

The polynomial degrees of Grassmann and Segre varieties over $\text{GF}(2)$

R. SHAW

r.shaw@hull.ac.uk

Department of Mathematics, University of Hull, Hull HU6 7RX, United Kingdom

Abstract

A recent proof that the Grassmannian $\mathcal{G}_{1,n,2}$ of lines of $\text{PG}(n, 2)$ has polynomial degree $\binom{n}{2} - 1$ is outlined, and is shown to yield a theorem about certain kinds of subgraphs of any (simple) graph $\Gamma = (\mathcal{V}, \mathcal{E})$ such that $|\mathcal{E}| < |\mathcal{V}|$. Somewhat similarly, the polynomial degree of the Segre variety $\mathcal{S}_{m,n,2}$, $m \leq n$, is shown to be $mn + m$, and in consequence a graph theory result is obtained about certain subgraphs of any graph Γ which is a subgraph of the complete bipartite graph $K_{m+1,n+1}$.

Keywords: polynomial degree, Grassmannian $\mathcal{G}_{1,n,2}$, Segre variety $\mathcal{S}_{m,n,2}$, subgraph enumerations

AMS Classification: 51E20, 05C30, 05C90, 14G25

1 The polynomial degree of a subset ψ of $\text{PG}(N, 2)$

In succeeding sections we will be interested in the polynomial degrees of the following varieties over the finite field $\text{GF}(2)$:

- (i) the Grassmann variety $\mathcal{G}_{1,n,2}$ of the lines of $\text{PG}(n, 2)$, considered as a subset of points of the finite projective space $\text{PG}(\binom{n+1}{2} - 1, 2) = \mathbb{P}(\wedge^2 V_{n+1,2})$;
- (ii) the Segre variety $\mathcal{S}_{m,n,2}$, considered as a subset of points of the finite projective space $\text{PG}(mn + m + n, 2) = \mathbb{P}(V_{m+1,2} \otimes V_{n+1,2})$.

However it will help to first consider material concerned with the polynomial degree of a general subset ψ of points of a general finite projective space $\text{PG}(N, 2) = \mathbb{P}(V)$, where $V = V_{N+1} = V(N + 1, 2)$.

For the most part the notation will be as in [9]. In particular $S = \text{PG}^{(0)}(N, 2)$ denotes the set of points (0-flats) of $\text{PG}(N, 2) = \mathbb{P}(V)$, and we identify S with the nonzero vectors $V \setminus \{0\}$ of the vector space V . The set $F(V)$ of all functions $V \rightarrow \text{GF}(2)$ is a vector space over $\text{GF}(2)$ of dimension $|V| = 2^{N+1}$, and its elements are the characteristic functions $\chi(\psi)$, also denoted χ_ψ , of the subsets $\psi \subseteq V$. In the case when ψ is a singleton set $\{a\}$, $a \in V$, we put $\chi_a := \chi_{\{a\}}$. In fact, rather than $F(V)$, *our main focus*

is on the vector subspace $F(S)$, of dimension $|S| = 2^{N+1} - 1$ over $\text{GF}(2)$, consisting of all functions $S \rightarrow \text{GF}(2)$.

Upon choosing a basis $\mathcal{B} = \{e_1, e_2, \dots, e_{N+1}\}$ for V an element $x \in V$ may be viewed as an $(N+1)$ -tuple $(x_1, x_2, \dots, x_{N+1}) \in \text{GF}(2)^{N+1}$. The basis \mathcal{B} for V gives rise to an associated *monomial basis* \mathcal{M} for $F(S)$, namely

$$\mathcal{M} = \Xi_1 \cup \Xi_2 \cup \dots \cup \Xi_{N+1}, \quad \text{where } \Xi_r = \{x_{i_1}x_{i_2}\dots x_{i_r}\}_{1 \leq i_1 < i_2 < \dots < i_r \leq N+1}. \quad (1.1)$$

If ψ^c denotes the complement *within the set* S of ψ then $\chi(\psi) + \chi(\psi^c) = I$, where I denotes that element of $F(S)$ such that $I(x) = 1$ for all $x \in S$. The characteristic functions $\chi_a, a \in S$, have the coordinate expression:

$$\chi_a(x) = \chi_0(a+x), \quad \text{where } \chi_0(x) = \prod_{i=1}^{N+1} (1+x_i), \quad (1.2)$$

and $I = \chi(S)$ has the coordinate expression

$$I(x) = 1 + \prod_{i=1}^{N+1} (1+x_i) = \sum_i x_i + \sum_{i < j} x_i x_j + \dots + x_1 x_2 \dots x_{N+1}. \quad (1.3)$$

This last expression (1.3) may be viewed as the special case $r = N+1$, $X^c = S$, of the following easily verified result:

if X is an $(N-r)$ -flat in $\text{PG}(N, 2)$ which is the intersection of the r hyperplanes $f_1(x) = 0, \dots, f_r(x) = 0$, then

$$\begin{aligned} \chi(X^c) &= 1 + \prod_{i=1}^r (1+f_i) \\ &= \sum_i f_i + \sum_{i < j} f_i f_j + \sum_{i < j < k} f_i f_j f_k + \dots + f_1 f_2 \dots f_r. \end{aligned} \quad (1.4)$$

For $r > 0$, let $F_r = F_r(S)$ denote the subspace of $F(S)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \dots, x_{N+1})$ with $\deg f \leq r$ and $f(0) = 0$; we put $F_0 := \{0\}$. The subspaces F_r are thus nested:

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_N \subset F_{N+1} = F(S), \quad (1.5)$$

with $F_r, r \geq 1$, possessing the monomial basis \mathcal{M}_r where

$$\mathcal{M}_r = \Xi_1 \cup \Xi_2 \cup \dots \cup \Xi_r, \quad 1 \leq r \leq N+1. \quad (1.6)$$

Observing that Ξ_{N+1} consists of the single monomial $m_{N+1} := x_1 x_2 \dots x_{N+1}$, it follows from (1.2) that F_N consists of the characteristic functions of all even ($|\psi| \equiv 0 \pmod{2}$) subsets ψ of S .

The subspace F_r of $F(S)$ has just been given an algebraic definition, but *there exists an equivalent geometric definition*, namely as that subspace of $F(S)$ which is generated by the characteristic functions $\chi(X^c)$ of the complements X^c of the $(N-r)$ -flats X of $\text{PG}(N, 2)$. For if we define subspaces $C_r, 0 \leq r < N$, of $F(S)$ by

$$C_r = \prec \chi(X^c) \succ_{X \in \text{PG}^{(r)}(N, 2)}, \quad (1.7)$$

then it can be shown, see [9, Theorem 1.5], cf. [1, Section 5.3], that

$$C_{N-r} = F_r, \quad r = 1, 2, \dots, N. \quad (1.8)$$

Setting $Q_\psi := \chi(\psi^c)$, a subset ψ of S has equation $Q_\psi(x) = 0$. If $Q_\psi \in F_r \setminus F_{r-1}$ we will say that ψ has *polynomial degree* r , and we write $\deg Q_\psi = r$ for the degree of Q_ψ . (Here $\deg Q_\psi$ is the *reduced* degree of Q_ψ ; if $\deg Q_\psi = r$ then of course, see (1.5), $Q_\psi \in F_s$ for each $s \geq r$.) Recall that the subspace $C_0 = F_N$ consists of the characteristic functions of all the even subsets of S . Consequently if ψ is an odd subset of S (and so ψ^c is an even subset) then ψ has polynomial degree $\leq N$. On the other hand, since $\chi(\psi) + \chi(\psi^c) = I$, and $\deg I = N + 1$, *an even subset always has polynomial degree $N + 1$.*

In general the determination of the polynomial degree of a subset $\psi \subset S$ is a formidable problem — and especially so if a direct algebraic approach is attempted, based for example upon (1.2). But quite often progress can be made by using a geometrical approach based upon the next theorem.

Theorem 1.1 *Let ψ be an odd subset of $S = \text{PG}^{(0)}(N, 2)$, and consider the following two conditions:*

- (A) *$|X \cap \psi|$ is odd for all r -flats X of $\text{PG}(N, 2)$;*
- (B) *there exists an $(r - 1)$ -flat X of $\text{PG}(N, 2)$ for which $|X \cap \psi|$ is even.*

Then

- (i) *$Q_\psi \in F_r$ if and only if ψ satisfies condition (A);*
- (ii) *ψ has polynomial degree r if and only if ψ satisfies both (A) and (B).*

Proof. (i) See [9, Theorem 1.7], or see [7].

(ii) From (B) it follows from (i) that $Q_\psi \notin F_{r-1}$. So both (A) and (B) will hold if and only if $Q_\psi \in F_r \setminus F_{r-1}$. ■

We adopt this geometric approach in the present paper and thereby aim to determine the polynomial degree in the cases (i) $\psi = \mathcal{G}_{1,n,2}$ and (ii) $\psi = \mathcal{S}_{m,n,2}$. Another of our aims is to demonstrate a tie-in with problems of enumeration of certain kinds of subgraphs of (i) graphs of order $n + 1$, and (ii) graphs which are subgraphs of the complete bipartite graph $K_{m+1,n+1}$.

To these ends, with respect to a choice of basis $\{e_1, e_2, \dots, e_{N+1}\}$ for V_{N+1} , let $X_{i_1 i_2 \dots i_s}$ denote that $(N - s)$ -flat with coordinate equation $x_{i_1} = x_{i_2} = \dots = x_{i_s} = 0$, and let $Y(j_1, j_2, \dots, j_{s+1})$ denote the s -flat $\langle e_{j_1}, e_{j_2}, \dots, e_{j_{s+1}} \rangle$. Observe that if $\{i_1, i_2, \dots, i_{N-s}\} = \{1, 2, \dots, N + 1\} \setminus \{j_1, j_2, \dots, j_{s+1}\}$ then

$$X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \dots, j_{s+1}). \quad (1.9)$$

Upon writing $\chi_{i_1 \dots i_s} := \chi(X_{i_1 \dots i_s}^c)$ observe that elements of the set $\mathcal{F}_s := \{\chi_{i_1 \dots i_s} \mid 1 \leq i_1 < i_2 < \dots < i_s \leq N + 1\}$ are in bijective correspondence with the faces of the simplex of reference of projective dimension $N - s$. Upon noting from (1.4) that the element $\chi_{i_1 i_2 \dots i_s}$ of C_{N-s} differs from $x_{i_1} x_{i_2} \dots x_{i_s}$ by elements of $F_{s-1} = C_{N-s+1}$, we derive from (1.6) the following theorem.

Theorem 1.2 (cf. [8, Section 2]) For $0 \leq r < N$ a basis for $C_r = F_{N-r}$ is

$$\mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_{N-r}. \quad (1.10)$$

By appeal to this *simplex basis* (1.10) for C_r it follows that in theorem 1.1 we can replace “for all r -flats X of $\text{PG}(N, 2)$ ” in condition (A) by “for all s -flats $X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \dots, j_{s+1})$, $s \geq r$ ”. In this manner we obtain from theorem 1.1 the following theorem. In this theorem a flat X of $\text{PG}(N, 2)$ is said to be ψ -odd or ψ -even according as X meets ψ in an odd or even number of points.

Theorem 1.3 Let ψ be an odd subset of $S = \text{PG}^{(0)}(N, 2)$ and consider the following two conditions:

(A*) for $s \geq r$ each of the s -flats $X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \dots, j_{s+1})$ is ψ -odd;

(B*) at least one of the $(r-1)$ -flats $Y(j_1, j_2, \dots, j_r)$ is ψ -even.

Then (i) $Q_\psi \in F_r$ if and only if ψ satisfies condition (A*);

(ii) ψ has polynomial degree r if and only if ψ satisfies (A*) and (B*).

2 The polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$

2.1 Introduction

For $V_{n+1} = V(n+1, 2)$ the bivector space $\wedge^2 V_{n+1}$ has vector space dimension $\binom{n+1}{2}$. We will be dealing with the associated projective space $\mathbb{P}(\wedge^2 V_{n+1}) = \text{PG}(N, 2)$, where $N := N_n = \binom{n+1}{2} - 1$, and, for $n \geq 3$, we will be interested in the Grassmannian $\mathcal{G}_{1,n,2} \subset S := \text{PG}^{(0)}(N, 2)$, which consists of the Grassmann images $m = a \wedge b$ of the lines $\mu = \langle a, b \rangle$ of $\text{PG}(n, 2) = \mathbb{P}V_{n+1}$. Observe that $|\mathcal{G}_{1,n,2}| = \frac{1}{3}(2^{n+1} - 1)(2^n - 1)$ is odd. The natural action of $A \in \text{GL}(n+1, 2)$ upon $V_{N+1} := \wedge^2 V_{n+1}$ is by $T_A = \wedge^2 A : a \wedge b \mapsto Aa \wedge Ab$. Now, for $n > 3$, the subgroup $G(\mathcal{G}_{1,n,2})$ of $\text{GL}(N+1, 2)$ which stabilizes $\mathcal{G}_{1,n,2}$ is the isomorphic image under T of $\text{GL}(n+1, 2)$. Consequently the space $F(S)$ will be viewed as a $\text{GL}(n+1, 2)$ -space under the action L defined by $(L_A f)(x) = f(T_A^{-1}x)$, $A \in \text{GL}(n+1, 2)$, $x \in S$. In the following we set

$$Q_{1,n,2} = \chi((\mathcal{G}_{1,n,2})^c), \quad \delta_n = \deg Q_{1,n,2}, \quad N_n = n + d_n, \quad (2.1)$$

where $d_n := N_n - n = \binom{n}{2} - 1$. Note therefore, from equation 1.8, that

$$F_{d_n} = C_n, \quad F_n = C_{d_n}. \quad (2.2)$$

Theorem 2.1 (i) $\delta_n \leq d_n + 1$ for all $n \geq 3$;

(ii) $\delta_n \geq d_n$ for all $n \geq 3$;

(iii) $\delta_n = d_n$ for n in the range $3 \leq n \leq 7$.

Proof. See [4] and [9, Theorems 3.1, 3.2, 3.5]. Important ingredients in the proof are: (i) the existence of (Latin) $(n-1)$ -flats internal to $\mathcal{G}_{1,n,2}$; (ii) the existence, see [3], of (d_n-1) -flats external to $\mathcal{G}_{1,n,2}$. Appeal is also made to theorem 1.1. Further, for (iii), the computer was needed in [9] to handle the cases $n = 5, 6, 7$. ■

The results in theorem 2.1(iii) provide good evidence for the following conjecture. This conjecture is just the $q = 2$ special case of a $\mathcal{G}_{1,n,q}$ conjecture put forward (but with almost no supporting evidence!) in [4].

Conjecture 2.2 *For all $n \geq 3$ the Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree $d_n = \binom{n}{2} - 1$. That is $Q_{1,n,2} \in F_{d_n} \setminus F_{d_n-1}$ ($= C_n \setminus C_{n+1}$).*

Theorem 2.3 *Conjecture 2.2 holds if and only if every d_n -flat of $\text{PG}(N, 2)$ is $\mathcal{G}_{1,n,2}$ -odd.*

Proof. Theorem 1.1(i) states that $\delta_n \leq d_n$ if and only if every d_n -flat is $\mathcal{G}_{1,n,2}$ -odd. But, theorem 2.1(ii), $\delta_n \leq d_n$ holds if and only if $\delta_n = d_n$. ■

2.2 Using simple graphs on $n+1$ vertices

The verification in [9] that conjecture 2.2 at least holds up if n is in the range $3 \leq n \leq 7$ was helped by making use of certain simple graphs. Let $\Gamma = (\mathcal{V}_n, \mathcal{E})$ denote a (simple) graph having vertex set $\mathcal{V}_n := \{1, 2, \dots, n+1\}$ and edge set \mathcal{E} . Along with Γ we also need its complement $\bar{\Gamma} = (\mathcal{V}_n, \bar{\mathcal{E}})$. In section 1 a point $x \in S = \text{PG}^{(0)}(N, 2)$ had coordinates $(x_1, x_2, \dots, x_{N+1})$ relative to a choice of basis $\{e_1, e_2, \dots, e_{N+1}\}$ for V_{N+1} . In our present area of concern a basis $\{e_1, e_2, \dots, e_{n+1}\}$ for V_{n+1} gives rise to a product basis $\{e_i \wedge e_j\}_{1 \leq i < j \leq n+1}$ for $V_{N+1} = \wedge^2 V_{n+1}$, and a point $x = \sum_{1 \leq i < j \leq n+1} x_{ij} e_i \wedge e_j \in S$ has coordinates $(x_{ij})_{1 \leq i < j \leq n+1}$.

Given such a graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ to each edge $E = ij := \{i, j\} \in \mathcal{E}$ let us, relative to our choice of basis, associate:

- (i) the coordinate $x_E := x_{ij} (= x_{ji})$, and hence the hyperplane $x_E = 0$;
- (ii) the basis element $e_E := e_i \wedge e_j$ for V_{N+1} .

Further let $X_{\mathcal{E}}$ denote the flat having coordinate equations $x_E = 0$, each $E \in \mathcal{E}$, and let $Y(\bar{\mathcal{E}})$ denote the flat $\langle \{e_E\}_{E \in \bar{\mathcal{E}}} \rangle$. So $x \in Y(\bar{\mathcal{E}})$ if and only if $x = \sum_{E \in \bar{\mathcal{E}}} x_E e_E$, that is if and only if $x_E = 0$, for each $E \in \mathcal{E}$. Hence, cf. eq. (1.9), we have result (i) of:

$$(i) X_{\mathcal{E}} = Y(\bar{\mathcal{E}}); \quad (ii) \chi((X_{\mathcal{E}})^c) = 1 + \prod_{E \in \mathcal{E}} (1 + x_E), \quad (2.3)$$

with the result (ii) being an instance of the result (1.4). If we define

$$\chi_{\mathcal{E}} := \chi((X_{\mathcal{E}})^c) = \chi(Y(\bar{\mathcal{E}})^c) \quad \text{and} \quad \mathcal{F}_s = \{\chi_{\mathcal{E}}\}_{|\mathcal{E}|=s} \quad (2.4)$$

then it follows from theorem 1.2 that, for $0 \leq r < N$, the set

$$\{\chi_{\mathcal{E}}\}_{|\mathcal{E}| \leq N-r} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_{N-r} \quad (2.5)$$

is a basis for $C_r = F_{N-r}$.

Upon noting that $Y(\bar{\mathcal{E}})$ is a d_n -flat if and only if $|\mathcal{E}| = n$, theorem 2.3 translates, see [10, Remark 2.5], into the following theorem.

Theorem 2.4 *The Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree d_n if and only if for each isomorphism class of graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ of size $|\mathcal{E}| = n$ the d_n -flat $X_{\mathcal{E}} = Y(\bar{\mathcal{E}})$ has odd intersection with $\mathcal{G}_{1,n,2}$.*

Direct application of theorem 2.3 would require a knowledge of the $\text{GL}(n+1, 2)$ -orbits of the d_n -flats of $\text{PG}(N, 2)$. But for $n > 4$ this knowledge will surely never be gained. For even in the case $n = 4$, of $\mathcal{G}_{1,4,2}$, much effort was expended in [11] to classify just one kind of flat in $\text{PG}(9, 2)$, namely those external to $\mathcal{G}_{1,4,2}$. In contrast, for small values of n application of theorem 2.4 is quite feasible, at least with computer help.

In order to apply theorem 2.4 we need to consider, for each isomorphism class of graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ of size $|\mathcal{E}| = n$, the intersection of the d_n -flat $X_{\mathcal{E}} = Y(\bar{\mathcal{E}})$ with $\mathcal{G}_{1,n,2}$. Setting $h(\mathcal{E}) := |Y(\bar{\mathcal{E}}) \cap \mathcal{G}_{1,n,2}|$, we would like to show that $h(\mathcal{E})$ is odd for all such graphs Γ . In [9] the points of $\mathbb{P}(\wedge^2 V_{n+1}) = \text{PG}(N, 2)$ were viewed as alternating matrices of size $n+1$, and, for $n \leq 7$, Magma [2] was used to compute the ranks of all the elements of a d_n -flat $Y(\bar{\mathcal{E}})$. A simpler procedure was adopted in [10], where it was shown that, rather than computing the ranks of the bivectors $x \in X_{\mathcal{E}} = Y(\bar{\mathcal{E}})$, the intersection numbers $h(E) := |X_{\mathcal{E}} \cap \mathcal{G}_{1,n,2}|$ can be computed *using simple combinatorial considerations*, as in the next lemma. For this lemma some further notation is needed.

In the following α, β, γ will denote subsets of $\mathcal{V}_n := \{1, 2, \dots, n+1\}$ which are pairwise disjoint and non-empty. Given the graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$, if two such subsets α, β satisfy

$$ij \notin \mathcal{E} \text{ (that is } ij \in \bar{\mathcal{E}} \text{) for all } i \in \alpha \text{ and for all } j \in \beta, \quad (2.6)$$

then we write $\alpha \perp_{\mathcal{E}} \beta$ (or equally $\beta \perp_{\mathcal{E}} \alpha$). Such a non-ordered pair $\{\alpha, \beta\}$ will be termed a *dyad* for the graph $(\mathcal{V}_n, \mathcal{E})$. Also if three (pairwise disjoint and non-empty) subsets α, β, γ of \mathcal{V}_n satisfy $\alpha \perp_{\mathcal{E}} \beta$, $\alpha \perp_{\mathcal{E}} \gamma$ and $\beta \perp_{\mathcal{E}} \gamma$, then we write $\alpha \perp_{\mathcal{E}} \beta \perp_{\mathcal{E}} \gamma$, and we will term the non-ordered triple $\{\alpha, \beta, \gamma\}$ a *triad* for the graph $(\mathcal{V}_n, \mathcal{E})$. For an agreed edge set \mathcal{E} we abbreviate $\perp_{\mathcal{E}}$ by \perp . We also put $e_{\alpha} := \sum_{i \in \alpha} e_i$, and (for $\alpha \cap \beta = \emptyset$) we put $e_{\alpha\beta} := e_{\alpha \cup \beta}$.

Lemma 2.5 *The flat $Y(\bar{\mathcal{E}})$ meets $\mathcal{G}_{1,n,2}$ in $h(\mathcal{E}) = p(\mathcal{E}) + q(\mathcal{E})$ points, where $p(\mathcal{E})$ is the number of dyads $\{\alpha, \beta\}$ for the graph $(\mathcal{V}_n, \mathcal{E})$; $q(\mathcal{E})$ is the number of triads $\{\alpha, \beta, \gamma\}$ for the graph $(\mathcal{V}_n, \mathcal{E})$.*

Proof. Every line λ in $\text{PG}(n, 2)$ is (with respect to the agreed choice of basis $\{e_1, e_2, \dots, e_{n+1}\}$ for V_{n+1}) of one of the following two kinds:

- (1) $\lambda_{\{\alpha, \beta\}} = \{e_{\alpha}, e_{\beta}, e_{\alpha\beta}\}$ for two disjoint non-empty subsets α, β of \mathcal{V}_n ;

(2) $\lambda_{\{\alpha,\beta,\gamma\}} = \{e_{\alpha\gamma}, e_{\beta\gamma}, e_{\alpha\beta}\}$ for three pairwise disjoint non-empty subsets α, β, γ of \mathcal{V}_n .

Now the Grassmann image $l_{\{\alpha,\beta\}} = e_\alpha \wedge e_\beta = \sum_{i \in \alpha} \sum_{j \in \beta} e_i \wedge e_j$ of $\lambda_{\{\alpha,\beta\}}$ lies in $Y(\bar{\mathcal{E}})$ if and only if $ij \in \bar{\mathcal{E}}$ for all $i \in \alpha$ and for all $j \in \beta$, that is if and only if $\alpha \perp \beta$. Similarly the Grassmann image

$$\begin{aligned} l_{\{\alpha,\beta,\gamma\}} &= e_{\alpha\gamma} \wedge e_{\beta\gamma} = e_\alpha \wedge e_\beta + e_\alpha \wedge e_\gamma + e_\beta \wedge e_\gamma \\ &= \sum_{i \in \alpha} \sum_{j \in \beta} e_i \wedge e_j + \sum_{i \in \alpha} \sum_{j \in \gamma} e_i \wedge e_j + \sum_{i \in \beta} \sum_{j \in \gamma} e_i \wedge e_j \end{aligned}$$

of $\lambda_{\{\alpha,\beta,\gamma\}}$ lies in $Y(\bar{\mathcal{E}})$ if and only if $\alpha \perp \beta \perp \gamma$. ■

For some further aspects of $p(\mathcal{E})$ and $q(\mathcal{E})$, and an example of their computation for a particular edge-set \mathcal{E} , see [10, Section 2.3].

2.3 The Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree d_n

By use of the foregoing combinatorial lemma 2.5 it is possible ([6]), with computer aid, to improve upon the result in theorem 2.1(iii) and verify that conjecture 2.2 holds also in the case $n = 8$. Nevertheless, in order to show that $\delta_n = d_n$ for all n , clearly some new idea is needed, as is provided, *cf.* [9, Section 4.2.3], by the following lemma.

Lemma 2.6 *Suppose that there exists a family Ψ of d_n -flats of $\text{PG}(N, 2)$ such that (i) $\prec \{\chi(X^c)\}_{X \in \Psi} \succ = C_{d_n} = F_n$, and (ii) each $X \in \Psi$ meets $\mathcal{G}_{1,n,2}$ in an odd number of points. Then $\mathcal{G}_{1,n,2}$ has polynomial degree d_n .*

Proof. See [9, P.3 of Theorem 3.4]. ■

By choosing Ψ in the preceding lemma to be the $\text{GL}(n+1, 2)$ -orbit of the d_n -flat $X_{\mathcal{P}_n}$, where \mathcal{P}_n denotes the path $\{12, 23, 34, \dots, n, n+1\}$ of length $|\mathcal{P}_n| = n$, we now prove that $\delta_n = d_n$ for all n .

Lemma 2.7 *The $\text{GL}(n+1, 2)$ -orbit of $\chi(X_{\mathcal{P}_n}^c)$ spans F_n .*

Proof. See [12, Lemma 2.3], where the proof proceeds by induction upon n , and makes use of a certain degree-lowering property of transvections. ■

Lemma 2.8 *The d_n -flat $X_{\mathcal{P}_n}$ meets $\mathcal{G}_{1,n,2}$ in an odd number of points.*

Proof. From Lemma 2.5 we have $|X_{\mathcal{P}_n} \cap \mathcal{G}_{1,n,2}| = p(\mathcal{P}_n) + q(\mathcal{P}_n)$. Setting $p_n := p(\mathcal{P}_n)$ and $q_n := q(\mathcal{P}_n)$ the following recurrence relations are easily derived (see [12, Section 2.3.1]):

$$p_n = 2p_{n-1} + p_{n-2} + 2^{n-1} - 1, \quad q_n = 2q_{n-1} + 2q_{n-2} + p_{n-2}. \quad (2.7)$$

Consequently $h_n := p_n + q_n$ satisfies $h_n = 2h_{n-1} + 2h_{n-2} + 2^{n-1} - 1$, from which it immediately follows that h_n is odd for all n . ■

Theorem 2.9 For all $n \geq 3$ the Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree $d_n = \binom{n}{2} - 1$.

Proof. We have just shown that if Ψ is the $\mathrm{GL}(n+1, 2)$ -orbit of the d_n -flat $X_{\mathcal{P}_n}$ then Ψ satisfies both (i) and (ii) of Lemma 2.6. ■

Remark 2.10 Upon communicating news of the above proof of conjecture 2.2 to Glynn, Maks & Casse, the author learned that these authors had also been able to prove the conjecture: see their revised document [5]. The two proofs have some features in common, namely use of the simplex basis (2.5) and, in effect, of the result in [9, P5 of Theorem 3.4]. Nevertheless the two proofs of the conjecture 2.2 are distinct. Indeed, the Glynn/Maks/Casse proof has the notable virtue of being much shorter than the one in [12]. The brevity of their proof arises from their appeal to a nice geometrical lemma, see [5, Lemma 15]. With the aid of this lemma they are able to deal with the intersections $X_{\mathcal{E}} \cap \mathcal{G}_{1,n,2}$ for edge sets \mathcal{E} of size $\geq d_n$ all in the same way, with no need for separate consideration of the different isomorphism classes of such edge sets. For more details concerning the history of the two proofs, see [12, Section 2.5].

2.4 Graph theory spin-off

For a simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ observe that two non-empty subsets α, β of \mathcal{V} , of sizes $|\alpha| = a$ and $|\beta| = b$, satisfy $\alpha \perp \beta$ if and only if $\bar{\Gamma} = (\mathcal{V}, \mathcal{E})$ contains as a subgraph (but not necessarily an induced subgraph) the complete bipartite graph $K_{a,b}$ with parts the a -set α and the b -set β . Similarly if $\gamma \subset \mathcal{V}$ is of size $|\gamma| = c \neq 0$, then $\alpha \perp \beta \perp \gamma$ holds if and only if $\bar{\Gamma}$ contains as a subgraph (but not necessarily an induced subgraph) the complete tripartite graph $K_{a,b,c}$ with parts the a -set α , the b -set β and the c -set γ . For any simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ let us define $p(\Gamma)$ to be the total number of subgraphs of $\bar{\Gamma}$ which are isomorphic to $K_{a,b}$ for some a, b satisfying $a \geq b > 0$. Similarly we define $q(\Gamma)$ to be the total number of subgraphs of $\bar{\Gamma}$ which are isomorphic to $K_{a,b,c}$ for some a, b, c satisfying $a \geq b \geq c > 0$.

From theorem 2.3 and eq. 1.5 the finite geometry result $\delta_n = d_n$ yields, via Lemma 2.5, the following theorem in graph theory.

Theorem 2.11 If $\Gamma = (\mathcal{V}, \mathcal{E})$ is any finite simple graph such that $|\mathcal{E}| < |\mathcal{V}|$ then $p(\Gamma) + q(\Gamma)$ is odd.

3 The polynomial degree of the Segre variety $\mathcal{S}_{m,n,2}$

In this section $V = V_{N+1}$ is the tensor product space $V_{m+1} \otimes V_{n+1}$ of two vector spaces V_{m+1} and V_{n+1} over $\mathrm{GF}(2)$; so $N = mn + m + n$. We deal

with the Segre variety $\mathcal{S}_{m,n,2} \subset S := \text{PG}^{(0)}(N, 2)$ which consists of all the decomposable elements $u \otimes v$, $u(\neq 0) \in V_{m+1}$, $v(\neq 0) \in V_{n+1}$, of V_{N+1} . Given a basis $\{e_1, \dots, e_{m+1}\}$ for V_{m+1} and a basis $\{f_1, \dots, f_{n+1}\}$ for V_{n+1} then for non-empty subsets α of $\mathcal{U} := \{1, 2, \dots, m+1\}$ and β of $\mathcal{V} := \{1, 2, \dots, n+1\}$ we set $e_\alpha := \sum_{i \in \alpha} e_i$ and $f_\beta := \sum_{j \in \beta} e_j$. So in this notation we have

$$\mathcal{S}_{m,n,2} = \{e_\alpha \otimes f_\beta \mid (\emptyset \neq) \alpha \subseteq \mathcal{U}, (\emptyset \neq) \beta \subseteq \mathcal{V}\}. \quad (3.1)$$

With respect to the product basis $\{e_i \otimes f_j\}_{1 \leq i \leq m+1, 1 \leq j \leq n+1}$ for V_{N+1} a point $x = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} x_{ij} e_i \otimes f_j \in S$ has coordinates $(x_{ij})_{1 \leq i \leq m+1, 1 \leq j \leq n+1}$, and the polynomial spaces $F_r = F_r(S)$ are accordingly defined.

Observe that the set $\mathcal{S}_{m,n,2}$ is the disjoint union of the $(2^{m+1} - 1)$ n -flats $Y_\alpha := \mathbb{P}(e_\alpha \otimes V_{n+1})$, $(\emptyset \neq) \alpha \subseteq \mathcal{U}$, and is also the disjoint union of the $(2^{n+1} - 1)$ m -flats $Z_\beta := \mathbb{P}(V_{m+1} \otimes f_\beta)$, $(\emptyset \neq) \beta \subseteq \mathcal{V}$. Without loss of generality we will assume that $m \leq n$; in which case we set $d_{m,n} := mn + m$ and aim to show that the polynomial degree of $\mathcal{S}_{m,n,2}$ is $d_{m,n}$.

Lemma 3.1 *There exists a $(d_{m,n} - 1)$ -flat X of $\text{PG}(N, 2)$ which is skew to the flat $Y_{\mathcal{U}}$ but which meets each of the $2^{m+1} - 2$ flats Y_α , $\alpha \subset \mathcal{U}$.*

Proof. We give an explicit example of such a $(d_{m,n} - 1)$ -flat X . Let X be that flat which is spanned by basis vectors of two kinds:

$$(a) \{e_i \otimes f_j\}_{1 \leq i \leq m, j \neq i} \quad (b) \{e_{m+1} \otimes f_j\}_{j \leq m}. \quad (3.2)$$

Since mn basis vectors are of the kind (a) and m are of the kind (b) it follows that X is a $(mn + m - 1)$ -flat as required. Alternatively described, X is that flat of $\text{PG}(N, 2)$ which satisfies the coordinate conditions

$$(a) x_{ii} = 0, 1 \leq i \leq m \quad (b) x_{(m+1)j} = 0, j \geq m+1. \quad (3.3)$$

(The total number of independent conditions is thus $m + (n+1 - m) = n+1$, so that X is a flat of projective dimension $N - n - 1 = d_{m,n} - 1$, as required.) Suppose α is strictly a subset of \mathcal{U} , and so $i \notin \alpha$ for some $i \in \mathcal{U}$; then, from (3.2), $e_\alpha \otimes f_i \in X$, and so X meets Y_α . On the other hand every point $(\sum_{i=1}^{m+1} e_i) \otimes v$ of $Y_{\mathcal{U}}$ violates the coordinate conditions (3.3). ■

Theorem 3.2 *For $m \leq n$ the polynomial degree $\delta_{m,n}$ of the Segre variety $\mathcal{S}_{m,n,2} \subset \text{PG}(mn + m + n, 2)$ is $d_{m,n} = mn + m$.*

Proof. Since $N - n = d_{m,n}$, every $d_{m,n}$ -flat X in $\text{PG}(N, 2)$ meets each of the n -flats Y_α in an odd number of points, and so meets $\mathcal{S}_{m,n,2}$ in an odd number of points. Hence, by theorem 1.1(i), $\delta_{m,n} \leq mn + m$. On the other hand in $\text{PG}(N, 2)$ there exists, lemma 3.1, a $(d_{m,n} - 1)$ -flat X which is skew to $Y_{\mathcal{U}}$ and which meets (in an odd number of points) the remaining

even number of n -flats Y_α , $\alpha \subset \mathcal{U}$; so X meets $\mathcal{S}_{m,n,2}$ in an even number of points. Hence, by theorem 1.1(ii), $\delta_{m,n} = d_{m,n}$. ■

Upon recalling the use of (simple) graphs in section 2.2 which were subgraphs of the complete graph Γ_{m+1} , let us in the present context consider instead subgraphs Γ of the complete bipartite graph $\Gamma_{m+1,n+1}$ whose parts are $\mathcal{U} = \{1, 2, \dots, m+1\}$ and $\mathcal{V}' = \{m+2, m+3, \dots, m+n+2\}$, of sizes $|\mathcal{U}| = m+1$ and $|\mathcal{V}'| = n+1$.

For such a bipartite graph $\Gamma = (\mathcal{U} \cup \mathcal{V}', \mathcal{E})$, each edge $E \in \mathcal{E}$ is of the form $E = ij' := \{i, j'\}$ for some $i \in \mathcal{U}$ and some $j' \in \mathcal{V}'$. Along with Γ we also consider its “bipartite complement” $\Gamma^* = (\mathcal{U} \cup \mathcal{V}', \mathcal{E}^*)$, where $\mathcal{E}^* = \{ij' \mid i \in \mathcal{U}, j' \in \mathcal{V}', ij' \notin \mathcal{E}\}$. For $j \in \mathcal{V}$ define $j' := j + m + 1$ and note that $j \longleftrightarrow j'$ establishes a bijective correspondence between the sets \mathcal{V} and \mathcal{V}' . To each edge $E = ij' \in \mathcal{E}$, let us associate:

- (i) the coordinate $x_E := x_{ij}$, and hence the hyperplane $x_E = 0$;
- (ii) the basis element $e_E := e_i \otimes f_j$ for V_{N+1} .

Further let $X_{\mathcal{E}}$ denote the flat having coordinate equations $x_E = 0$, each $E \in \mathcal{E}$, and let $Y(\mathcal{E})$ denote the flat $\langle \{e_E\}_{E \in \mathcal{E}} \rangle$. Take note that $X_{\mathcal{E}} = Y(\mathcal{E}^*)$, cf. eq. (2.3)(i).

Next let us look at the analogue for $\mathcal{S}_{m,n,2}$ of lemma 2.5. Things are simpler in the present context: since every point of $\mathcal{S}_{m,n,2}$ is of the form $e_\alpha \otimes f_\beta$ we will have no need of “triads”. Given a bipartite graph $\Gamma = (\mathcal{U} \cup \mathcal{V}', \mathcal{E})$ then, for (non-empty) subsets $\alpha \subseteq \mathcal{U}$, $\beta' \subseteq \mathcal{V}'$, we write $\alpha \perp \beta'$ whenever $ij' \notin \mathcal{E}$ (that is $ij' \in \mathcal{E}^*$) for all $i \in \alpha$ and for all $j' \in \beta'$. Such an ordered pair (α, β') will be termed a *dyad* for the graph $\Gamma = (\mathcal{U} \cup \mathcal{V}', \mathcal{E})$.

The following lemma now follows in a straightforward way.

Lemma 3.3 *The flat $X_{\mathcal{E}} = Y(\mathcal{E}^*)$ meets $\mathcal{S}_{m,n,2}$ in $p(\mathcal{E})$ points, where $p(\mathcal{E})$ is the number of dyads for the bipartite graph $\Gamma = (\mathcal{U} \cup \mathcal{V}', \mathcal{E})$.*

Note that (α, β') is a dyad for the graph $\Gamma = (\mathcal{U} \cup \mathcal{V}', \mathcal{E})$ if and only if the graph $\Gamma^* = (\mathcal{U} \cup \mathcal{V}', \mathcal{E}^*)$ contains as a subgraph the complete bipartite graph having parts α and β' . Bearing this in mind we may now translate our finite geometry results concerning the polynomial degree of the Segre variety $\mathcal{S}_{m,n,2}$ into the following bipartite graph theorem. (In this theorem a, b play the roles of the preceding $m+1, n+1$, and $(\mathcal{A} \cup \mathcal{B}, \mathcal{E})$ plays the role of $(\mathcal{U} \cup \mathcal{V}', \mathcal{E}^*)$.)

Theorem 3.4 *Let $\Gamma = (\mathcal{A} \cup \mathcal{B}, \mathcal{E})$ be any finite bipartite graph which is a subgraph of the complete bipartite graph $K_{a,b}$ whose parts \mathcal{A}, \mathcal{B} have sizes $|\mathcal{A}| = a$, $|\mathcal{B}| = b$ with $a \leq b$. Let $N(\Gamma)$ denote the total number of subgraphs of Γ which are isomorphic to $\Gamma_{a',b'}$ for some $a' \leq a$, $b' \leq b$. Then $N(\Gamma)$ is odd for all such graphs Γ of size $|\mathcal{E}| > ab - b$. Moreover there exists at least one such graph Γ of size $|\mathcal{E}| = ab - b$ for which $N(\Gamma)$ is even.*

References

- [1] E.F. Assmus and J.D. Key, *Designs and their Codes*, Cambridge University Press, Cambridge (1993).
- [2] W. Bosma, J. Cannon and C. Playoust, The MAGMA algebra system I: The user language, *J. Symbol. Comput.* **24** (1997), 235-265.
- [3] B.N. Cooperstein, External flats to varieties in $\text{PG}(\wedge^2 V)$ over finite fields. *Geom. Dedicata* **69** (1998), 223-235.
- [4] David G. Glynn, Johannes G. Maks, L.R.A. (Rey) Casse, The polynomial degree of the Grassmannian $\mathcal{G}(n, 1, q)$ of lines in finite projective space $\text{PG}(n, q)$, preprint (July, 2003).
- [5] Revised version of [4], (July, 2005). A further revision has now been published: *Des. Codes Cryptogr.* **40** (2006), 335-341.
- [6] N.A. Gordon, private communication, Spring 2005.
- [7] R. Shaw, A characterization of the primals in $\text{PG}(m, 2)$, *Des. Codes Cryptogr.* **2** (1992), 253-256.
- [8] R. Shaw, Finite geometries and Clifford algebras III, see *Proc. of the 2nd Workshop on Clifford Algebras and their Applications in Mathematical Physics*, Montpellier, France, (1989); eds. A. Micali et al., Kluwer Acad. Pubs. (1992), pp. 121-132.
- [9] R. Shaw and N.A. Gordon, The polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$, *Des. Codes Cryptogr.* **39** (2006), 289-306.
- [10] R. Shaw, Grassmann and Segre varieties over $\text{GF}(2)$: some graph theory links, preprint prior to BCC20, Durham (July 2005), accessible from: <http://www.hull.ac.uk/php/masrs/> .
- [11] R. Shaw, J.G. Maks and N.A. Gordon, The classification of flats in $\text{PG}(9, 2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$, *Des. Codes Cryptogr.* **34** (2005) 203-227.
- [12] R. Shaw, The Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree $\binom{n}{2} - 1$, (July 2005). (Accessible from: <http://www.hull.ac.uk/php/masrs/>).