line) and the other in 0 points. Similarly if $\psi_3 = X \cup Y \cup P_0$ the $\mathcal{G}_{\psi_3}$-orbit $\Omega_7^{(3)}(333)$ consists of those 7 solids which meet each of the planes $X, Y, P_0$ in a line.*

We view the present paper only as of the nature of a reconnaissance expedition, and so in most cases we will look in detail at just a few selected $\mathcal{G}_\psi$-orbits. Nevertheless our partial survey does bring out a surprising variety of behaviours of the $\psi$-associate!

## 2.2 The case $\psi_1 = X$

Here the orbits of the 651 lines, 651 solids and 1395 planes are seen to be:

$$\begin{gathered}\Omega_7^{(1)}(3),\ \Omega_{196}^{(1)}(1),\ \Omega_{448}^{(1)}(0); \quad \Omega_7^{(3)}(7),\ \Omega_{196}^{(3)}(3),\ \Omega_{448}^{(3)}(1); \\ \Omega_1^{(2)}(7),\ \Omega_{98}^{(2)}(3),\ \Omega_{784}^{(2)}(1),\ \Omega_{512}^{(2)}(0).\end{gathered} \tag{2.8}$$

In addition we have of course two orbits $\Omega_7^{(0)}(1)$, $\Omega_{56}^{(0)}(0)$ of points and two orbits $\Omega_7^{(4)}(7)$, $\Omega_{56}^{(4)}(3)$ of hyperplanes. In this exceptionally simple case it is easy to describe the full story for the $\psi_1$-associates. From Theorem 1.17(v) we see that a line $L$ is singular, or non-singular, according as $L$ meets $X$, or is external to $X$. So every point of $X$ is a singular point, and indeed $\mathrm{rad}\,\psi_1 = X$. So $X \subseteq W^\#$ for any flat $W$ of $\mathrm{PG}(n,2)$.

From Theorem 1.17(i) the associates of planes are seen to behave as follows:

| $P \in$ | $\Omega_1^{(2)}(7)$ | $\Omega_{98}^{(2)}(3)$ | $\Omega_{784}^{(2)}(1)$ | $\Omega_{512}^{(2)}(0)$ |
|---|---|---|---|---|
| $P^\# =$ | $\mathrm{PG}(5,2)$ | $\mathrm{PG}(5,2)$ | $\langle P, X\rangle$ | $X$ |

(2.9)

Also the associates of solids and of hyperplanes are

$$\begin{array}{c|ccc|} D\in & \Omega_7^{(3)}(7) & \Omega_{196}^{(3)}(3) & \Omega_{448}^{(3)}(1) \\ \hline D^{\#}= & \mathrm{PG}(5,2) & \langle D,X\rangle & X \end{array} \qquad \begin{array}{c|cc|} H\in & \Omega_7^{(4)} & \Omega_{56}^{(4)} \\ \hline H^{\#}= & H & X \end{array}. \tag{2.10}$$

Alternatively we may make use of the explicit coordinate form of $T$ which for $\psi_1$ is particularly simple. For if $X=\langle e_4,e_5,e_6\rangle$, and so $Q(x)=x_1x_2x_3+$(terms of degree $\leq 2$), then, see (1.4), $T(a,b,x)$ is just a single $3\times 3$ determinant:

$$T(a,b,x)=(a_2b_3+a_3b_2)x_1+(a_1b_3+a_3b_1)x_2+(a_1b_2+a_2b_1)x_3. \tag{2.11}$$

Then the results (2.9) quickly follow from Theorem 1.17(ii).

Of course in such a simple setting one hardly expects much of interest, but the present case may give some indication of what can be expected when $\operatorname{rad}\psi$ is non-trivial.

## 2.3 The case $\psi_2=(X\cup Y)^c$

Observe that the $\psi_2$-odd planes are those which are either disjoint from both $X$ and $Y$ or meet both $X$ and $Y$. It follows from Theorem 1.17(v) that the singular lines are those which meet both $X$ and $Y$, and thus constitute a single $\mathcal{G}_{\psi_2}$-orbit $\Omega_{49}^{(1)}(11)$. It then follows that there are no singular points: $\operatorname{rad}\psi_2=\emptyset$.

Let us concentrate upon the $\psi_2$-even planes, namely those which meet one, but not both, of $X$ and $Y$. There are thus three orbits of even planes, namely $\Omega_2^{(2)}(70)=\{X,Y\}$, $\Omega_{98}^{(2)}(30)$ and $\Omega_{588}^{(2)}(10)$. Since for $y\in Y$ and $L$ a line of $X$ the plane $\langle y,L\rangle$ is odd it follows from Theorem 1.17(i),(iii) that $X^{\#}=Y$; similarly $Y^{\#}=X$. Thus planes $P\in\Omega_2^{(2)}(70)$ satisfy $P^{\#\#}=P$. For a plane $P\in\ \Omega_{98}^{(2)}(30)$ one finds that $P^{\#}\in\Omega_2^{(2)}(70)$; thus $P^{\#\#\#}=P^{\#}$. For a plane $P\in\ \Omega_{588}^{(2)}(10)$ one finds that $P^{\#}\in\Omega_{98}^{(2)}(70)$; thus $P^{\#\#\#\#}=P^{\#\#}$. Observe therefore that, in the case of $\psi_2$, all even planes are faithfully-even, and give rise to #-sequences of types $(0,1)$, $(1,2)$ and $(2,3)$.

## 2.4 The case $\psi_3=X\cup Y\cup P_0=\mathcal{S}_{1,2,2}$

It is worth noting that $\psi_3$ may be viewed as the Segre variety

$$\mathcal{S}_{1,2,2}=\{v\otimes w: v\in\mathbb{P}V_2,\ w\in\mathbb{P}V_3\}, \tag{2.12}$$

which lies in $\mathrm{PG}(5,2)=\mathbb{P}V_6$, and which consists of the 21 decomposable elements of a tensor product space $V_6=V_2\otimes V_3$. For note that $\mathcal{S}_{1,2,2}$ contains three internal planes, namely $\mathbb{P}(v\otimes V_3)$, $v\in\mathbb{P}V_2$, which may be identified with the three internal planes $X,Y,P_0$ of $\psi_3$. The seven internal lines $\mathbb{P}(V_2\otimes w)$, $w\in\mathbb{P}V_3$, of $\mathcal{S}_{1,2,2}$ are transversals of the planes $X,Y,P_0$. The stabilizer $\mathcal{G}$ of $\psi_3=\mathcal{S}_{1,2,2}$ consists of elements of $\mathrm{GL}(V_2\otimes V_3)$ of the form $C\otimes D$, $C\in\mathrm{GL}(V_2)$, $D\in\mathrm{GL}(V_3)$ and so $\mathcal{G}$ is isomorphic to $\mathrm{GL}(2,2)\times\mathrm{GL}(3,2)$, of order $6\times 168=1008$.

Worthy of note are two particular elements $J,K\in\mathrm{GL}(V_3\oplus V_3)$ defined by $J(x,y)=(y,x)$, $K(x,y)=(y,x+y)$, $x,y\in V_3$, and satisfying $J^2=I$ and $K^3=I$. Observe that $J$ and $K$ effect the following permutations

$$J:(XY)(P_0)(P_1P_6)(P_2P_5)(P_4P_3),\quad K:(XYP_0)(P_1P_2P_4)(P_3P_5P_6). \tag{2.13}$$

of the members of $\Sigma_9$. For example, $K$ maps the element $(x, Ax) \in P_1$ to $(Ax, (I + A)x) = (Ax, A^3x) \in P_2$. So *both* $J$ *and* $K$ *are elements of* $\mathcal{G}_{\psi_3}$, *and also of* $\mathcal{G}(\Sigma_9)$.

There are two $\mathcal{G}_{\psi_3}$-orbits of internal lines, namely the 7 transversals $\Omega_7^{(1)}(111)$ of the three planes $X,\ Y,\ P_0$, and the $(3 \times 7 =)21$ lines $\Omega_{21}^{(1)}(300)$ which lie inside one of the planes $X,\ Y,\ P_0$. Through a line $L \in \Omega_{21}^{(1)}(300)$ pass three lines $\in \Omega_7^{(1)}(111)$, and these three generate a solid of an orbit $\Omega_7^{(3)}(333)$. A solid $D \in \Omega_7^{(3)}(333)$ is of the form $D = \mathcal{H} \cup L' \cup L''$, where the 9-set $\mathcal{H} := D \cap \psi_3$ is a hyperbolic quadric in the 3-flat $D$ and where $L', L''$ are the two lines of $D$ which are external to $\mathcal{H}$. Such external lines $L', L''$ form a $\mathcal{G}_{\psi_3}$-orbit $\Omega_{14}^{(1)}(000)$. (There exist in fact a further 168 external lines and these form an orbit $\Omega_{168}^{(1)}(000)$.)

Of course it is fairly easy to determine all the $\mathcal{G}_{\psi_3}$-orbits of flats, and then presumably to obtain the full story for their $\psi_3$-associates. However we content ourselves with reporting here just a couple of particularly interesting aspects, one concerning the singular lines and the other the external planes.

With the aid of Theorem 1.17(v) one can derive the next theorem, from which it follows that *there are no singular points:* $\operatorname{rad}\psi_3 = \emptyset$. Another consequence is that a plane of $\mathrm{PG}(5, 2)$ can contain at most one singular line.

**Theorem 2.4** *There are precisely* 21 $\psi_3$*-singular lines in* $\mathrm{PG}(5, 2)$, *namely the lines* $L \in \Omega_7^{(1)}(111) \cup \Omega_{14}^{(1)}(000)$. *Moreover these constitute a line-spread for* $\mathrm{PG}(5, 2)$.

Concerning planes $P$ which are external to $\psi_3 = X \cup Y \cup P_0$, without loss of generality, see [11, Section 4.1], [12, Section 4], we may assume that $P$ is one of the planes $P_r = \{(x, A^r x),\ x \in \mathbb{P}V_3\},\ r = 1, 2 \ldots, 6$, in Eq. (2.3), for some element $A \in \mathrm{GL}(3, 2)$ satisfying $A^3 = I + A$. Since there exist precisely eight $Z_7$-subgroups of $\mathrm{GL}(3, 2)$ there are $8 \times 6 = 48$ planes external to $\psi_3$, forming a $\mathcal{G}_{\psi_3}$-orbit $\Omega_{48}^{(2)}(000)$.

**Theorem 2.5** *If* $\psi_3$ *and* $P_r$ *are as in (2.7) and (2.3) then the associates of the external planes* $P_r$ *are as follows:*

$$(P_1)^\# = P_2, \qquad (P_2)^\# = P_4, \qquad (P_4)^\# = P_1, \tag{2.14}$$

$$(P_6)^\# = P_5, \qquad (P_5)^\# = P_3, \qquad (P_3)^\# = P_6. \tag{2.15}$$

*So for any external plane* $P$ *we have* $P^{\#\#\#} = P$.

**Proof.** Knowing from Theorem 1.17(iii) that $(P_1)^\#$ is a plane, to show that $(P_1)^\# = P_2$ we need to show, see Theorem 1.17(i), that the plane $\langle z, L\rangle$ is $\psi$-odd for each point $z \in P_2$ and each line $L \subset P_1$. Because the planes of the spread 2.2 are invariant under the $Z_7$ subgroup of $\mathcal{G}_{\psi_3}$ generated by $\Phi_A = A \oplus A$, it suffices to consider just one choice of line $L \subset P_1$, say $L_1 := \{(a_1, a_2), (a_2, a_3), (a_4, a_5)\}$ (in the notation of Equation (2.4)). Then one sees that:

*(a)* if $z = (a_1, a_3) \in P_2$ then the plane $\langle z, L_1\rangle$ meets $\mathcal{S}_{1,2}$ in the three points $(0, a_5) \in Y,\ (a_4, 0) \in X$ and $(a_2, a_2) \in P_0$;

*(b)* for the other six points $z \in P_2$ the plane $\langle z, L_1\rangle$ meets just one of the planes $X, Y, P_0$.

So indeed $\langle z, L\rangle$ is $\psi$-odd for each point $z \in P_2$ and each line $L \subset P_1$, whence

$P_2 = (P_1)^\#$. On applying to this last relation the element $K \in \mathcal{G}_{\psi_3}$ of (2.13) we obtain the other two relations in (2.14). Further the relations (2.15) follow from the relations (2.14) upon applying the involution $J \in \mathcal{G}_{\psi_3}$, see (2.13). ∎

**Corollary 2.6** *The* 48 *planes external to the Segre variety* $\mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$ *fall into eight pairs of ordered triplets* $\{(P_1, R_1, S_1), (P_2, R_2, S_2)\}$ *such that* $\psi^c = P_1 \cup R_1 \cup S_1 \cup P_2 \cup R_2 \cup S_2$ *and*

$$P_i^\# = R_i, \quad R_i^\# = S_i, \quad S_i^\# = P_i, \quad i = 1, 2. \tag{2.16}$$

**Remark 2.7** *One may be tempted to believe that the existence of ordered triplets* $(P, R, S)$ *of distinct planes in* $\mathrm{PG}(5,2)$ *satisfying* $P^\# = R,\ R^\# = S,\ S^\# = P$ *must somehow originate from* $d = 3$. *In fact such a belief does not hold up, for, as mentioned in Section 1.4.1, in the* $d = 5$ *case* $\psi = G_{1,4,2}$ *there exist triples* $(P, R, S)$ *of distinct 4-flats in* $\mathrm{PG}(9,2)$ *such that* $P^\# = R,\ R^\# = S,\ S^\# = P$.

## 2.5 The case $\psi_4 = Y \cup P_1 \cup P_2 \cup P_4 \cup P_0 = (X \cup P_6 \cup P_5 \cup P_3)^c$

We now consider the 35-set $\psi_4$ underlying the non-maximal partial spread $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$. By appeal to the results in [11, Section 4] we see that the following lemma holds.

**Lemma 2.8** *There exists a subgroup* $\mathcal{A} \cong \mathrm{Alt}(4)$ *of* $\mathcal{G}_{\psi_4}$ *which fixes the plane* $P_0$ *and effects all even permutations of the four planes* $\{Y, P_1, P_2, P_4\}$. *Let* $\mathcal{N}$ *denote the normal subgroup* $\cong (Z_2)^2$ *of* $\mathcal{A}$. *Then the subgroup* $\mathcal{B} := \langle \Phi_A, \mathcal{N} \rangle$ *of* $\mathcal{G}_{\psi_4}$, *of order* 28, *has the structure* $Z_7 \times (Z_2)^2$. *It acts transitively upon the* $7 \times 4 = 28$ *internal lines, and also upon the* 28 *internal points, of the four planes* $Y, P_1, P_2, P_4$.

**Theorem 2.9** *If* $\psi_4$ *is the* 35*-set underlying the non-maximal partial spread* $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$ *then*
*(i) every line* $L$ *of the plane* $P_0$ *is* $\psi_4$*-singular;*
*(ii) the lines of the other four planes of* $\Sigma_5$ *are all non-singular.*

**Proof.** (i) Let $D$ be the solid $\langle (0, a_1), (0, a_2), (a_1, a_2), (a_2, a_3) \rangle$ and observe that $D$ is disjoint from the line $L = \langle (a_2, a_2), (a_3, a_3) \rangle \subset P_0$. Hence the fifteen planes $P$ which contain $L$ are the fifteen planes $\langle p, L \rangle, p \in D$. By a straightforward check we find that all fifteen planes are $\psi_4$-odd, with twelve meeting $\psi_4$ in 5 points, one in 3 points and two ($P_0$ and one other) being internal to $\psi_4$. So, by theorem 1.17(v), $L$ is singular. The same is true for all 7 lines of $P_0$, since they form a single orbit under the action of the subgroup $\langle \Phi_A \rangle \cong Z_7$ of $\mathcal{G}_{\psi_4}$.

(ii) By lemma 2.8 the 28 internal lines of the four planes $Y, P_1, P_2, P_4$ lie on the same $\mathcal{G}_{\psi_4}$-orbit, so it suffices to check that just one of them is non-singular. Consider the line $L = \langle (0, a_1), (0, a_2) \rangle \subset Y$ and observe that the plane $\langle (a_3, a_0), L \rangle$ is even, meeting $\psi_4$ in the four points $\{(a_3, a_4)\} \cup L$. ∎

The author was on the verge of being surprised by the fact that for any non-maximal partial plane-spread $\Sigma_5$ in $\mathrm{PG}(5,2)$ one plane of $\Sigma_5$ is thus singled out as being that unique plane of $\Sigma_5$ which is singular. But then he recalled that eleven years ago he had, in talks delivered at the Pythagorean and Italian Combinatorics conferences in 1996 (see [11, Theorem 4.3] and [12, Theorem 4.2]), uncovered the existence of a *privileged plane*, as described in part (i) of the next theorem. It was then hardly a surprise to find that this privileged plane of $\Sigma_5$ coincided with the singular plane of $\Sigma_5$.

**Theorem 2.10** *(i) Any non-maximal partial spread* $\Sigma_5$ *of* 5 *planes in* $\mathrm{PG}(5,2)$ *possesses a privileged member, say* $W_* \in \Sigma_5$*, such that each of the 7 further internal planes* $S_i$ *of* $\psi_4$ *(see after (2.6)) meets* $W_*$ *in a line and meets each of the four other planes* $W \in \Sigma_5$ *in a point. The* 7 *lines* $S_i \cap W_*$ *are distinct, as are the* 28 *points* $S_i \cap W$, $i = 0, 1, \ldots 6$, $W \in \Sigma_5 \setminus \{W_*\}$.

*(ii) If* $\psi$ *is the underlying* 35*-set of* $\Sigma_5$ *then the privileged plane of* $\Sigma_5$ *is that plane of* $\Sigma_5$ *which is* $\psi$*-singular.*

**Proof.** (i) See [12, Theorem 4.2].

(ii) If $\psi_4$ and $\Sigma_5$ is as in theorem 2.9 then $P_0$ must be the privileged plane of $\Sigma_5$ since it is the only plane of $\Sigma_5$ fixed by the subgroup $\mathcal{A}$ of $\mathcal{G}_{\psi_4}$. (*Aliter:* see [12, after eq. (4.23)].) ■

**Remark 2.11** *Consider the following* $4+1+4$ *partition of the spread* $\Sigma_9$*:*

$$\Sigma_9 = \Sigma_4 \cup \{P_0\} \cup \Sigma_4', \text{ where } \Sigma_4 = \{Y, P_1, P_2, P_4\},\ \Sigma_4' = \{X, P_6, P_5, P_3\}. \quad (2.17)$$

*It is known, see [11, Section 4], that* $P_0$ *is the privileged plane of both* $\Sigma_5 = \Sigma_4 \cup \{P_0\}$ *and* $\Sigma_5' = \Sigma_4' \cup \{P_0\}$*, and that the symmetry groups* $\mathcal{G}(\Sigma_4), \mathcal{G}(\Sigma_5), \mathcal{G}(\Sigma_4')$ *and* $\mathcal{G}(\Sigma_5')$ *are all equal to* $\mathcal{G}_{\psi_4}$*. Further the full stabilizer of* $\psi_4$ *is*

$$\mathcal{G}_{\psi_4} = \langle \Phi_A, \mathcal{A} \rangle \cong (Z_7 \times (Z_2)^2) \rtimes Z_3; \quad (2.18)$$

*moreover the subgroup* $\mathcal{G}(\Sigma_4, \Sigma_4')$ *of* $\mathcal{G}(\Sigma_9)$ *consisting of all those elements which preserve the* $4+1+4$ *partition (2.17) is the following direct product:*

$$\mathcal{G}(\Sigma_4, \Sigma_4') = \mathcal{G}_{\psi_4} \times \langle J \rangle. \quad (2.19)$$

*Here the involution* $J$*, see (2.13), effects the interchanges* $\Sigma_4 \rightleftarrows \Sigma_4'$ *and* $\Sigma_5 \rightleftarrows \Sigma_5'$ *via its effect* $(XY)(P_0)(P_1P_6)(P_2P_5)(P_4P_3)$*, on the planes of* $\Sigma_9$*.*

**Theorem 2.12** *As previously let* $\psi_4$ *be the* 35*-set underlying the non-maximal partial spread* $\Sigma_5 = \{Y, P_1, P_2, P_4, P_0\}$ *whose (unique) extension to a complete spread* $\Sigma_9$ *is as in (2.17). Then*

*(i)* $(P_0)^\# = \mathrm{PG}(5,2)$*;*

*(ii) if* $P \in \Sigma_4 := \Sigma_5 \setminus \{P_0\}$ *then* $P$ *is self-associate:* $P^\# = P$*;*

*(iii) if* $P \in \Sigma_4' := \Sigma_9 \setminus \Sigma_5$ *then* $P^\# = J(P)$*, where* $J$ *is as in (2.19).*

**Proof.** (i) This follows immediately from theorems 2.9(i) and 1.18(iv).

(ii) This follows immediately from theorems 2.9(ii) and 1.18(i).

(iii) First let us check that $X^\# = Y$. Since $X$ is an even plane, we know from theorem 1.17(iii) that $X^\#$ is a plane disjoint from $X$. Now $X$ is stabilized by the element $\Phi_C \in \mathcal{G}_{\psi_4}$; moreover the only other two planes in $\mathrm{PG}(5,2)$ which are stabilized by $\Phi_C$ are $P_0$ and $Y$. Hence either $X^\# = Y$ or $X^\# = P_0$. The last possibility is easily ruled out: for example the plane $R = \langle p, L \rangle$ joining the point $p = (a_7, a_7)$ of $P_0$ to the line $L = \langle (a_1, a_0), (a_2, a_0) \rangle$ of $X$ is even, meeting $\psi_4$ in the 4-set $\{(a_7, a_7), (a_3, a_7), (a_5, a_7), (a_6, a_7)\}$. Consequently $P^\# = J(P)$ holds in the case of $X \in \Sigma_4$. Since $J$ centralizes the subgroup $\mathcal{N}$ of $\mathcal{G}_{\psi_4}$, and since $\mathcal{N}$ is transitive on the four planes $\Sigma_4$, result (iii) follows. ■

Note that in the present $\psi = \psi_4$ case the even planes $P \in \Sigma_4'$ are thus very far from being faithfully even, since their first associate $P^\#$ is odd.

The next theorem relates the privileged plane of a non-maximal partial spread $\Sigma_5$ to the $\psi_3$-associate of section 2.4.

**Theorem 2.13** *Suppose that $\Sigma_4$ is any partial spread of four planes in* $\mathrm{PG}(5,2)$. *Let $\Sigma_9 = \Sigma_4 \cup \Sigma_5$ be its unique extension to a complete spread. Choosing $P \in \Sigma_4$, let $\psi_3$ denote the underlying 21-set of the three planes $\Sigma_4 \setminus \{P\}$ and use $\psi_3$ to define the associate $X^{\#}$ of a flat $X$ of* $\mathrm{PG}(5,2)$. *Then the privileged plane of $\Sigma_5$ is $P^{\#\#}$.*

**Proof.** If $\Sigma_4 = \{X, Y, P_0, P_1\}$ then, see [12, Theorem 4.3, Proof], the privileged plane of $\Sigma_5$ is $P_4$. But, see theorem 2.5, for $\psi_3 = X \cup Y \cup P_0$ we have $P_4 = P_1^{\#\#}$. ■

## 2.6 The case $\psi_5$: a double-five of planes

In [10] the 35-set $\psi_5 \subset \mathrm{PG}(5,2)$ was constructed starting out from a regular icosahedron whose faces were coloured in two enantiomorphic ways — the two colourings being related to two decompositions $\psi_5 = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5$ and $\psi_5 = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$ of $\psi_5$ into mutually disjoint planes, each set of five planes being a maximal partial spread. The symmetry group of $\psi_5$ is icosahedral: $\mathcal{G}_{\psi_5} \cong \mathrm{Alt}(5) \times Z_2$. As noted after Eq. (2.7) it has a cubic equation $Q(x) = 0$. In [11, Section 2.2] can be found the explicit cubic polynomial $Q$, and its associated alternating trilinear form $T$.

At first glance the set $\psi_5$ seems to hold considerable promise for interesting results concerning the $\psi_5$-associate of flats. Certainly $\psi_5$ has a more interesting and intricate geometric structure than the preceding sets $\psi_1$ - $\psi_4$. Furthermore there is the added excitement of the existence of a $\mathcal{G}_{\psi_5}$-invariant non-degenerate symplectic form $x.y$ on $V_6$: see [11, Lemma 2.1]. So one has hopes for interesting relations between the $\psi_5$-associate $X^{\#}$ of a flat $X$ and the polar $X^{\perp}$ of $X$ with respect to the symplectic polarity on $\mathrm{PG}(5,2)$ determined by $x.y$. The fact that $V = V_6$ is equipped with both a non-degenerate bilinear form $x.y$ and a trilinear form $T(x,y,z)$ gives rise to a further feature of interest, namely that $V$ can be made into a 6-dimensional non-associative algebra $\mathcal{A}$, as follows. From the non-degeneracy of the symplectic form $x.y$ we have a linear isomorphism $\phi : V \to V^*$ defined by $a \mapsto \phi_a$, where $\phi_a(x) = a.x$. Upon defining the product $ab$ of elements $a, b \in V$ by $ab = \phi^{-1}(f_{a,b})$, where $f_{a,b}(x) := T(a,b,x),\ x \in V$, the vector space $V$ is thereby made into a (non-associative) algebra $\mathcal{A}$ over $\mathrm{GF}(2)$. Since $T$ is symmetric, $\mathcal{A}$ is commutative, and since $T$ is alternating, we have $a^2 = 0$, for all $a \in \mathcal{A}$. Also, since both the symplectic form $x.y$ and the trilinear form $T(x,y,z)$ are $\mathcal{G}_{\psi_5}$-invariant, the automorphism group $\mathrm{Aut}\,\mathcal{A}$ contains $\mathcal{G}_{\psi_5}$.

Nevertheless these initial hopes appear not to be borne out, the chief 'culprit' for this failure being the existence of a (unique) fixed point $u$ of $\mathcal{G}_{\psi_5}$, termed the *nucleus* of $\psi_5$. The nucleus $u$ also gives rise to the hyperplane $\varpi$ whose equation is $u.x = 0$ and which meets $\psi_5$ in a parabolic quadric $\mathcal{P}_4$ (the unique parabolic quadric lying on $\psi_5$), the nucleus of $\mathcal{P}_4$ being $u$. Now, as noted in [11, Section 2.2, Remark (i)], $T(x,y,u) = 0$ for all $x, y \in V$ — that is, in our present language, $u$ is a singular point for $\psi_5$. Consequently, see Lemma 1.20, *there exist no faithfully $\psi_5$-even planes in* $\mathrm{PG}(5,2)$. Indeed if $P$ is any even plane then $P^{\#}$ is a plane which contains $u$ and so is odd. (In fact $u$ is the only singular point: $\mathrm{rad}\,\psi_5 = \{u\}$: for $\mathrm{rad}\,\psi_5$ is a flat which is stabilized by $\mathcal{G}_{\psi_5}$, and the only other such flat is the hyperplane $\varpi$, which is easily seen to contain non-singular points.

Moreover the 6-dimensional non-associative algebra $\mathcal{A}$ does not appear to have much interest, again due to the properties of the point $u$. For example, as well as the property $x^2 = 0$, we have $ux = 0$, for all $x \in \mathcal{A}$. Also if we consider the 5-dimensional subalgebra $\mathcal{A}^{\text{even}}$ of $\mathcal{A}$ consisting of vectors of even weight with respect to the basis $\mathcal{B}$ employed in [11], then for all $x, y \in \mathcal{A}^{\text{even}}$ the algebra product $xy$ is either 0 or $u$.

We conclude by giving a relationship between $P^\perp$ and $P^\#$ which holds if the plane $P$ is one of the five internal planes $\alpha_r$, or one of the five planes $\beta_r$: $P^\perp = P, \quad P^\# = \langle u, P\rangle$.

# 3 Other hypersurfaces worthy of attention?

Leaving aside the cubic hypersurfaces in $\mathrm{PG}(5,2)$ of Section 2, and also the Grassmannians $\mathcal{G}_{1,m,2}$ of Section 1.4.1, out of the truly enormous number of other choices available for $\psi$ we list here just a few of the many cases where the study of the $\psi$-associate (of at least some kinds of flats) would appear to hold out promise:
*(a)* the Segre variety $\psi_a = \mathcal{S}_{1,3,2} \subset \mathrm{PG}(7,2)$, for which $d = 4$;
*(b)* the Segre variety $\psi_b = \mathcal{S}_{2,2,2} \subset \mathrm{PG}(8,2)$, for which $d = 6$;
*(c)* the hypersurface $\psi_c \subset \mathrm{PG}(8,2) = \mathbb{P}(\operatorname{End} V_3)$ with equation $\det A = 0$, $A \in \operatorname{End}(V_3)$; so $d = 3$;
*(d)* the hypersurface $\psi_d \subset \mathrm{PG}(15,2) = \mathbb{P}(\operatorname{End} V_4)$ with equation $\det A = 0$, $A \in \operatorname{End}(V_4)$; so $d = 4$.

Concerning the first two of these, the polynomial degree of the Segre variety $\mathcal{S}_{r,s,2} \subset \mathrm{PG}(rs + r + s, 2)$, $r \leq s$, is known, [15], to be $d = rs + r$. (Recall that the cubic $\psi_3$ in Section 2.4 is in effect the $(r, s) = (1, 2)$ case of a Segre variety.) The symmetry groups $\mathcal{G}_\psi$ in these four cases are quite large:

| $\psi =$ | $\psi_a$ | $\psi_b$ | $\psi_c$ | $\psi_d$ |
|---|---|---|---|---|
| $\mathcal{G}_\psi \cong$ | $L_2(2) \times L_4(2)$ | $(L_3(2) \times L_3(2)).2$ | $(L_3(2) \times L_3(2)).2$ | $(L_4(2) \times L_4(2)).2$ |
| $\lvert\mathcal{G}_\psi\rvert =$ | $120,960$ | $56,448$ | $56,448$ | $812,851,200$ |

.

(See [1] for the structure of $\mathcal{G}_\psi$ in the cases *(c)* and *(d)*; concerning $\mathcal{G}_{\psi_b} \cong \mathcal{G}_{\psi_c}$, the 49 points of $\psi_b$ can be identified with the 49 elements of $\psi_c$ of rank 1.) However as the projective dimension $n$ increases the number of lines, planes, solids, ..., in $\mathrm{PG}(n, 2)$ of course increases enormously:

| Number of $r$-flats | $\mathrm{PG}(7,2)$ | $\mathrm{PG}(8,2)$ | $\mathrm{PG}(15,2)$ |
|---|---|---|---|
| $r = 0$ (points) | 255 | 511 | $65,535$ |
| $r = 1$ (lines) | $10,795$ | $43,435$ | $715,795,115$ |
| $r = 2$ (planes) | $97,155$ | $788,035$ | $> 1.6 \times 10^{12}$ |
| $r = 3$ (solids) | $200,787$ | $3,309,747$ | $> 9.1 \times 10^{14}$ |

So it would be a formidable task to fully carry out the last two steps of the plan in Section 1.4! Nevertheless let us at least determine $X^\#$ for two kinds of flat $X$ in the case of $\psi_c$.

## 3.1 The case $\psi = \psi_c \subset \mathrm{PG}(8,2) : \det A = 0,\ A \in \operatorname{End}(V_3)$

In the case $\psi = \psi_c$ even planes are of interest since we know from Corollary 1.16 that the $\psi_c$-associate $P^\#$ of an even plane $P \subset \mathrm{PG}(8,2)$ is necessarily a disjoint 5-flat. For our first kind of $\psi_c$-even plane $P \subset \mathrm{PG}(8,2)$ choose a basis $\mathcal{B}$ for

$V_3$ and let $P$ consist of those elements of $\operatorname{End} V_3$ whose matrices are diagonal. Then $P$ is even, since $\det A = 0$ for the six elements $A \in P$ other than $A = I$. Let $S$ denote the 5-flat $\subset \mathrm{PG}(8,2)$ consisting of those elements of $\mathrm{GL}(V_3)$ which have matrices whose diagonal entries are all zero.

**Theorem 3.1** $P^{\#} = S$.

**Proof.** Let $e_{ij}$ denote that element of $\operatorname{End} V_3$ whose matrix in the basis $\mathcal{B}$ has for its sole nonzero entry a 1 in the $ij$ place. So $P = \langle e_{11}, e_{22}, e_{33} \rangle$. and $S = \langle \{e_{kl}\}_{k \neq l} \rangle$. Now for $i \neq j$, $k \neq l$ the plane $\langle e_{ii}, e_{jj}, e_{kl} \rangle$, is odd since all seven of its elements have zero determinant. From Theorem 1.17(i) it follows that $P^{\#} = S$. ■

**Remark 3.2** *This result has an obvious generalization to the hypersurface* $\psi \subset \mathrm{PG}(n^2-1, 2) = \mathbb{P}(\operatorname{End} V_n)$ *with equation* $\det A = 0$; *here* $d = n$, $P = \langle e_{11}, \dots, e_{nn} \rangle$ *is an* $(n-1)$*-flat and* $S$ *is a disjoint* $(n^2 - n - 1)$*-flat. In the* $n = 2$ *case* $\psi$ *is a hyperbolic quadric* $\mathcal{H}_3$ *in* $\mathrm{PG}(3,2)$ *and* $P = \langle e_{11}, e_{22} \rangle$ *is a bisecant of* $\mathcal{H}_3$, *one of an orbit* $\Omega$ *of 18 bisecants; observe that* $P^{\#} = \langle e_{12}, e_{21} \rangle$ *also* $\in \Omega$.

For another kind of $\psi_c$-even plane in $\mathrm{PG}(8,2)$, choose an element $A \in \mathrm{GL}(V_3)$ of order 7. Then the Singer subgroup $Z = \{I, A, A^2, \dots, A^6\} \cong \mathbb{Z}_7$ of $\mathrm{GL}(V_3)$ is a plane in $\mathrm{PG}(8,2)$ which is external to $\psi_c$ and so is $\psi_c$-even. The normalizer $N(Z)$ in $\mathrm{GL}(V_3)$ is a subgroup $\mathcal{F} \cong Z_7 \rtimes Z_3$ of order 21 which is generated by $A$ and $C$ where $C \in \mathrm{GL}(V_3)$, of order 3, satisfies $CAC^{-1} = A^2$. Consider the coset decomposition $\mathcal{F} = X \cup Y \cup Z$, where $X = CZ$ and $Y = C^2 Z$ comprise the 14 elements of $\mathcal{F}$ which are of order 3. Observe that $X$ and $Y$ are also $\psi_c$-even planes in $\mathrm{PG}(8,2)$.

**Theorem 3.3** *(i)* $Z^{\#} = \langle X, Y \rangle$ *(ii)* $X^{\#} = \langle Y, Z \rangle$ *(iii)* $Y^{\#} = \langle X, Z \rangle$.

**Proof.** (i) Consider the plane $P = \langle L, D \rangle$ where $L$ is a line in $Z$ and $D \in X \cup Y$. Then $P$ contains 4 external points $L \cup \{D\}$ and 3 further points $\{B + D\}_{B \in L}$, each of the latter points $B + D$ being of the form $A^r(I + A^s C)$ or $A^r(I + A^s C^2)$. Now $\det(I + A^s C) = 0$ and $\det(I + A^s C^2) = 0$, since $A^s C$ and $A^s C^2$, being of order 3, possess a fixed point in $\mathrm{PG}(3,2)$; so the 3 points $\{B + D\}_{B \in L}$ are all internal, and hence all of the planes $\langle L, D \rangle$ are odd. By Lemma 1.11(i) it follows that $X \subset Z^{\#}$ and $Y \subset Z^{\#}$, and hence that $\langle X, Y \rangle \subseteq Z^{\#}$. But $X$ and $Y$ are disjoint planes, and so $\langle X, Y \rangle$ is a 5-flat. Hence, by the choice $(d, n) = (3, 8)$ in Corollary 1.16, $Z^{\#} = \langle X, Y \rangle$.

(ii), (iii) An entirely similar proof applies. ■

**Remark 3.4** *The foregoing planes* $X, Y, Z$ *are, in vector space language, examples of 3-dimensional* linear sections *of* $\mathrm{GL}(3,2)$. *For* $n = 3$ *and* $n = 4$ *all linear sections of* $\mathrm{GL}(n,2)$ *were classified in [3]. For* $n = 3$ *they are all of* Singer type *as exemplified by the foregoing* $Z$. *However* $\mathrm{GL}(4,2)$ *possesses several kinds of linear sections, not all of Singer type, see [3, Theorem 5.1], and it may be of interest to determine the* $\psi_d$*-associates of each of these.*

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