The cubic Segre variety $\mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$

Ron Shaw and Neil A. Gordon
r.shaw@hull.ac.uk, n.a.gordon@hull.ac.uk

Abstract

The Segre variety $\mathcal{S}_{1,2,2}$ is a 21-set ψ of points in $\mathrm{PG}(5,2)$ which has a cubic equation $Q(x) = 0$. If $T(x,y,z)$ denotes the alternating trilinear form obtained by completely polarizing the cubic polynomial Q, then the *ψ-associate* $U^{\#}$ of an r-flat $U \subset \mathrm{PG}(5,2)$ is defined to be

$$U^{\#} = \{z \in \mathrm{PG}(5,2) \,|\, T(u_1,u_2,z) = 0 \text{ for all } u_1,u_2 \in U\},$$

and so is an s-flat for some s. Those lines L of $\mathrm{PG}(5,2)$ which are *singular,* satisfying that is $L^{\#} = \mathrm{PG}(5.2)$, are shown to form a complete spread of 21 lines. For each r-flat $U \subset \mathrm{PG}(5,2)$ its associate $U^{\#}$ is determined. Examples are given of four kinds of planes P which are self-associate, $P^{\#} = P$, and three kinds of planes for which $P, P^{\#}, P^{\#\#}$ are disjoint planes such that $P^{\#\#\#} = P$.

Keywords cubic hypersurfaces in $\mathrm{PG}(n,2)$; associate of a projective flat; Segre variety $\mathcal{S}_{1,2,2}$; alternating trilinear form

1 Introduction

1.1 The Segre variety $\mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$

We deal with vector spaces $V_{n+1} := V(n+1,2)$ over the field $\mathrm{GF}(2)$ and so we can identify the points $\langle x\rangle$ of the associated projective spaces $\mathrm{PG}(n,2) = \mathbb{P}(V_{n+1})$ with the nonzero vectors $x \in V_{n+1}$. We use $\prec u,v,\ldots\succ$ for the vector subspace spanned by vectors $u,v,\ldots$, and $\langle u,v,\ldots\rangle$ for the flat (projective subspace) generated by projective points $u, v,$ ... . Consider the Segre variety

$$\psi = \mathcal{S}_{1,2,2} = \{v \otimes w : v \in \mathbb{P}V_2,\ w \in \mathbb{P}V_3\}, \tag{1.1}$$

which lies in $\mathrm{PG}(5,2) = \mathbb{P}V_6$, and which consists of the 21 decomposable elements of the tensor product space $V_6 = V_2 \otimes V_3$. Lying on $\mathcal{S}_{1,2,2}$ is the partial 2-spread $\Sigma_3 = \{\alpha_v\}$ of the three projective planes $\alpha_v := \mathbb{P}(v \otimes V_3)$, $v \in \mathbb{P}V_2$, and the partial 1-spread $S_7 = \{\lambda_w\}$ of the seven projective lines $\lambda_w = \mathbb{P}(V_2 \otimes w)$, $w \in \mathbb{P}V_3$. The line λ_w meets the plane α_v in the single point $v \otimes w$, and the lines λ_w will be referred to as *transversals* of the three internal planes α_v. The stabilizer $\mathcal{G}_\psi < \mathrm{GL}(V_6)$ of ψ is

$$\mathcal{G}_\psi = \{A \otimes B \,|\, A \in \mathrm{GL}(V_2),\ B \in \mathrm{GL}(V_3)\}, \tag{1.2}$$

isomorphic to $\mathrm{GL}(2,2) \times \mathrm{GL}(3,2)$ and so is of order $6 \times 168 = 1008$.

Actually *we will have no need in the following to make explicit reference to the tensor product structure.* Instead we will view $\mathcal{S}_{1,2,2}$ simply as the 21-set

$$\psi = \mathcal{S}_{1,2,2} = X \cup Y \cup P_0 \tag{1.3}$$

where $\{X, Y, P_0\}$ is a partial spread Σ_3 of three planes of $\mathrm{PG}(5,2)$. The partial spreads Σ_3 of size 3 constitute a single $\mathrm{GL}(6,2)$-orbit, and each Σ_3 determines a partial line-spread S_7 consisting of the seven transversals of the planes of Σ_3.

Often it will prove convenient to view V_6 as a direct sum $V_3 \oplus V_3$, and so a general element $z \in V_6$ is

$$z = (x, y) \in V_6 = V_3 \oplus V_3, \quad x, y \in V_3. \tag{1.4}$$

We then take the planes X, Y, P_0 of Σ_3 to be

$$X = (x, 0), \quad Y = (0, x), \quad P_0 = (x, x), \quad 0 \neq x \in V_3. \tag{1.5}$$

Let I_1 be the $\mathrm{GL}(V_3)$-invariant function on V_3 such that $I_1(0) = 0$ and $I_1(x) = 1$ if $x \neq 0$, and note that in any coordinate system I_1 takes the cubic form

$$I_1(x) = x_1 + x_2 + x_3 + x_2x_3 + x_3x_1 + x_1x_2 + x_1x_2x_3. \tag{1.6}$$

The first polarization $I_2(x, y) := I_1(x+y)+I_1(x)+I_1(y)$ of I_1 then, *cf.* [5, §2.2], satisfies

$$I_2(x, y) = \begin{cases} 1, & \text{if } x, y \text{ are linearly independent,} \\ 0, & \text{if } x, y \text{ are linearly dependent.} \end{cases} \tag{1.7}$$

Lemma 1.1 *In the foregoing notation the Segre variety $\psi = \mathcal{S}_{1,2,2}$ has the cubic equation $Q(z) = 0$, where $Q(x, y) = I_2(x, y)$.*

Proof. From (1.7), if $z = (x, y) \neq (0, 0)$, then $I_2(x, y) = 0$ if and only if either (i) $x = 0$, and so $z \in Y$, or (ii) $y = 0$, and so $z \in X$, or (iii) $x = y$, and so $z \in P_0$. ■

Corollary 1.2 *Choosing a basis $\mathcal{B} = \{e_1, e_2, \dots, e_6\}$ such that $X = \langle e_1, e_2, e_3\rangle$, and $Y = \langle e_4, e_5, e_6\rangle$, and setting $\mathcal{T} := \{126, 135, 156, 234, 246, 345\}$, then Q has the coordinate form*

$$Q(x) = \Sigma_{ijk\in\mathcal{T}}\ x_i x_j x_k + \text{(terms of degree } \leq 2),$$
$$\text{where } \mathcal{T} := \{126, 135, 156, 234, 246, 345\}. \tag{1.8}$$

The polynomial degree d of the Segre variety $\mathcal{S}_{m,n,2}$ is in fact known for general m, n; see [9] for a proof that $d = mn + m,\ m \leq n$.

1.2 The associate $U^{\#}$ of a flat $U \subset \mathrm{PG}(5,2)$

Let $T(v_1, v_2, v_3)$, $v_i \in V_6$, denote the trilinear form obtained by completely polarizing the cubic polynomial Q :

$$T(v_1, v_2, v_3) = \textstyle\sum_{i=1}^{3} Q(v_i) + \sum_{1\leq i<j\leq 3} Q(v_i + v_j) + Q(v_1 + v_2 + v_3). \tag{1.9}$$

Note that T is an alternating (and hence symmetric) function of v_1, v_2, v_3. Moreover

$$T(v_1, v_2, v_3) = \begin{cases} \sum_{v\in P} Q(v), & \text{if } \langle v_1, v_2, v_3\rangle \text{ is a plane } P, \\ 0, & \text{if } v_1, v_2, v_3 \text{ are dependent.} \end{cases} \tag{1.10}$$

Of course the invariance $Q(G^{-1}x) = Q(x)$ of Q under elements $G \in \mathcal{G}_\psi$ entails a corresponding $\mathcal{G}_\psi$-invariance of T.

Definition 1.3 *The* associate $U^{\#}$ *of an r-flat* $U \subset \mathrm{PG}(5,2)$ *is*

$$U^{\#} = \{v \in \mathrm{PG}(5,2) \,|\, T(u_1, u_2, v) = 0 \textit{ for all } u_1, u_2 \in U\}. \tag{1.11}$$

Since T is trilinear, $U^{\#}$ is an s-flat for some s. Since $U^{\#}$ is uniquely determined by U it follows that $\mathcal{G}_U \le \mathcal{G}_{U^{\#}}$, where $\mathcal{G}_U$ denotes that subgroup of $\mathcal{G}_\psi$ which stabilizes the flat U. Further, if W is also a flat, then from $W \subset U$ it follows that $U^{\#} \subseteq W^{\#}$. Moreover, since T is $\mathcal{G}_\psi$-invariant, if $U' = G(U), G \in \mathcal{G}_\psi$, then $(U')^{\#} = G(U^{\#})$.

Remark 1.4 *Definition 1.3 is the special case* $(d, n) = (3, 5)$ *of the general definition, see [8], of the* ψ*-associate of a flat in* $\mathrm{PG}(n, 2)$ *with respect to a given subset* ψ *of polynomial degree d. (Note that if* $d = 2$ *then* $U^{\#}$ *is the flat, usually denoted* $U^{\perp}$*, orthogonal to U.)*

Note that the associate $\langle u\rangle^{\#}$ of a point $u \in \mathrm{PG}(5,2)$ is the whole of $\mathrm{PG}(5,2)$. Concerning the associate of a line L of $\mathrm{PG}(5,2)$ we make the following definition.

Definition 1.5 *The* associated linear form *of a line* $L = \langle a_1, a_2\rangle$ *of* $\mathrm{PG}(5,2)$ *is* f_L*, where*

$$f_L(x) = T(a_1, a_2, x), \quad x \in V_6. \tag{1.12}$$

This is a good definition: since T is alternating and trilinear, any choice of independent points $a_1, a_2 \in L$ yield the same linear form.

The associate $L^{\#}$ of a line $L \subset \mathrm{PG}(5,2)$ is thus the flat H_L :

$$H_L := \{x \in \mathrm{PG}(5,2) |\ f_L(x) = 0\}. \tag{1.13}$$

So $L^{\#}$ is a hyperplane whenever f_L is not the zero form, while $L^{\#} = \mathrm{PG}(5,2)$ if f_L is the zero form.

Definition 1.6 *If* f_L *is the zero form the line L will be termed* singular*. Also, for* $r \ge 1$*, an r-flat will be termed* singular *if it contains one or more singular lines, and* non-singular *if it contains no singular lines.*

In the following, a flat $U \subset \mathrm{PG}(5,2)$ is termed *odd* or *even* according as $|U \cap \psi|$ is odd or even.

Lemma 1.7 *(i) A plane* $P = \langle a, b, c\rangle$ *is odd or even according as* $T(a, b, c) = 0$ *or* $T(a, b, c) = 1$.

(ii) If a plane P is singular then P is odd.

Proof. (i) Recall from (1.10) that $T(a, b, c) = \sum_{v \in P} Q(v)$.

(ii) Since T is alternating and trilinear we need to check that $T(a, b, c) = 0$ for just one generating set $\{a, b, c\}$ for P. Choosing $L = \langle a, b\rangle$ to be a singular line in P, it follows from (1.12) that $T(a, b, c) = f_L(c) = 0$. ■

Theorem 1.8 *Every solid (3-flat), and also every hyperplane (4-flat), in* $\mathrm{PG}(5,2)$ *is odd, but some planes in* $\mathrm{PG}(5,2)$ *are even.*

Proof. This follows from $\deg Q = 3$: see [3]. ■

Plan. In [8, §1.4] a four-point plan was described for the study of the ψ-associate in the case of a general hypersurface $\psi \subset \mathrm{PG}(n,2)$. The aim of the present work is to carry out this plan in the particular case where ψ is the subset (1.3) of $\mathrm{PG}(5,2)$. The first two points of the general plan have already been achieved, see Lemma 1.1 and Equation (1.2). The third point of the general plan is dealt with in Section 2 where we determine, for each $r \geq 0$, the $\mathcal{G}_\psi$-orbits of r-flats of $\mathrm{PG}(5,2)$. The fourth point of the general plan is dealt with in Section 3 where, for a representative U of each $\mathcal{G}_\psi$-orbit of r-flats, we determine its associate $U^\#$ and list the orbit to which $U^\#$ belongs.

It is hoped that the study of the ψ-associate in the present case of $\psi = \mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$ will be of help in future investigations of the ψ-associate for various other choices of subset $\psi \subset \mathrm{PG}(5,2)$, and indeed of $\psi \subset \mathrm{PG}(n,2)$. Also, although the choice (1.3) of ψ has a very simple geometric structure, nevertheless the present investigation does appear to illuminate in an interesting way some of the geometry of ψ, and possibly even bring to light some new geometrical aspects: see in particular Corollary 3.5, and also Theorem 2.3 and Corollary 3.6.

1.3 General results for a cubic hypersurface ψ in $\mathrm{PG}(5,2)$

Before carrying out the foregoing plan for the particular choice (1.3) of ψ, we list in the next theorem some results which hold for any cubic hypersurface ψ in $\mathrm{PG}(5,2)$. Part (i) of the theorem, which is an easy consequence of (1.10), is particularly noteworthy, since it rephrases the definition of the associate $U^\#$ solely in terms of certain incidence properties. Because a coordinate form for T is usually quite complicated, use of Theorem 1.10(i), and of its offshoots, is very often the best way to determine $U^\#$.

Remark 1.9 *In the case where ψ is the 21-set (1.3) the coordinate form for T is readily obtained. For, with respect to the basis $\mathcal{B}$, it follows from the coordinate expression (1.8) that the completely polarized form $T = T_Q$ of Q is a sum of six 3×3 determinants:*

$$T(x,y,z) = \sum_{ijk \in \mathcal{T}} \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}. \tag{1.14}$$

Incidentally if $\{f_1, f_2, \ldots, f_6\}$ is the basis in V_6^* dual to the basis $\mathcal{B}$ for V_6 in Corollary 1.2 then, upon identifying T with that trivector $t \in \wedge^3 V_6^*$ such that $T(x,y,z) = < t|x \wedge y \wedge z >$, Eq. (1.14) can be expressed

$$t = f_{126} + f_{135} + f_{156} + f_{234} + f_{246} + f_{345}, \quad \textit{where } f_{ijk} := f_i \wedge f_j \wedge f_k. \tag{1.15}$$

Theorem 1.10 *The following hold for any cubic hypersurface ψ in $\mathrm{PG}(5,2)$.*
(i) If U is an r-flat in $\mathrm{PG}(5,2)$ then

(a) for $r \geq 1$ a point $y \in U^c$ is in the associate $U^\#$ of U if and only if for each line L of U the plane $\langle y, L\rangle$ is odd;

(b) for $r \geq 2$ a point $y \in U$ is in the associate $U^\#$ of U if and only if every plane P of U which contains y is odd.

(ii) A line L in $\mathrm{PG}(5,2)$ is singular if and only if every plane P which contains L is odd.

(iii) The associate of a plane $P = \langle a_1, a_2, a_3\rangle$ is $P^{\#} = H_{L_{12}} \cap H_{L_{13}} \cap H_{L_{23}}$, where $L_{ij} = \langle a_i, a_j\rangle$. Further, $P^{\#}$ is an s-flat for some $s \geq 2$, and a point $y \in P^c$ lies in $P^{\#}$ if and only if each of the three planes $\langle y, L_{ij}\rangle$ is odd.
(iv) If P is an even plane then P is non-singular and $P^{\#}$ is a disjoint plane.
(v) If P is an odd plane then $P^{\#} \supseteq P$; moreover $P^{\#}$ is odd.

Proof. See [8, Theorem 1.17], ■

Part (v) of the theorem can in fact be strengthened as follows.

Lemma 1.11 *([8, Theorem 1.18]) For any cubic $\psi \subset \mathrm{PG}(5,2)$ suppose that P is an odd plane. Then $P^{\#} \supseteq P$ and $P^{\#}$ is odd; further*
(i) if P is non-singular then P is self-associate: $P^{\#} = P$;
(ii) if P contains just one singular line then $P^{\#}$ is a solid;
(iii) if P contains just one pencil of singular lines then $P^{\#}$ is a hyperplane;
(iv) if every line $L \subset P$ is singular then $P^{\#} = \mathrm{PG}(5,2)$.

However it is the even planes P which are particularly enticing, for by Theorem 1.10(iv) $P^{\#}$ is then also a plane, which is moreover necessarily disjoint from P. If $P^{\#}$ also turns out to be even, then $P^{\#\#}$ $(= (P^{\#})^{\#})$ is a further plane, which is moreover disjoint from $P^{\#}$ (but not necessarily from P).

Definition 1.12 *An even plane P is termed* faithfully-even *if each of $P_1 = P^{\#},\ P_2 = P^{\#\#},\ \ldots,\ P_{r+1} = (P_r)^{\#}$ ... is also even.*

For any faithfully-even plane P we can define an ordered pair (r, s) of integers r, s, with $0 \leq r < s$, such that the members of the finite "#-sequence" $P_0(= P), P_1, \ldots P_s$ are distinct and such that $(P_s)^{\#} = P_r$. For example, for a #-sequence of type $(1, 2)$ the planes $P, P^{\#}, P^{\#\#}$ are distinct and $P^{\#\#\#}$ coincides with $P^{\#}$.

Lemma 1.13 *Let $P = \langle a_1, a_2, a_3\rangle$ be an even plane $\subset \mathrm{PG}(5,2)$, and suppose that $N = \langle n_1, n_2, n_3\rangle$ is a plane disjoint from P such that each of the nine planes $\langle a_i, a_j, n_k\rangle,\ 1 \leq i < j \leq 3,\ k = 1, 2, 3,$ is odd. Then $N = P^{\#}$.*

Proof. By Theorem 1.10(iii),(iv) these properties entail that $P^{\#} = N$. ■

2 $\mathcal{G}_\psi$-orbits of flats

2.1 Preliminaries

2.1.1 Planes external to $\psi = \mathcal{S}_{1,2,2}$

At times it helps to view the partial plane-spread $\Sigma_3 = \{X, Y, P_0\}$ as part of a complete plane-spread Σ_9. To this end, if the planes X, Y, P_0 are as in equation (1.5), choose $A \in \mathrm{GL}(V_3)$ to be of order 7 and satisfying $A^3 = I + A$. Since A is fixed-point-free on $\mathbf{P}V_3 = \mathrm{PG}(2,2)$ it gives rise, *cf.* [4, §4.1], to the following plane-spread Σ_9 for $\mathbf{P}(V_3 \oplus V_3) = \mathrm{PG}(5,2)$:

$$\Sigma_9 = \{X, Y, P_0, P_1,\ \ldots\ , P_6\} \tag{2.1}$$

upon defining the further six planes $P_1, P_2,\ \ldots\ , P_6$ of $\mathrm{PG}(5,2)$ by:

$$P_r = \{(x, A^r x),\ x \in \mathbb{P}V_3\}, \quad r = 1, 2\ \ldots\ , 6. \tag{2.2}$$

Worthy of note in this context are the elements $J, R \in \mathrm{GL}(V_3 \oplus V_3)$ defined by

$$J(x,y) = (y,x), \qquad R(x,y) = (y, x+y), \qquad x,y \in V_3, \tag{2.3}$$

and satisfying $J^2 = I$ and $R^3 = I$.

Lemma 2.1 *Both J and R are elements of $\mathcal{G}_\psi$, and also of $\mathcal{G}(\Sigma_9)$.*

Proof. This follows since J and R effect the following permutations

$$J : (XY)(P_0)(P_1P_6)(P_2P_5)(P_4P_3), \qquad R : (XYP_0)(P_1P_2P_4)(P_3P_5P_6) \tag{2.4}$$

of the members of Σ_9. For example, R maps the element $(x, Ax) \in P_1$ to $(Ax, (I+A)x) = (Ax, A^3x)$, which last is an element of P_2. ■

Lemma 2.2 *There are precisely 48 planes in* PG(5, 2) *which are external to a Segre variety ψ. They form a single $\mathcal{G}_\psi$-orbit.*

Proof. Any plane external to ψ in (1.3) must be of the form $\{(x, Bx),\ x \in \mathbb{P}V_3\}$ for some $B \in \mathrm{GL}(V_3)$ which is fixed-point-free on PG(3, 2). So B has to be of order 7. There are precisely 48 such elements in $\mathrm{GL}(V_3)$, falling into two conjugacy classes 7A and 7B, each of size 24, associated with the two minimal polynomials t^3+t+1 and t^3+t^2+1. If A is in class 7A then so are A^2 and A^4, while $A^6 = A^{-1}, A^5 = A^{-2}$ and $A^3 = A^{-4}$ are in class 7B. But, see (2.4), the planes P_r in (2.2) nevertheless belong to the same $\mathcal{G}_\psi$-orbit. ■

At times it will help to deal in more detail with the spread (2.1). Let us choose a basis $\{a_1, a_2, a_3\}$ for V_3 and define $A \in \mathrm{GL}(V_3)$ by $Aa_1 = a_2,\ Aa_2 = a_3,\ Aa_3 = a_1 + a_2$. Then A is of order 7 and satisfies $A^3 = I + A$. Upon setting $a_{r+1} = A^r a_1,\ r = 0, 1, \ldots, 6$, the 49 points underlying the seven planes $P_0, P_1, \ldots, P_6$ are the points $(a_i, a_j),\ 1 \le i, j \le 7$. Observe that

$$P_r = \{(a_i, a_{i+r}),\ i = 1, 2, \ldots, 7\}, \qquad r = 0, 1, 2, 3, 4, 5, 6, \tag{2.5}$$

$$X = \{(a_i, 0),\ i = 1, 2, \ldots, 7\},\ Y = \{(0, a_i),\ i = 1, 2, \ldots, 7\}, \tag{2.6}$$

where in (2.5) the value of the index $i + r$ is taken mod 7.

Shorthand notation. It helps at times to adopt the shorthand notation ij for $(a_i, a_j),\ i0$ for $(a_i, 0)$ and $0j$ for $(0, a_j)$.

2.1.2 Notation and terminology

We denote by $\Omega_N^{(r)}(n_1n_2n_3)$ a $\mathcal{G}_\psi$-orbit of r-flats in PG(5, 2) of length $|\,\Omega_N^{(r)}| = N$ which consists of r-flats U having *intersection pattern* $n_1n_2n_3 := \{n_1, n_2, n_3\}$: that is U meets one plane of $\Sigma_3 = \{X, Y, P_0\}$ in n_1 points, another in n_2 points and the third in n_3 points. Of course each $n_i \in \{0, 1, 3, 7\}$. Thus the orbit $\Omega_3^{(2)}(700)$ consists of the 3 internal planes X, Y, P_0, the orbit $\Omega_7^{(1)}(111)$ consists of the 7 transversals of the internal planes and the orbit $\Omega_{21}^{(1)}(300)$ consists of the 3×7 lines internal to one of the planes X, Y, P_0. Also, by Lemma 2.2 the planes external to ψ form a single orbit $\Omega_{48}^{(2)}(000)$.

If $L \in \Omega_{21}^{(1)}(300)$ then L determines a 9-set $\mathcal{H} \subset \psi$, the union of the three transversals of X, Y, P_0 which pass through the three points of L, and so also determines a solid $D_L = \langle \mathcal{H} \rangle$. The seven solids arising in this way will be referred to as *special solids;* they form a $\mathcal{G}_\psi$-orbit $\Omega_7^{(3)}(333)$. Each special solid is of the form $D = \mathcal{H}_D \cup M_D \cup M'_D$, where the 9-set $\mathcal{H}_D := D \cap \psi$ is a hyperbolic quadric in the 3-flat D and where M_D, M'_D are two lines which are external to $\mathcal{H}_D$ and to ψ. Such lines M_D, M'_D will be termed *special external lines.* These $7 \times 2 = 14$ special external lines are distinct since two special solids meet in a line $L \in \Omega_7^{(1)}(111)$. So we have a $\mathcal{G}_\psi$-orbit $\Omega_{14}^{(1)}(000)$. For $M \in \Omega_{14}^{(1)}(000)$ we denote by D_M that special solid which contains M. Tangents to ψ which lie in a special solid will be termed *special tangents.* Each special solid $D = \mathcal{H}_D \cup M_D \cup M'_D$ determines 9 special tangents, namely the 9 lines $\{\langle a, a' \rangle | a \in M_D,\ a' \in M'_D\}$, and the $7 \times 9 = 63$ special tangents thereby determined form an orbit $\Omega_{63}^{(1)}(100)$. Further, each special solid D contains 15 *special planes*, these being of two types (I) and (II): the 9 special planes in D of type (I) meet $\mathcal{H}_D$ in two intersecting lines, while the 6 special planes in D of type (II) meet $\mathcal{H}_D$ in a conic (3-arc). The $7 \times 9 = 63$ special planes of type I form an orbit $\Omega_{63}^{(2)}(311)$, and the $7 \times 6 = 42$ special planes of type II form an orbit $\Omega_{42}^{(2)}(111)$, which needs to be distinguished from another orbit $\Omega_{42^*}^{(2)}(111)$ of planes which, see section 2.4(c), meet ψ in a line. A flat which is not special will be termed *general.* Each bisecant of ψ lies in one and only one of the special solids — for example, in shorthand notation, see after (2.6), the bisecant $\langle 10, 02 \rangle$ lies in the special solid determined by the line $\{10, 20, 40\}$. (Thus every bisecant is "special".) Since there are 18 bisecants in each special solid, the bisecants form an orbit $\Omega_{126}^{(1)}(110)$ of length $7 \times 18 = 126$.

2.2 $\mathcal{G}_\psi$-orbits of points and lines

There are of course two $\mathcal{G}_\psi$-orbits of points:

$$\Omega_{21}^{(0)}(100) = \{\text{internal points}\} = \psi, \qquad \Omega_{42}^{(0)}(000) = \{\text{external points}\} = \psi^c.$$

Concerning the lines of $PG(5,2)$, recall the five $\mathcal{G}_\psi$-orbits encountered in section 2.1.2. Left to consider are (i) general tangents and (ii) general external lines.
(i) *General tangents.* Consider the lines which pass through a point $p \in \psi$. Since p belongs to 3 special solids, through p pass 3 special tangents and $3 \times 4 = 12$ bisecants, as well as one transversal and 3 other internal lines. The remaining $31 - 3 - 12 - 1 - 3 = 12$ lines through p are therefore general tangents. So there are $|\psi| \times 12 = 252$ general tangents, and these are seen to form a single $\mathcal{G}_\psi$-orbit $\Omega_{252}^{(1)}(100)$.
(ii) *General external lines.* Each of the six planes $P_1, P_2, \ldots, P_6$ contains seven lines, thus accounting for 42 general external lines. Also each of the $\binom{6}{3} = 20$ triples of the planes $P_1, P_2, \ldots, P_6$ produces 7 transversals. But 14 of these $20 \times 7 = 140$ transversals are the special external lines $\in \Omega_{14}^{(1)}(000)$. Hence altogether there are $42 + 140 - 14 = 168$ general external lines, and these form a single $\mathcal{G}_\psi$-orbit $\Omega_{168}^{(1)}(000)$.

Consequently the 651 lines of $PG(5,2)$ fall into the seven $\mathcal{G}_\psi$-orbits listed in the following table. (*Check:* $7 + 21 + 126 + 252 + 63 + 168 + 14 = 651.$✓) The second column records the order $|\mathcal{G}_L| = |\mathcal{G}_\psi|/|\Omega|$ of that subgroup $\mathcal{G}_L$ of $\mathcal{G}_\psi$

which stabilizes L. Concerning the third column, see Theorem 2.3(i) below.

Orbit	$\lvert\mathcal{G}_L\rvert$	Singular?	Notes
$\Omega_7^{(1)}(111)$	144	Yes	transversals of the three internal planes
$\Omega_{21}^{(1)}(300)$	48	No	lines of an internal plane
$\Omega_{126}^{(1)}(110)$	8	No	bisecants
$\Omega_{252}^{(1)}(100)$	4	No	general tangents
$\Omega_{63}^{(1)}(100)$	16	No	special tangents
$\Omega_{168}^{(1)}(000)$	6	No	general external lines
$\Omega_{14}^{(1)}(000)$	72	Yes	special external lines

(2.7)

2.3 Singular flats

2.3.1 Singular lines

Theorem 2.3 *(i) The set $\mathcal{L}$ of ψ-singular lines in* PG(5, 2) *consists of two $\mathcal{G}_\psi$-orbits: $\mathcal{L} = \Omega_7^{(1)}(111) \cup \Omega_{14}^{(1)}(000)$. Consequently the* 21 *singular lines $\mathcal{L}$ constitute a line-spread $\mathcal{L}$ for* PG(5, 2).

(ii) A plane P of PG(5, 2) *contains at most one singular line.*

(iii) A solid D of PG(5, 2) *which contains more than one singular line contains precisely five singular lines, such solids forming two $\mathcal{G}_\psi$-orbits, namely the orbit $\Omega_7^{(3)}(333)$ consisting of the special solids, and a second orbit $\Omega_{14}^{(3)}(111)$ consisting of solids of the form $D = \langle L, M\rangle$ where $L \in \Omega_7^{(1)}(111)$, $M \in \Omega_{14}^{(1)}(000)$ and $L \notin D_M$. A solid $D \in \Omega_7^{(3)}(333)$ contains three lines $\in \Omega_7^{(1)}(111)$ and two lines $\in \Omega_{14}^{(1)}(000)$, while a solid $D \in \Omega_{14}^{(3)}(111)$ contains one line $\in \Omega_7^{(1)}(111)$ and four lines $\in \Omega_{14}^{(1)}(000)$.*

Proof. (i) If $L \in \Omega_7^{(1)}(111)$ then the planes P which contain L are of two kinds: either P intersects ψ in the line L, or P intersects ψ in two intersecting lines (P being therefore a special plane of type (I)). In either case P is odd, and so, by Theorem 1.10(ii), L is singular. Next consider a line $M \in \Omega_{14}^{(1)}(000)$, and let $D_M = \mathcal{H}_M \cup M \cup M'$ be the special solid which contains M. The 3 planes $P \subset D_M$ which contain M are special planes of type (II), meeting ψ in a 3-arc. The remaining 12 planes P which contain M meet ψ in a single point, thus accounting for the $12(= 21-9)$ points of $\psi \setminus \mathcal{H}_M$. So, again by Theorem 1.10(ii), M is singular. On the other hand if a line L belongs to an orbit other than $\Omega_7^{(1)}(111)$ or $\Omega_{14}^{(1)}(000)$, see table (2.7), then it is easy to find an even plane which contains L. Finally note that the 21 lines $L \in \Omega_7^{(1)}(111) \cup \Omega_{14}^{(1)}(000)$ are pairwise disjoint, and so together cover the 63 points of PG(5, 2): $21 \times 3 = 63$.

(ii) As just noted, two singular lines are disjoint.

(iii) Consider a solid $D = \langle L, M\rangle$ which contains two singular lines L, M.

Case 1. First suppose that $L \in \Omega_7^{(1)}(111)$ and $M \in \Omega_7^{(1)}(111)$. Then D is a special solid and so contains, as a spread, three lines $\in \Omega_7^{(1)}(111)$ (a set of generators of $\mathcal{H}_D := D \cap \psi$) and two lines $\in \Omega_{14}^{(1)}(000)$.

Case 2. Secondly consider a solid $D = \langle L, M\rangle$ where $L \in \Omega_7^{(1)}(111)$, $M \in \Omega_{14}^{(1)}(000)$ and where, to avoid a repetition of Case 1, $L \notin D_M$. Without loss of

generality, it suffices to consider the case where, *in shorthand notation,* see after (2.6), $L = \{10, 01, 11\}$, $M = \{23, 35, 52\}$. Then we see that $D = L \cup M \cup M_1 \cup M_2 \cup M_3$ where $M_1 = \{43, 36, 64\}$, $M_2 = \{54, 47, 75\}$ and $M_3 = \{76, 62, 27\}$. So D contains, as a spread, one line $\in \Omega_7^{(1)}(111)$ and four lines $\in \Omega_{14}^{(1)}(000)$. For the given L there are 4 choices of special solid D such that $L \notin D$, and each such D contains 2 lines $M \in \Omega_{14}^{(1)}(000)$. The solid $D = L \cup M \cup M_1 \cup M_2 \cup M_3$ already considered accounts for 4 of these $4 \times 2 = 8$ lines M, and the only other Case 2 solid which contains the given L is $D' = L \cup M' \cup M_1' \cup M_2' \cup M_3'$, where $M' = \{32, 53, 25\}, .M_1' = \{34, 63, 46\}$, $M_2' = \{45, 74, 57\}$ and $M_3' = \{67, 26, 72\}$, and thus accounts for the remaining 4 lines M. Consequently Case 2 solids form a $\mathcal{G}_\psi$-orbit $\Omega_{14}^{(3)}$ of length $7 \times 2 = 14$.
Case 3. A straightforward check confirms that all solids D which contain two lines $L, M \in \Omega_{14}^{(1)}(000)$ have already been found under Cases 1 and 2. ■

2.3.2 Singular planes and super-singular solids

The line-spread $\mathcal{L}$ in the theorem is in fact a Desarguesian one, its 21 lines being derivable from the 21 points of a Desarguesian plane PG(2, 4). Since the 21 points of PG(2, 4), being 1-dimensional subspaces of $V(3, 4)$ and hence 2-dimensional subspaces of $V(6, 2)$, they give rise to a spread S of 21 lines of PG(5, 2). Further the 21 lines of PG(2, 4) give rise to a set of 21 solids of PG(5, 2), and since each line has five points, the spread S is a *normal spread* (see [1]) in that it has the following property: given any two members $L, L' \in S$ then the solid $D = \langle L, L' \rangle$ contains five members of S, these thus forming a spread for D. The line-spread $\mathcal{L}$ in the theorem is of this kind, each of the 21 solids $D \in \Omega_7^{(3)}(333) \cup \Omega_{14}^{(3)}(111)$ containing, see part (iii) of the theorem, five members of $\mathcal{L}$. To tie in S with the spread $\mathcal{L}$ in the theorem arising from ψ in (1.3), consider a Baer subplane $\mathcal{B} = \{Q_1, Q_2, \ldots, Q_7\}$ of PG(2, 4). Setting $\mathrm{GF}(4) = \{0, 1, \rho, \rho^2)$ each point $Q \in \mathcal{B}$ gives rise to three nonzero vectors $a, \rho a, \rho^2 a \in V(3, 4)$ and hence to a line $L_Q = \{a, Ra, R^2 a\}$ in PG(5, 2), where R is an element of GL(6, 2) of order 3 whose orbits in PG(5, 2) are the lines of the spread S. Points $a_i \in L_{Q_i}$ can be chosen so that $X := \{a_1, a_2, \ldots, a_7\}$ is a plane in PG(5, 2), and thereby we arrive at three disjoint planes X, $Y := R(X)$ and $P_0 = R^2(X)$. Setting $\psi = X \cup Y \cup P_0$, the seven lines $\{L_{Q_i}\}$ form the $\mathcal{G}_\psi$-orbit $\Omega_7^{(1)}(111)$.

In addition to the set $\mathcal{L} = \Omega_7^{(1)}(111) \cup \Omega_{14}^{(1)}(000)$ of singular lines in PG(5, 20 let us now consider the set $\mathcal{P}$ of singular planes and the set

$$\mathcal{D} = \Omega_7^{(3)}(333) \cup \Omega_{14}^{(3)}(111) \tag{2.8}$$

consisting of those solids which contain (as a spread) five lines $L \in \mathcal{L}$. Now through each $L \in \mathcal{L}$ pass 15 planes, and these $21 \times 15 = 315$ planes $P \in \mathcal{P}$ are distinct since, see Theorem 2.3(ii), a singular plane contains just one singular line. So $|\mathcal{P}| = 315$; we will see in the next section that $\mathcal{P}$ in fact consists of four $\mathcal{G}_\psi$-orbits.

Lemma 2.4 *If P is any ψ-singular plane in* PG(5, 2) *there exists a unique line $L_P \in \mathcal{L}$ and a unique solid $D_P \in \mathcal{D}$ such that $L_P \subset P \subset D_P$.*

Proof. The uniqueness of L_P has already been noted in Theorem 2.3(ii). If a solid $D \supset P$ is such that $D \in \mathcal{D}$ then it is unique, since the four singular lines of D other than L_P must be precisely those four lines $L' \in \mathcal{L}$ which meet P in a point $\in P \setminus L_P$. But since $\mathcal{L}$ is a normal spread such a solid $D_P = \langle L, L' \rangle \in \mathcal{D}$ exists. ■

Lemma 2.5 *If $D \in \mathcal{D}$ then every plane $P \subset D$ is singular. Consequently*

$$T(a,b,c) = 0 \qquad \textit{for all } a,b,c \in D. \tag{2.9}$$

Proof. The 15 planes in D comprise three planes through each of the five singular lines in D. Hence (2.9) holds on account of Lemma (1.7). ■

In view of this lemma we will refer to the 21 solids $D \in \mathcal{D}$ as *super-singular* solids. Also, on occasion, we will refer to the super-singular solid D_P in Lemma 2.4 as the *cloak* of the singular plane P.

Lemma 2.6 *If L is non-singular line then there exists a unique super-singular solid $D_L \in \mathcal{D}$ such that $L \subset D_L$.*

Proof. By the lead-in to this section, a non-singular line $L = \langle a, b \rangle$ lies inside a unique super-singular solid, namely the solid $D_L = \langle L_a, L_b \rangle \in \mathcal{D}$, where L_a, L_b denote the singular lines passing through the points a, b. (*Aliter:* each solid $D \in \mathcal{D}$ contains $35 - 5 = 30$ non-singular line, and these $21 \times 30 = 630$ lines are distinct since the non-singular lines in $\mathrm{PG}(5,2)$ number $651 - 21 = 630$.) ■

2.4 The eleven $\mathcal{G}_\psi$-orbits of planes

In Section 2.1.2 we encountered three $\mathcal{G}_\psi$-orbits of planes. In total there are in fact eleven orbits, as in the following table. In this table, and also in the table in section 2.5 for orbits of solids, an entry $r + s$ under the heading 'Singular lines' indicates the presence of r lines $\in \Omega_7^{(1)}(111)$ and s lines $\in \Omega_{14}^{(1)}(000)$.

Orbit	$\lvert\mathcal{G}_P\rvert$	Singular lines	Notes
$\Omega_3^{(2)}(700)$	336	0	internal planes
$\Omega_{63}^{(2)}(311)$	16	$1+0$	special planes of type (I): 9 in each special solid
$\Omega_{168}^{(2)}(310)$	6	0	see note (a)
$\Omega_{63}^{(2)}(300)$	16	0	see note (b)
$\Omega_{42}^{(2)}(111)$	24	$0+1$	special planes of type (II): 6 in each special solid
$\Omega_{42^*}^{(2)}(111)$	24	$1+0$	see note (c): $P \cap \psi$ is a line $\in \Omega_7^{(1)}(111)$
$\Omega_{168}^{(2)}(111)$	6	0	see note (d)
$\Omega_{504}^{(2)}(110)$	2	0	see note (e)
$\Omega_{168}^{(2)}(100)$	6	$0+1$	see note (f): contains one special external line
$\Omega_{126}^{(2)}(100)$	8	0	see note (g): contains one special tangent
$\Omega_{48}^{(2)}(000)$	21	0	external planes: see Lemma 2.2(i).

(2.10)

Notes.

(a) These planes are of the kind $P = \langle L, p \rangle$ where $L \in \Omega_{21}^{(1)}(300)$ and p is a point of one of the two internal planes other than that which contains L and such that $p \notin D_L$. Since there are 21 choices for L and $4 + 4 = 8$ choices for p, these planes form a $\mathcal{G}_\psi$-orbit $\Omega_{168}^{(2)}(310)$ of length $21 \times 8 = 168$.

(b) In addition to an internal plane, through a given $L \in \Omega^{(1)}_{21}(300)$ there pass three planes $\in \Omega^{(2)}_{63}(311)$ and, see note (a), eight planes $\in \Omega^{(2)}_{168}(310)$. The remaining 3 planes through L are of pattern 300, and the $21 \times 3 = 63$ planes of this kind form an orbit $\Omega^{(2)}_{63}(300)$.

(c) Through a given $L \in \Omega^{(1)}_{7}(111)$ there pass $63 \div 7 = 9$ planes $\in \Omega^{(2)}_{63}(311)$. The remaining six planes through L are thus of pattern 111, and the $7 \times 6 = 42$ planes of this kind form an orbit $\Omega^{(2)}_{42^*}(111)$.

(d) Other than planes of the orbits $\Omega^{(2)}_{42}(111)$ and $\Omega^{(2)}_{42^*}(111)$ there is a third kind of plane of pattern 111, namely planes P such that $P \cap \psi$ is a 3-arc *whose points do not lie in a common special solid.* These three points can be chosen in $7 \times 6 \times 4 = 168$ ways, and so the planes P generated by such a 3-arc form a $\mathcal{G}_\psi$-orbit of length 168.

(e) To construct a plane of this kind choose a bisecant $L = \langle a, b\rangle$ joining points $a, b \in \psi$, and let D denote the special solid which contains L. Fix upon one of the points a, b as preferred *"pivot"*, and let U be a special tangent through the pivot. If we restrict the choice of U so that it does not lie in D, then the plane $P = \langle L, U\rangle$ meets ψ in $\{a, b\}$, and so has the intersection pattern 110. There are 126 choices of $L \in \Omega^{(1)}_{126}(110)$, 2 choices of pivot point and 2 choices of special solid containing the pivot other than D. Consequently planes P of this kind form a $\mathcal{G}_\psi$-orbit of length $126 \times 2 \times 2 = 504$. (See also Remark **??** below.)

(f) There are 14 special lines L and 12 points $p \in \psi \setminus \mathcal{H}_L$, yielding an orbit of $14 \times 12 = 168$ planes of this kind.

(g) In shorthand notation consider a plane $P = \langle ij, M\rangle$ which contains the special tangent $M = \{10, 21, 41\} \in \Omega^{(1)}_{63}(100)$. Observe that P is of pattern 100 if and only if $i \in \{3, 5, 6, 7\}$ and $j \in \{2, 4\}$. So through each special tangent M pass two planes of pattern 100, each containing four of the eight points ij. Such planes P form an orbit of length $63 \times 2 = 126$.

Remark 2.7 *In conformity with Theorem 1.10(iv), observe that the even planes contain no singular lines.*

Remark 2.8 *Observe that the singular planes $P \in \mathcal{P}$ are those belonging to one of the four orbits $\Omega^{(2)}_{63}(311)$, $\Omega^{(2)}_{42}(111)$, $\Omega^{(2)}_{42^*}(111)$, $\Omega^{(2)}_{168}(100)$. So $|\mathcal{P}| = 315$, as was more simply obtained in Section 2.3.2.*

2.5 $\mathcal{G}_\psi$-orbits of hyperplanes and solids

If a subspace $W \subset V_6$ has dimension $r + 1$ then its annihilator, defined by $W^{\mathrm{O}} := \{f \in V_6^* | f(w) = 0, \text{ for all } w \in W\}$, is a subspace of the dual space V_6^* of dimension $5 - r$. *We use the same notation also for projective subspaces.* Thus if U is an r-flat in $\mathrm{PG}(5, 2)$ then U^{O} is a $(4-r)$-flat in the dual projective space $\mathrm{PG}(5, 2)^* = \mathbb{P}V_6^*$. In particular if $P \subset \mathrm{PG}(5, 2)$ is a plane then $P^{\mathrm{O}} \subset \mathrm{PG}(5, 2)^*$ is also a plane. *Moreover the partial spread $\Sigma_3 = \{X, Y, P_0\}$ of planes in* $\mathrm{PG}(5, 2)$ *gives rise to a partial spread $\Sigma_3^* = \{X^{\mathrm{O}}, Y^{\mathrm{O}}, P_0^{\mathrm{O}}\}$ of planes in the dual space* $\mathrm{PG}(5, 2)^*$. In consequence, elementary duality considerations enable us to determine immediately the orbits of hyperplanes and of solids from those of points and of lines, respectively.

Concerning hyperplanes, a point p lies on, or off, a plane P according as the hyperplane $\langle p\rangle^{\mathrm{O}}$ contains the plane P^{O}, or meets P^{O} in a line. So knowing that points of PG(5, 2) fall into two orbits $\Omega_{21}^{(0)}(100)$ and $\Omega_{42}^{(0)}(000)$ it follows that the hyperplanes of PG(5, 2) fall into the two orbits $\Omega_{21}^{(4)}(733)$ and $\Omega_{42}^{(4)}(333)$. In more detail, if $H \in \Omega_{21}^{(4)}(733)$ then $H = \langle D, P\rangle$, for some special solid D and $P \in \Sigma_3$, and $H \cap \psi = \mathcal{H}_D \cup P$. And if $H \in \Omega_{42}^{(4)}(333)$ then $H \cap \psi$ is a *non-regulus* (see [6, §2.2.1], [2, §A.2(i)]) whose transversal is a line $\in \Omega_7^{(1)}(111)$.

Concerning the orbits of solids, note that in PG(5, 2) a line L meets a plane P in 3, 1 or 0 point(s) according as the solid L^{O} meets the plane P^{O} in 7, 3 or 1 point(s), respectively. So the existence of a $\mathcal{G}_\psi$-orbit $\Omega_7^{(1)}(111)$ of lines (the seven transversals) implies the existence of a $\mathcal{G}_\psi$-orbit $\Omega_7^{(3)}(333)$ of solids (the seven special solids). Similarly the existence of an orbit $\Omega_{126}^{(1)}(110)$ of lines implies the existence of an orbit $\Omega_{126}^{(3)}(331)$ of solids. In this manner we obtain from the seven orbits of lines in table (2.7) the following seven orbits of solids.

Orbit	$\lvert\mathcal{G}_D\rvert$	Singular lines	Notes
$\Omega_7^{(3)}(333)$	144	$3+2$	special solids, see section 2.1.2
$\Omega_{21}^{(3)}(711)$	48	$1+0$	$D = \langle P, L\rangle,\ P \in \Omega_3^{(2)},\ L \in \Omega_7^{(1)}$
$\Omega_{126}^{(3)}(331)$	8	$1+0$	see note (a)
$\Omega_{252}^{(3)}(311)$	4	$0+1$	see note (b)
$\Omega_{63}^{(3)}(311)$	16	$1+0$	see note (c)
$\Omega_{168}^{(3)}(111)$	6	$0+1$	see note (d)
$\Omega_{14}^{(3)}(111)$	72	$1+4$	see Theorem 2.3(iii).

(2.11)

Notes.

(a) A solid $D \in \Omega_{126}^{(3)}(331)$ is of the form $\langle L, M\rangle$ where L, M are lines which lie in two different internal planes and where $D_L \neq D_M$. Having chosen one of the 21 lines $\in \Omega_{21}^{(1)}(300)$ for L there are 12 choices for M, and so the orbit has length $\frac{1}{2}(21 \times 12) = 126$. It follows from Theorem 2.3 that D has no singular line other than that line $\in \Omega_7^{(1)}(111)$ which is a transversal of L, M and the third internal plane.

(b) A solid $D \in \Omega_{252}^{(3)}(311)$ is of the form $\langle L, M\rangle$ where $L \in \Omega_{21}^{(1)}(300)$, $M \in \Omega_{14}^{(1)}(000)$ and $D_L \neq D_M$. Since each of the 6 special solids $D \neq D_L$ contains 2 lines $M \in \Omega_{14}^{(1)}(000)$, the orbit has length $21 \times 6 \times 2 = 252$. It follows from Theorem 2.3 that D has no singular line other than $M \in \Omega_{14}^{(1)}(000)$.

(c) Of the seven extensions of a plane $P \in \Omega_{63}^{(2)}(311)$ to a solid only one solid D has the same intersection pattern 311 as P. So such D form an orbit of length 63. Further the only singular line in such a solid D is that which is in P.

(d) Given a plane $P \in \Omega_{168}^{(2)}(111)$ then the 3-arc $\{a, b, c\} := P \cap \psi$ is stabilized by $\mathcal{G}_P$ and so $n := a + b + c \in P$ is a fixed point of $\mathcal{G}_P$. By Theorem 2.3(i) n lies on a unique singular line, L_n say. Hence L_n is stabilized by $\mathcal{G}_P$, as is the solid $D = \langle P, L_n\rangle$. A simple check confirms that D has the same intersection pattern 111 as P, and in fact is the only extension of P to a solid of pattern 111. Consequently solids D of the kind $D = \langle P, L_n\rangle$ form an orbit $\Omega_{168}^{(3)}(111)$ and satisfy $\mathcal{G}_D = \mathcal{G}_P$. By Theorem 2.3(iii) the line L_n is the only singular line contained in $D = \langle P, L_n\rangle$.

Remark 2.9 *Observe that every solid D in* $\mathrm{PG}(5,2)$ *contain at least one singular line, and so, in accordance with Theorem 2.3(iii), contains either one or five singular lines. Note also that for any solid D we have $|D \cap \psi| \in \{3,5,7,9\}$; of course the fact that every solid is odd is in accordance with Theorem 1.8.*

3 Associates of flats

Throughout this section bear in mind the fact, see after Eq. (1.11), that if flat U' is on the same $\mathcal{G}_\psi$-orbit as flat U then $(U')^\#$ is on the same orbit as $U^\#$.

3.1 The associate of a line

Besides the 21 singular lines $\mathcal{L} = \Omega_7^{(1)}(111) \cup \Omega_{14}^{(1)}(000)$ in $\mathrm{PG}(5,2)$ there are 630 non-singular lines. The latter are of two kinds, namely the 210 special lines considered in Section 2.1.2 and the 420 general lines considered in Section 2.2. We denote these two sets of non-singular lines by $\mathcal{L}_{210}$ and $\mathcal{L}_{420}$:

$$\begin{aligned}\mathcal{L}_{210} &:= \Omega_{21}^{(1)}(300) \cup \Omega_{126}^{(1)}(110) \cup \Omega_{63}^{(1)}(100),\\ \mathcal{L}_{420} &:= \Omega_{252}^{(1)}(100) \cup \Omega_{168}^{(1)}(000).\end{aligned}$$

Recall from Lemma 2.6 that a non-singular line L lies inside a unique super-singular solid $D_L \in \mathcal{D}$.

Theorem 3.1 *(i) If $L \in \mathcal{L}$ then $L^\# =$* $\mathrm{PG}(5,2)$.
(ii) If $L \in \mathcal{L}_{210}$ then $L^\# \in \Omega_{21}^{(4)}(733)$.
(iii) If $L \in \mathcal{L}_{420}$ then $L^\# \in \Omega_{42}^{(4)}(333)$.

Proof. (i) If $L \in \mathcal{L}$ then f_L is the zero form.
(ii), (iii) Given that $L = \langle a, b\rangle$ is non-singular consider, see Lemma 2.6 the super-singular solid $D_L \supset L$. Since, see Eq. (2.9), $T(a,b,p) = 0$ for all $p \in D_L$, it follows that $L^\# \supset D_L$. So in case (ii) the hyperplane $L^\#$ contains $D_L \in \Omega_7^{(3)}(333)$, whence $L^\# \in \Omega_{21}^{(4)}(733)$, while in case (iii) the hyperplane $L^\#$ contains $D_L \in \Omega_{14}^{(3)}(111)$, whence $L^\# \in \Omega_{42}^{(4)}(333)$. ∎

Remark 3.2 *The* $651 - 21 = 630$ *non singular lines are mapped* $10:1$ *by $L \mapsto L^\#$ onto the* 63 *hyperplanes. A solid $D \in \mathcal{D}$ lies in three hyperplanes, say H_1, H_2, H_3, and the* 30 *non-singular lines L in D fall into three 10-sets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ such that $L \in \mathcal{N}_i$ if $L^\# = H_i, i = 1,2,3$. Given a non-singular line $L \in D_L$ its 10-set $\mathcal{N}_L$ is seen to be determined in the following interesting way. If $D_L = \mathbb{P}V_4$ then one checks that the space V_4 can be equipped with* $\mathrm{Sp}(4,2)$*-geometry* such that the five singular lines in D_L are self-polar *in precisely three ways. If we further demand that L is also self-polar then the geometry is uniquely determined, and one finds that $\mathcal{N}_L$ consists of those* $(15-5=)$ 10 *self-polar lines in D_L other than the* 5 *singular lines.*

3.2 The associate of a plane

3.2.1 Odd planes

From the Table (2.10) there exist eight orbits of odd planes, four consisting of non-singular planes and four of singular planes:

The eight orbits of odd planes		
Non-singular	$\Omega_3^{(2)}(700),\ \Omega_{63}^{(2)}(300),\ \Omega_{168}^{(2)}(111),\ \Omega_{126}^{(2)}(100)$	(3.1)
Singular	$\Omega_{63}^{(2)}(311),\ \Omega_{42}^{(2)}(111),\ \Omega_{42^*}^{(2)}(111),\ \Omega_{168}^{(2)}(100)$	

Recall from Lemma 2.4 that a singular plane P lies inside a unique super-singular solid, namely its cloak D_P.

Theorem 3.3 *Suppose that P is an odd plane in* PG(5, 2).
(i) If $P \in \Omega_3^{(2)}(700) \cup \Omega_{63}^{(2)}(300) \cup \Omega_{168}^{(2)}(111) \cup \Omega_{126}^{(2)}(100)$ then $P^\# = P$.
(ii) If $P \in \Omega_{63}^{(2)}(311) \cup \Omega_{42}^{(2)}(111) \cup \Omega_{42^*}^{(2)}(111) \cup \Omega_{168}^{(2)}(100)$ then $P^\# = D_P$.
(iii) If $P \in \Omega_{63}^{(2)}(311) \cup \Omega_{42}^{(2)}(111)$ then $P^\# \in \Omega_7^{(3)}(333)$, while if $P \in \Omega_{42^*}^{(2)}(111) \cup \Omega_{168}^{(2)}(100)$ then $P^\# \in \Omega_{14}^{(3)}(111)$.

Proof. (i) By Lemma 1.11(i) $P^\# = P$. for any odd non-singular plane P.

(ii) Here P is singular and is contained in its cloak $D_P \in \mathcal{D}$. It follows immediately from Eq. (2.9) that $D_P \subseteq P^\#$. But since P contains just one singular line, by Lemma 1.11(ii) $P^\#$ is a solid, whence $P^\# = D_P$.

(iii) A special solid $D \in \Omega_7^{(3)}(333)$ contains planes from the orbits $\Omega_{63}^{(2)}(311)$ and $\Omega_{42}^{(2)}(111)$, but no planes from the orbits $\Omega_{42^*}^{(2)}(111)$ and $\Omega_{168}^{(2)}(100)$, these latter planes therefore all being contained in the orbit $\Omega_{14}^{(3)}(111)$. (*Checks:* $7 \times 15 = 63 + 42$✓ and $14 \times 15 = 210 = 42 + 168$✓) ■

3.2.2 Even planes

To complete our treatment of the ψ-associate of a planes in PG(5, 2) we need to deal with the even planes, which from (2.10) are those belonging to one of the three orbits

$$\Omega_{168}^{(2)}(310), \quad \Omega_{504}^{(2)}(110), \quad \Omega_{48}^{(2)}(000). \tag{3.2}$$

Recall from Theorem 1.10 that if P is any even plane then $P^\#$ is a plane, and is disjoint from P.

First we treat the orbit $\Omega_{48}^{(2)}(000)$ by finding the associate of the planes $P_r,\ r = 1, 2\ \ldots\ , 6$, in Eq. (2.2).

Theorem 3.4 *The associates of the external planes P_r are as follows:*

$$(P_1)^\# = P_2, \qquad (P_2)^\# = P_4, \qquad (P_4)^\# = P_1, \tag{3.3}$$

$$(P_6)^\# = P_5, \qquad (P_5)^\# = P_3, \qquad (P_3)^\# = P_6. \tag{3.4}$$

So for any external plane P we have $P^{\#\#\#} = P$.

Proof. To show that $(P_1)^\# = P_2$ we need to show, see theorem 1.10(i), that the plane $\langle z, L\rangle$ is ψ-odd for each point $z \in P_2$ and each line $L \subset P_1$. Because the planes of the spread (2.1) are invariant under the Z_7 subgroup of

$\mathcal{G}_\psi$ generated by $\Phi_A = A \oplus A$, it suffices to consider just one choice of line $L \subset P_1$, say $L_1 := \{12, 23, 34\}$ (in shorthand notation, see after (2.6)). Then one sees that:
(a) if $z = 13 \in P_2$ then the plane $\langle z, L_1 \rangle$ meets ψ in the three points $05 \in Y$, $40 \in X$ and $22 \in P_0$;
(b) for the other six points $z \in P_2$ the plane $\langle z, L_1 \rangle$ meets just one of the planes X, Y, P_0.
So indeed $\langle z, L \rangle$ is ψ-odd for each point $z \in P_2$ and each line $L \subset P_1$, whence $P_2 = (P_1)^{\#}$. On applying to this last relation the element $R \in \mathcal{G}_\psi$ of lemma 2.1 we obtain the other two relations in (3.3). Further the relations (3.4) follow from the relations (3.3) upon applying the involution $J \in \mathcal{G}_\psi$, see (2.4). ■

Corollary 3.5 *The* 48 *planes external to the Segre variety* $\psi = \mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$ *fall into eight pairs of ordered triplets* $\{(R_1, S_1, U_1), (R_2, S_2, U_2)\}$ *satisfying* $\psi^c = R_1 \cup S_1 \cup U_1 \cup R_2 \cup S_2 \cup U_2$ *and*

$$R_i^{\#} = S_i, \quad S_i^{\#} = U_i, \quad U_i^{\#} = R_i, \quad i = 1, 2. \tag{3.5}$$

For the next corollary consult [4, §4.1], [5, §4] for the *privileged plane* of a non-maximal partial plane-spread Σ_5 in $\mathrm{PG}(5,2)$.

Corollary 3.6 *Suppose that* Σ_4 *is any partial spread of four planes in* $\mathrm{PG}(5,2)$. *Let* $\Sigma_9 = \Sigma_4 \cup \Sigma_5$ *be its unique extension to a complete spread. Choosing* $P \in \Sigma_4$, *let* ψ *denote the underlying* 21*-set of the three planes* $\Sigma_4 \setminus \{P\}$ *and use* ψ *to define the associate* $X^{\#}$ *of a flat* X *of* $\mathrm{PG}(5,2)$. *Then the privileged plane of* Σ_5 *is* $P^{\#\#}$.

Proof. If $\Sigma_4 = \{X, Y, P_0, P_1\}$ then, see [5, Theorem 4.3, Proof], the privileged plane of Σ_5 is P_4. But, see (3.3), $P_4 = P_1^{\#\#}$. ■

In order to deal with the planes belonging to the orbits $\Omega_{168}^{(2)}(310$ and $\Omega_{504}^{(2)}(110)$ it will prove convenient to adopt the temporary notation W_1, W_2, W_3 for the three internal planes X, Y, P_0 of ψ. Further we will make a choice of a solid $D \in \Omega_7^{(3)}(333)$, and also choose a line $M \in \Omega_7^{(1)}(111)$ which is disjoint from D. So $M = \{m_1, m_2, m_3\}$ where $\{m_i\} = M \cap W_i$.

Theorem 3.7 *In the foregoing notation, denote by* $L_i = D \cap W_i$, $i = 1, 2, 3$, *the three lines of* D *which belong to the orbit* $\Omega_{21}^{(1)}(300)$. *Define planes* N_1, N_2, N_3 *and* N_1', N_2', N_3' *as follows:*

$$N_1 = \langle m_2, L_3 \rangle, \quad N_2 = \langle m_3, L_1 \rangle, \quad N_3 = \langle m_1, L_2 \rangle, \tag{3.6}$$

$$N_1' = \langle m_3, L_2 \rangle, \quad N_2' = \langle m_1, L_3 \rangle, \quad N_3' = \langle m_2, L_1 \rangle. \tag{3.7}$$

Then N_1, N_2, N_3 *are pairwise disjoint planes of the orbit* $\Omega_{168}^{(2)}(310)$, *and so also are* N_1', N_2', N_3'. *Moreover*

$$(N_1)^{\#} = N_2, \quad (N_2)^{\#} = N_3, \quad (N_3)^{\#} = N_1, \tag{3.8}$$

$$(N_1')^{\#} = N_3', \quad (N_2')^{\#} = N_1', \quad (N_3')^{\#} = N_2'. \tag{3.9}$$

Proof. Since N_3 meets ψ in the four points $\{m_1\} \cup L_2$, then $N_3 \in \Omega_{168}^{(2)}(310)$; similarly $N_1, N_2, N_1', N_2', N_3' \in \Omega_{168}^{(2)}(310)$. In order to prove the results (3.8), (3.9) we may without loss of generality suppose that, in shorthand notation, $L_1 = \{10, 20, 40\}$, $L_2 = \{01, 02, 04\}$, $L_3 = \{11, 22, 44\}$ and $M = \{30, 03, 33\}$. Then a straightforward application of Lemma 1.13 quickly confirms the result $(N_1)^\# = N_2$. From this last result the remaining five results (3.8), (3.9) follow upon applying the symmetries J and R of Lemma 2.1. For R effects the permutations $(L_1L_2L_3)$ and $(m_1m_2m_3)$, and hence effects $(N_1N_2N_3)$ and $(N'_1N'_2N'_3)$, while the involution J effects the permutations $(L_1L_2)(L_3)$ and $(m_1m_2)(m_3)$, and hence effects $(N_1N'_2)(N_2N'_1)(N_3N'_3)$. ■

Theorem 3.8 *In the foregoing notation, let $L = \{p_1, p_2, p_3\}$, where $\{p_i\} = L \cap W_i$, be one of the three lines of D which belong to the orbit $\Omega^{(1)}_7(111)$. Denote by T_i, $i = 1, 2, 3$, that tangent to ψ which contains p_i and which lies in D. Define planes Q_1, Q_2, Q_3 and Q'_1, Q'_2, Q'_3 as follows:*

$$Q_1 = \langle m_2, T_3\rangle, \quad Q_2 = \langle m_3, T_1\rangle, \quad Q_3 = \langle m_1, T_2\rangle, \tag{3.10}$$

$$Q'_1 = \langle m_3, T_2\rangle, \quad Q'_2 = \langle m_1, T_3\rangle, \quad Q'_3 = \langle m_2, T_1\rangle. \tag{3.11}$$

Then Q_1, Q_2, Q_3 are pairwise disjoint planes of the orbit $\Omega^{(2)}_{504}(110)$, and so are Q'_1, Q'_2, Q'_3. Moreover

$$(Q_1)^\# = Q_2, \quad (Q_2)^\# = Q_3, \quad (Q_3)^\# = Q_1, \tag{3.12}$$

$$(Q'_1)^\# = Q'_3, \quad (Q'_2)^\# = Q'_1, \quad (Q'_3)^\# = Q'_2. \tag{3.13}$$

Proof. Since Q_3 meets ψ in the two points $\{m_1, p_2\}$, then $Q_3 \in \Omega^{(2)}_{504}(110)$; similarly $Q_1, Q_2, Q'_1, Q'_2, Q'_3 \in \Omega^{(2)}_{504}(110)$. In order to prove the results (3.12), 3.13 we may, without loss of generality, suppose that, in shorthand notation, $D = \langle 20, 30, 02, 03\rangle$, $M = \{10, 01, 11\}$ and $L = \{20, 02, 22\}$, whence $T_1 = \{20, 32, 52\}$, $T_2 = \{02, 23, 25\}$, $T_3 = \{22, 35, 53\}$. Then a straightforward application of Lemma 1.13 quickly confirms the result $(Q\ _1)^\# = Q_2$. The remaining five results then follow upon applying, *cf.* the preceding proof, the symmetries J and R of Lemma 2.1. ■

The next theorem summarizes results in the preceding three theorems.

Theorem 3.9 *There are three $\mathcal{G}_\psi$-orbits of even planes in* PG(5, 2)*:*

$$\Omega^{(2)}_{168}(310, \quad \Omega^{(2)}_{504}(110), \quad \Omega^{(2)}_{48}(000). \tag{3.14}$$

If P is any even plane then $P, P^\#, P^{\#\#}$ are pairwise disjoint planes belonging to the same $\mathcal{G}_\psi$-orbit; further $P^{\#\#\#} = P$. Thus every even plane is faithfully-even, with #-sequence (0, 2).

Remark 3.10 *One should not attribute this property $P^{\#\#\#} = P$ of even planes solely to the fact that ψ has degree $d = 3$. For consider the case of the Grassmannian variety $\psi = \mathcal{G}_{1,4,2}$ in* PG(9, 2)*, where $d = 5$ and so where, see [7, Corollary 3.2] the analogues of the ψ-even planes are the $\mathcal{G}_{1,4,2}$-even* 4*-flats in* PG(9, 2)*. Now it was found in [7, Section 3] (using the definition of $X^\#$ appropriate to $d = 5$) that there exist $\mathcal{G}_{1,4,2}$-even 4-flats X such that $X, X^\#, X^{\#\#}$ are pairwise disjoint even 4-flats satisfying $X^{\#\#\#} = X$.*

3.3 The associate of a 3-flat

We denote by $\bar{\mathcal{D}}$ the set of solids in PG(5, 2) which are not super-singular. From Table (2.11) $\bar{\mathcal{D}}$ consists of five $\mathcal{G}_\psi$-orbits:

$$\bar{\mathcal{D}} = \Omega^{(3)}_{21}(711) \cup \Omega^{(3)}_{126}(331) \cup \Omega^{(3)}_{252}(311) \cup \Omega^{(3)}_{63}(311) \cup \Omega^{(3)}_{168}(111).$$

But first, in the next theorem, we treat the remaining two orbits of solids, the super-singular solids $\mathcal{D}$.

Theorem 3.11 *If $D \in \mathcal{D} = \Omega^{(3)}_7(333) \cup \Omega^{(3)}_{14}(111)$ then $D^\# = D$.*

Proof. It follows from Eq. (2.9) that $D^\# \supseteq D$. On the other hand $D^\# \subseteq P^\#$ for any plane $P \subset D$; but P is singular and so by Theorem 3.3(ii) $P^\# = D$. ■

Lemma 3.12 *A solid D is super-singular if and only if*

$$T(a, b, c) = 0, \quad \textit{for all } a, b, c \in D. \tag{3.15}$$

Proof. The 'only if' implication has already been proved: see Lemma 2.5. The 'if' implication will be proved if we can show that for all $D \in \bar{\mathcal{D}}$ (3.15) does not hold. Suppose to the contrary that a solid $D \in \bar{\mathcal{D}}$ satisfies property (3.15) and let L be the unique singular line in D, see Remark 2.9. Given any line $M \neq L$ in D then M is non-singular and so $M^\#$ is a hyperplane. There are 34 lines $M \neq L$ in D and by the hypothesized property (3.15) each hyperplane $M^\#$ contains D. But, see Remark 3.2, these 34 lines give rise to at least four distinct hyperplanes, contradicting the fact that a solid D in PG(5, 2) is contained in just three hyperplanes. This contradiction shows that only the 21 solids $D \in \mathcal{D}$ satisfy the property (3.15). ■

An alternative proof can be arrived at by looking separately at each of the five orbits of solids $D \in \bar{\mathcal{D}}$ and in each case finding an even plane $\langle a, b, c\rangle \subset D$, whence, see Lemma 1.7, $T(a, b, c) = 1$.

Lemma 3.13 *A solid $D \in \bar{\mathcal{D}}$ contains a unique point n_D, the* nucleus *of D, such that*

$$T(a, b, n_D) = 0, \quad \textit{for all } a, b \in D. \tag{3.16}$$

Further n_D lies on the unique singular line L_D in D.

Proof. Given a solid $D = \mathbb{P}V_4$ let T_4 denote the restriction of T to $\times^3 V_4$. If $D \in \bar{\mathcal{D}}$ then by the preceding lemma, $T_4 \neq 0$ and so, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$ for V_4, we will have

$$T_4(a, b, x) = \begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix}, \qquad a, b, x \in D.$$

Hence the unique choice $n_D = e_4$ satisfies (3.16). To see that $n_D \in L_D$ we will show that the contrary assumption $n_D \notin L_D$ leads to a contradiction. For if $L_D = \langle b_1, b_2\rangle$ and $n_D \notin L_D$ then for suitable b_3 we would have $D = \langle b_1, b_2, b_3, n_D\rangle$.But from (3.16) we would have $T_4(b_i, b_j, n_D) = 0$, $ij = 12, 13, 23$, and since $\langle b_1, b_2\rangle$ is singular we would also have $T_4(b_1, b_2, b_3) = 0$, whence $T(a, b, c) = 0$, for all $a, b, c \in D$, contradicting $D \in \bar{\mathcal{D}}$. ■

Theorem 3.14 *If $D \in \bar{\mathcal{D}}$ then $D^{\#}$ is a point, namely the nucleus n_D of D.*

Proof. From (3.16) we see that $n_D \in D^{\#}$. To see that $D^{\#}$ is no larger than $\{n_D\}$ let $P = \langle a, b, n_D \rangle$ be any plane $P \subset D$ which meets L_D in n_D. Then P is non-singular, since P does not contain the only singular line L_D in D. But from (3.16) and Lemma 1.7 it follows that P is odd, and hence, by Lemma 1.11(i), $P^{\#} = P$. Consequently any point $p \in D^{\#}$ must lie on the intersection of such planes P, and so $p = n_D$. ∎

To complete the treatment of solids, let us in the following identify the point n_D for each of the five constituent orbits of $\bar{\mathcal{D}}$.

(i) $D \in \Omega_{21}^{(3)}(711)$. Here $D = \langle L, P \rangle$, where $L = L_D \in \Omega_7^{(1)}(111)$ and $P \in \Omega_3^{(2)}(700)$, and one sees that n_D is the point $L \cap P$.

(ii) $D \in \Omega_{126}^{(3)}(331)$. Setting $\Omega_3^{(2)}(700) = \{W_1, W_2, W_3\}$ then if D is such that $|D \cap W_1| = |D \cap W_2| = 3$, one finds that n_D is the point $D \cap W_3$.

(iii) $D \in \Omega_{252}^{(3)}(311)$. Here, see section 2.5, note (b), $D = \langle L, M \rangle$ for unique lines $L \in \Omega_{21}^{(1)}(300)$, $M \in \Omega_{14}^{(1)}(000)$, $D_L \neq D_M$. Moreover, if $D \cap \psi = L \cup \{a_1, a_2\}$, then $a_1 + a_2 \in M$ and one finds that $n_D = a_1 + a_2$.

(iv) $D \in \Omega_{63}^{(3)}(311)$. Here D contains a unique line $L \in \Omega_7^{(1)}(111)$ and a unique line $M \in \Omega_{21}^{(1)}(300)$ and one finds that $D^{\#}$ is the point $L \cap M$.

(v) $D \in \Omega_{168}^{(3)}(111)$ If $D \cap \psi = \{a, b, c\}$, then $n_D = a + b + c$.

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Prof. R. Shaw, Centre for Mathematics,
University of Hull, Hull HU6 7RX, UK

Dr. N. A. Gordon, Department of Computer Science,
University of Hull, Hull HU6 7RX, UK