The classification of flats in $\mathrm{PG}(9,2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$

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Abstract

Constructions are given of different kinds of flats in the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2)$. In particular it is shown that there exist precisely two $\mathrm{GL}(5,2)$-orbits of external 4-flats, each with stabilizer group $\cong 31{:}5$. (No 5-flat is external.) For each $k = 1,2,3$, two distinct kinds of external k-flats are simply constructed out of certain partial spreads in $\mathrm{PG}(4,2)$ of size $k+2$. A third kind of external plane, with stabilizer $\cong 2^3{:}(7{:}3)$, is also shown to exist. With the aid of a certain 'key counting lemma', it is proved that the foregoing amounts to a complete classification of external flats.

1 Introduction

1.1 Linearity in the presence of nonlinearity

Let $\mathcal{H}$ be a hypersurface in $\mathrm{PG}(N,q)$ determined by a (homogeneous) polynomial equation $F(x_0, x_1, ..., x_N) = 0$, say. Then one can pose the problem of classifying those flats which are maximal subject to being *internal* to $\mathcal{H}$, and those which are maximal subject to being *external* to $\mathcal{H}$. In the case $\deg F = 2$ the internal problem was solved a long time ago. In particular a non-singular hyperbolic quadric $\mathcal{H}$ in $\mathrm{PG}(N,q)$ is known to have precisely two distinct families of maximal internal flats, each consisting of flats of dimension $\frac{1}{2}(N-1)$.

For $\deg F > 2$, however, the problem becomes in general much harder. Nevertheless, precisely because it is difficult for linearity to secure a place in a highly nonlinear situation, we feel that any such success carries with it the prospect of some interesting mathematics. As an example, consider the problem of classifying the *linear sections* of $\mathrm{GL}(n,q)$. Here we deal with the hypersurface $\mathcal{H}$, whose equation $\det A = 0,\ A \in \mathrm{End}\, V(n,q)$, has degree n, which lies in the projective space $\mathrm{PG}(n^2-1,q) = \mathbb{P}(\mathrm{End}\, V(n,q))$, and we seek to classify those flats which are *external* to $\mathcal{H}$. It seems to us that this is a genuinely tough problem! For even for the small n and q values

$(n,q) = (4,2)$ the complete story was not easy to unravel (see [4]). For the description of a particularly interesting maximal linear section of $\mathrm{GL}(4,2)$, which lies inside an $\mathrm{Alt}(7)$ subgroup of $\mathrm{GL}(4,2) \cong \mathrm{Alt}(8)$, see [14].

In the present paper we will be chiefly concerned with those flats in $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are *external* to the Grassmannian $\mathcal{G}_{1,4,2}$ arising from the 155 lines of $\mathrm{PG}(4,2) = \mathbb{P}(V(5,2))$. The general Grassmannian $\mathcal{G}_{r,n,q}$ of the r-spaces of $\mathrm{PG}(n,q)$ is usually thought of as the algebraic variety arising from certain well-known quadratic Grassmann conditions. In particular, in the case $(n,q) = (4,2)$, the Grassmannian $\mathcal{G}_{1,4,2}$ is usually thought of as the variety in $\mathrm{PG}(9,2)$ determined by the simultaneous solutions of the five distinct quadratic conditions $b_{ij}b_{kl} + b_{ik}b_{jl} + b_{il}b_{jk} = 0,\ \{i,j,k,l\} \subset \{1,2,3,4,5\}$. Here the $b_{ij} (= b_{ji})$ are the coordinates of a bivector $b \in V_{10} = V_5 \wedge V_5,\ V_5 = V(5,2)$, with respect to a product basis$\{e_i \wedge e_j\}_{1\leq i<j\leq 5}$. Since *any* algebraic variety in $\mathrm{PG}(N,q)$ is in fact a hypersurface, see [7, Lemma 3.2], the Grassmannian $\mathcal{G}_{1,4,2}$ is alternatively described as the locus $F(b_{12}, b_{13}, ..., b_{45}) = 0$ for a certain single polynomial F in the 10 coordinates b_{ij}. It was proved in [9] that in this case $\deg F = 5$. The next theorem is a consequence of this fact. (For a different proof, without appeal to the quintic equation of $\mathcal{G}_{1,4,2}$, see theorem 1.3.)

Theorem 1.1 *No 5-flat in* $\mathrm{PG}(4,2)$ *is external to* $\mathcal{G}_{1,4,2}$.

Proof. By the particular case $r = 5$ of the theorem in [13], every 5-flat intersects $\mathcal{G}_{1,4,2}$ in an odd number of points. ■

1.2 Bivectors

For the moment let V be any finite-dimensional vector space and consider its associated space $\wedge^2 V$ of bivectors. Recall the natural action of $\mathcal{G} := \mathrm{GL}(V)$ upon $\wedge^2 V$: each $A \in \mathcal{G}$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\mathrm{GL}(\wedge^2 V)$ whose effect on the *decomposable bivectors* $u \wedge v \in V \wedge V$ is given by $T_A(u \wedge v) = Au \wedge Av$. If X is any object in $\wedge^2 V$ its stabilizer $\mathcal{G}_X$ is the subgroup of $\mathcal{G}$ defined by $\mathcal{G}_X = \{A \mid A \in \mathcal{G},\ T_A(X) = X\}$. Under this action T of $\mathrm{GL}(V)$ the orbit structure of the bivector space $\wedge^2 V$ is well-known: if V has dimension $2k$ or $2k+1$, then there are $k+1$ distinct orbits $\{\mathrm{Rk}_{2r};\ r = 0, 1,\ \dots\ , k\}$. Here $\mathrm{Rk}_{2r} = \mathrm{Rk}_{2r}(V)$ consists of those bivectors having rank $2r$, a bivector b having rank $2r > 0$ if and only if there exist linearly independent vectors $e_1,\ \dots\ , e_{2r}$ such that $b = e_1 \wedge e_2 + \dots + e_{2r-1} \wedge e_{2r}$.

From now we work over $\mathrm{GF}(2)$, *and so we may identify the nonzero elements of a vector space* $V = V(n+1,2) = V_{n+1}$ *with the points of the associated projective space* $\mathbb{P}V = \mathrm{PG}(n,2)$. Consequently, for $r > 0$, we may view the orbit $\mathrm{Rk}_{2r}(V)$ as a subset, to be denoted $\mathrm{Rk}_{2r}(n,2)$, of the projective space $\mathrm{PG}(N,2) = \mathbb{P}(\wedge^2 V)$ where $N = \frac{1}{2}n(n+1) - 1$.

We use $\prec u, v, \ldots \succ$ for the *vector* subspace spanned by $u, v, \ldots$. The Grassmann map $\prec u, v \succ \mapsto \prec u \wedge v \succ$ sends the 2-spaces of V to those 1-spaces of $\wedge^2 V$ which are spanned by decomposable bivectors. Projectively the lines of $\mathrm{PG}(n,2)$ are mapped onto the points of $\mathrm{Rk}_2(V) = \mathcal{G}_{1,n,2}$. *Throughout this paper the images in* $\mathcal{G}_{1,n,2} \subset \mathrm{PG}(N,2)$ *of lines* λ, μ, ν, τ *in* $\mathrm{PG}(n,2)$ *will be denoted* l, m, n, t. It is easy to see that $\{m_1, m_2, m_3\}$ is a line lying on $\mathcal{G}_{1,n,2}$ if and only if $\{\mu_1, \mu_2, \mu_3\}$ is a *pencil* (a, α) of lines in $\mathrm{PG}(n,2)$, consisting of those three lines μ_i of the plane α which pass through the point $a \in \alpha$.

Note. In order to keep the paper to a manageable size, some of the longer proofs have been omitted — but they can all be accessed in the research reports [10] and [11].

1.3 The Grassmannian $\mathcal{G}_{1,4,2}$

The case $n = 3$ of the foregoing, namely that of the Grassmannian $\mathcal{G}_{1,3,2} \subset \mathrm{PG}(5,2)$ arising from the 35 lines of $\mathrm{PG}(3,2)$, is of course extremely well-known, [6]. See appendix A.1 for some aspects of relevance to our present concerns.

But from now onwards our starting space will be $V_5 = V(5,2)$, along with its associated projective space $\mathrm{PG}(4,2)$, and we will be concerned with the corresponding bivector space $V_{10} = V(10,2) = V_5 \wedge V_5$, and with its associated projective space $\mathrm{PG}(9,2) = \mathbb{P}(V_{10})$. Note that under the action T of $\mathcal{G} := \mathrm{GL}(5,2)$, the projective space $\mathrm{PG}(9,2)\,(= V_{10} \setminus \{0\})$ is the disjoint union of just two $\mathcal{G}$-orbits:

$$\mathrm{PG}(9,2) = \mathrm{Rk}_2(4,2) \cup \mathrm{Rk}_4(4,2) \qquad (\text{where } \mathrm{Rk}_2(4,2) = \mathcal{G}_{1,4,2}). \tag{1.1}$$

Usually these two orbits will be abbreviated as Rk_2 and Rk_4 . Of course $|\,\mathrm{Rk}_2\,| = 155$ (the number of lines in $\mathrm{PG}(4,2)$), and so $|\,\mathrm{Rk}_4\,| = 1023 - 155 = 868$. In the following, the terms *internal, external, tangent* and *bisecant* usually refer to the Grassmannian $\mathrm{Rk}_2 = \mathcal{G}_{1,4,2}$.

1.3.1 A useful partition of Rk_4

As well as the $155 + 868$ partition of $\mathrm{PG}(9,2)$ given by (1.1), of great importance (see for example theorem 1.3 below) will be a certain partition of the 868 points of Rk_4 into 31 subsets of size 28, which arises from the 31 solids of $\mathrm{PG}(4,2)$. For observe that each solid $\sigma = \mathbb{P}(V_4)$ in $\mathrm{PG}(4,2)$ defines a 5-flat $\Pi(\sigma) = \mathbb{P}(\wedge^2 V_4)$ in $\mathrm{PG}(9,2)$, and so, upon defining

$$\mathcal{H}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_2 \qquad \text{and} \qquad \mathcal{W}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_4, \tag{1.2}$$

the 63 points of $\Pi(\sigma)$ consist of the 35 points of a hyperbolic quadric $\mathcal{H}(\sigma) = \mathrm{Rk}_2(V_4) = \mathcal{G}_{1,3,2}$ together with the 28 points $\mathcal{W}(\sigma)$ which are external to

$\mathcal{H}(\sigma)$. Now an element $b \in \mathrm{Rk}_4$ determines a solid $\sigma = \operatorname{im} b \subset \mathrm{PG}(4,2)$ and also a point $k = \ker b$ of the dual space $\mathrm{PG}(4,2)^* = \mathbb{P}(V_5^*)$, where

$$\operatorname{im} b := \{b \llcorner f \mid f \in \mathrm{PG}(4,2)^*\}, \qquad \ker b := \{f \in \mathrm{PG}(4,2)^* \mid b \llcorner f = 0\}. \tag{1.3}$$

(Here $\llcorner$ is 'right interior multiplication'; in terms of coordinates, using a basis in V^* dual to the basis in V, the equation $v = b \llcorner f$ reads $v_i = \sum_j b_{ij} f_j$.) Note that the solid σ has equation $k(x) = 0$.

Since $b \in \mathcal{W}(\sigma)$ if and only if $\operatorname{im} b = \sigma$, it follows that the subsets $\mathcal{W}(\sigma)$, σ a solid in $\mathrm{PG}(4,2)$, *yield a partition of the 868 points of* Rk_4 *into 31 subsets of size 28:* $868 = 31 \times 28$.

Definition 1.2 *An external line which lies inside one of the 28-sets $\mathcal{W}(\sigma)$ will be called a* special line; *the other external lines, which meet three distinct $\mathcal{W}(\sigma)$, will be referred to as* general lines. *An external plane which contains a special line will be called a* special plane; *the other external planes, which meet seven distinct $\mathcal{W}(\sigma)$, will be referred to as* general planes. *Similarly* general solids *are external solids which contain no special lines, and* special solids *contain at least one special line.*

We do not apply this special/general terminology to external 4-flats because, by the next theorem, no external 4-flat contains a special line: every external 4-flat is 'general'.

Theorem 1.3 *(i) An external 4-flat E meets each $\mathcal{W}(\sigma)$ in precisely one point.*

(ii) A special line possesses no extensions to an external 4-flat E.

(iii) External 5-flats do not exist; external 4-flats do exist.

Proof. (i) In $\mathrm{PG}(9,2)$ a 4-flat meets a 5-flat in at least one point. So E meets each of the thirty-one $\mathcal{W}(\sigma)$ in at least one point. But $|E| = 31$.

(ii) By (i), E cannot meet $\mathcal{W}(\sigma)$ in a line.

(iii) Within $\mathrm{PG}(9,2)$ any 5-flat meets each 5-flat $\Pi(\sigma)$ in at least a line. So an external 5-flat F would meet each of the thirty-one $\mathcal{W}(\sigma)$ in at least three points. The assumption that F exists thus leads to a contradiction: $31 \times 3 = 93$, yet $|F| = 63$. (See theorem 1.1 for another proof.) See section 3 below for two constructions of external 4-flats. ■

1.4 Classification of the lines in $\mathbb{P}(\wedge^2 V_5)$

First of all we consider the lines in $\mathrm{PG}(9,2)$ through a particular point $b \in \mathrm{Rk}_4$. Recall that b lies in $\mathcal{W}(\sigma)$ for precisely one choice of solid $\sigma = \operatorname{im} b \subset \mathrm{PG}(4,2)$. It should also be noted that the stabilizer $\mathcal{G}_b$ of b within $\mathrm{GL}(5,2)$ has the structure $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$. This comes about because $\mathcal{G}_b$ contains a normal subgroup $\cong (Z_2)^4$, consisting of those transvections which fix σ pointwise, and since the stabilizer of b within $\mathrm{GL}(4,2)$ is, see remark A.2, a subgroup $\mathrm{Sp}(4,2) \cong \mathrm{Sym}(6)$. The following lemma is relevant to our immediate concerns, but it will also be of use in the proof of theorem 2.8. To prove it one makes use of the symplectic polarities in σ and σ' induced by b and b', and considers the polar points α^b and $\alpha^{b'}$, see (A.2), of the plane $\alpha = \sigma \cap \sigma'$; see [10] for further details.

Lemma 1.4 *([10]) Given* $b \in \mathcal{W}(\sigma)$, *and given* $\sigma' \neq \sigma$, *consider the* 28 *lines* $\langle b, b'\rangle$ *through* b *which arise from the* 28 *choices of* $b' \in \mathcal{W}(\sigma')$. *For precisely four of these choices the line* $\langle b, b'\rangle$ *is a tangent to* Rk_2, *and for the remaining twenty-four choices the line* $\langle b, b'\rangle$ *is a general line.*

The 511 lines in $\mathrm{PG}(9,2)$ which pass through the point $b \in \mathcal{W}(\sigma)$ are:
(a) 10 bisecants (the 10 bisecants to $\mathcal{H}(\sigma)$);
(b) 15 tangents to $\mathcal{H}(\sigma)$;
(c) 120 tangents to the 120 points of $\mathrm{Rk}_2 \setminus \mathcal{H}(\sigma)$;
(d) 6 special lines (the other 6 lines through b lying in $\Pi(\sigma)$;
(e) $360\,(= 511 - 10 - 15 - 120 - 6)$ general lines.
Concerning (a), (b), (d), see appendix A.1. Concerning (c), (e), the stated numbers follow from lemma 1.4. For firstly there exist 4 points b' in each of the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, such that $\langle b, b'\rangle$ is a tangent: $4 \times 30 = 120$. Secondly there exist 24 points b' in each of the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, such that $\langle b, b'\rangle$ is a general line: $(24 \times 30) \div 2 = 360$. (The fact that the 120 tangents (c) have uniform intersection number 4 with the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, follows also from the fact that $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$ is transitive on the 30 solids $\sigma' \neq \sigma$.)

Next consider the 511 lines through a point $m \in \mathrm{Rk}_2$. Note that m will lie in $\mathcal{H}(\sigma)$ for 7 choices of σ (since a line μ in $\mathrm{PG}(4,2)$ lies in 7 solids). For m in $\mathcal{H}(\sigma)$ consider the 31 lines through m which lie in $\Pi(\sigma)$. Of these, 9 are internal to $\mathcal{H}(\sigma)$, since μ belongs to 9 pencils in σ. So there are $35 - 18 - 1 = 16$ points of $\mathcal{H}(\sigma)$ not on these 9 internal lines. Hence m lies on 16 bisecants in $\Pi(\sigma)$. The remaining $31 - 9 - 16 = 6$ lines are tangents in $\Pi(\sigma)$. It follows that through m pass:
(a) 21 internal lines (since μ belongs to 21 pencils in $\mathrm{PG}(4,2)$);
(b) 112 bisecants, 16 for each of the 7 $\mathcal{H}(\sigma)$ in the preamble;
(c) 42 tangents which lie in some $\Pi(\sigma)$, 6 for each of the 7 $\mathcal{H}(\sigma)$;
(d) $336\,(= 511 - 21 - 112 - 42)$ other tangents.
As an arithmetical check, note that the bisecants (b) and the tangents (c) account for all 28 points of each of seven $\mathcal{W}(\sigma)$, for σ as in the preamble. Considering the remaining 24 $\mathcal{W}(\sigma)$, note that each of the 336 tangents (d) accounts for 2 of their points. Now we have the check $336 \times 2 = 24 \times 28$.

In consequence of the foregoing, the 174,251 lines in $\mathrm{PG}(9,2)$ can be classified as follows:
(i) 1085 internal lines; (ii) 8680 bisecants;
(iii) 6510 tangents which lie in some $\Pi(\sigma)$; (iv) 52080 other tangents;
(v) 1736 special external lines; (vi) 104,160 general external lines.

Remark 1.5 *These six classes are in fact the six distinct* $\mathrm{GL}(5,2)$*-orbits of lines in* $\mathrm{PG}(9,2)$. *Concerning the lines (*v*), recall from lemma A.1 that each of the 31* $\mathcal{W}(\sigma)$ *contributes 56 special lines:* $31\times 56=1736$. *Noting, by the same lemma, that the stabilizer within* $\mathrm{GL}(4,2)$ *of a special line* $L\subset\mathcal{W}(\sigma)$ *has structure* $(\mathrm{Alt}(5)\times 3).2$, *the stabilizer* $\mathcal{G}_L$ *within* $\mathrm{GL}(5,2)$ *is seen to have the structure* $\mathcal{G}_L\cong 2^4:\{(\mathrm{Alt}(5)\times 3).2\}$, *since* $\mathcal{G}_L$ *contains a normal subgroup* $\cong (Z_2)^4$ *consisting of those transvections which fix* σ *pointwise. Concerning the general lines (*vi*), see theorem 2.8: they form a single* $\mathrm{GL}(5,2)$*-orbit, and have stabilizer group* $\cong \mathrm{Sym}(4)\times Z_2$.

2 The even hyperplane construction of external k-flats, $0\le k\le 3$, from partial spreads

The internal flats of $\mathcal{G}_{1,4,2}$ are well understood. The maximal ones are of two kinds, *Greek planes* and *Latin solids.* The 7 lines of a plane $\alpha\subset\mathrm{PG}(4,2)$ map onto the 7 points of a 'Greek' plane $P(\alpha)\subset\mathcal{G}_{1,4,2}$, and the 15 lines of $\mathrm{PG}(4,2)$ which pass through the point a, forming let us say the *star* $\mathrm{st}(a)$, map onto the 15 points of a 'Latin' solid, say $\mathrm{St}(a)\subset\mathcal{G}_{1,4,2}$.

So from now onwards we concentrate upon the external flats. Our normal notation for a chain of flats in $\mathrm{PG}(9,2)$ of (projective) dimensions $1,2,3,4,5$ will be $L\subset P\subset D\subset E\subset F$. Usually the 4-flat E in such a chain will be external; in which case, because of theorem 1.3, L, P, D will be general k-flats, $k=1,2,3$.

Given a partial spread $\mathcal{S}_r=\{\mu_1,\ldots,\mu_r\}$ in $\mathrm{PG}(4,2)$, let the corresponding r-set of points of $\mathrm{Rk}_2\subset\mathrm{PG}(9,2)$ be $\mathcal{C}_r=\{m_1,\ldots.,m_r\}$. For $1\le i<j<\ \ldots\ \le r$ we define $m_{ij}=m_i+m_j$, $m_{ijk}=m_i+m_j+m_j$, $\ldots.$, $m_\Sigma=m_{12\ldots r}=\sum_{i=1}^r m_i$. One should be aware that some of these vectors may be zero, and that coincidences may therefore occur amongst the points (= nonzero vectors). For example, if $\mathcal{S}_5=\{\mu_1,\ldots,\mu_5\}$ is a spread for a solid $\sigma\subset\mathrm{PG}(4,2)$ then, see lemma A.1(iv), $m_\Sigma=0$, whence $m_{123}=m_{45}$, $m_{1234}=m_5$, etc.

Lemma 2.1 *([10]) (i)* $\mathcal{C}_r$ *is a* r*-cap: no three points of* $\mathcal{C}_r$ *are collinear.*

(ii) All of the points m_{ij} *are external points.*

(iii) All of the points m_{ijk} *are external points.*

(iv) The $\binom{r}{2}$ *points* m_{ij} *are distinct.*

In the foregoing setting we will be particularly concerned with the projective space $\mathcal{E}(\mathcal{C}_r):=\langle\mathcal{C}_r\rangle_{\text{even}}$ which is generated by the $\binom{r}{2}$ points m_{ij}. In cases where $m_1,\ \ldots\ ,m_r$ are linearly independent, note that $\langle\mathcal{C}_r\rangle_{\text{even}}$ is the *even hyperplane* of the basis $\{m_1,\ldots,m_r\}$ for $\langle\mathcal{C}_r\rangle$; it is the unique hyperplane in $\langle\mathcal{C}_r\rangle$ which contains no element of $\mathcal{C}_r$, and so, amongst the hyperplanes in $\langle\mathcal{C}_r\rangle$, *only* $\mathcal{E}(\mathcal{C}_r)$ *is a candidate for being an external flat.*

In seeking those partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ for which $\mathcal{E}(\mathcal{C}_r)$ is an external flat we are in the fortunate position of having available a complete classification of all the partial spreads: see [5], where tables can be found giving details of all of the 64 $\mathrm{GL}(5,2)$-orbits (classes) of partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ of size $r \geq 1$. (The corresponding classification in $\mathrm{PG}(4,q)$ is not known for any other value of q.) One invariant of a partial spread $\mathcal{S}_r$ is the number of reguli which it contains. In [5] the reguli (any 3 mutually skew lines lying in some solid) are allocated to class IIIa.1, while the *non-reguli* (any 3 mutually skew lines which generate the whole space $\mathrm{PG}(4,2)$) are allocated to class IIIb.1. If $\mathcal{S}_r$ contains just one regulus it said to be of type I, and if it is regulus-free it said to be of type O. These two kinds of partial spread will be of particular concern to us, see after lemma 2.4. Concerning type O, we read off from [5, Tables B.2a, B.2b] that there are 12 classes of regulus-free partial spreads $\mathcal{S}_r$ of size $r \geq 3$, namely those labelled IIIa.1; IVa.1, IVb.1; Va.1, Vb.1, Vc.1, Vd.1, Ve.1; VIa.1, VIb.1, VIIa.1, VIIIa.1 (the size r being indicated by the corresponding Roman numeral). [*Note: In the following we will also use the same labels for the corresponding* $\mathrm{GL}(5,2)$-*orbits (classes) of r-caps.*]

Remark 2.2 *The elements m_i of an r-cap $\mathcal{C}_r = \{m_1, \ldots, m_r\}$ which arises as the Grassmann image of a partial spread $\mathcal{S}_r$ are not always linearly independent. A smallest cap $\mathcal{C}_r$ where the m_i are dependent will have $m_\Sigma = m_{12\ldots r} = 0$. Now, since all of the m_{ij} and m_{ijk} are external points, $m_\Sigma \neq 0$ for any $\mathcal{C}_r$ with $r < 5$. For the ten classes of partial spreads $\mathcal{S}_5$ listed in [5] we find that $m_\Sigma = 0$ in just two cases*
(i) class Va.1: $\mathcal{S}_5$ is a spread on a parabolic quadric $\subset \mathrm{PG}(4,2)$;
(ii) class Vj.1: $\mathcal{S}_5$ is a spread on a solid $\subset \mathrm{PG}(4,2)$; see lemma A.1(iv).
If $\mathcal{S}_r, r > 5$, contains a $\mathcal{S}_5 \in$ class Va.1, then accordingly there exist five elements of $\mathcal{C}_r$ which satisfy a linear relation. (No $\mathcal{S}_r, r > 5$, contains a $\mathcal{S}_5 \in$ class Vj.1: a spread on a solid $\subset \mathrm{PG}(4,2)$ is clearly a maximal partial spread in $\mathrm{PG}(4,2)$.) In fact, with one exception, *linear relations for a $\mathcal{C}_r$ with $r > 5$ exist only if $\mathcal{S}_r$ is an extension of a spread on a parabolic quadric. The exception occurs for $r = 8$ with $\mathcal{S}_8$ regulus-free, of class VIIIa.1; for this case the eight elements of $\mathcal{C}_8$ satisfy, see [12, Lemma 3.1], the single linear relation $m_\Sigma = 0$.*

Lemma 2.3 *If the 3-cap $\mathcal{C}_3$ arises from a partial spread $\mathcal{S}_3$ then $\mathcal{E}(\mathcal{C}_3)$ is an external line which is special or general according as $\mathcal{S}_3$ is a regulus or a non-regulus .*

Proof. The three points m_{12}, m_{13}, m_{23} form an external line L. If $\mathcal{S}_3$ is a regulus lying in a solid σ, each of $\mathrm{im}(m_{ij}) = \sigma$, so each $m_{ij} \in \mathcal{W}(\sigma)$. If $\mathcal{S}_3$ is a non-regulus then the $\mathrm{im}(m_{ij})$ are three distinct solids. ■

Lemma 2.4 *An external flat X contains at most one special line.*

Proof. Suppose to the contrary that X contains two distinct special lines, $L \subset \mathcal{W}(\sigma)$ and $L' \subset \mathcal{W}(\sigma')$. Then $\sigma \neq \sigma'$, since $\mathcal{W}(\sigma)$ contains no planes. Consider the plane $\alpha = \sigma \cap \sigma' \subset \mathrm{PG}(4,2)$. Recall (A.2) and note that the three polar points α^b, $b \in L$, form a line $\lambda \subset \alpha$, and the three polar points $\alpha^{b'}$, $b' \in L'$, form a line $\lambda' \subset \alpha$. Consequently there exists $b \in L$ and $b' \in L'$ such that $\alpha^b = \alpha^{b'}$, whence, see [10, proof of lemma 2.3], $\langle b, b' \rangle$ is a tangent — contradicting the assumption that X is external. ■

As an immediate consequence of these last two lemmas we have: *for $\mathcal{E}(\mathcal{C}_r)$ to be an external flat it is necessary for the partial spread $\mathcal{S}_r$ to contain at most one regulus, that is to be of type* O *or of type* I.

First let us look at those $\mathcal{E}(\mathcal{C}_r), r \geq 2$, for which $\mathcal{S}_r$ is regulus-free, such $\mathcal{E}(\mathcal{C}_r)$ being the only candidates for a general flat.

Theorem 2.5 *Given a regulus-free partial spread $\mathcal{S}_r, r \geq 2$, in* $\mathrm{PG}(4,2)$, *let $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{even}$ be defined as previously. Then $\mathcal{E}(\mathcal{C}_r)$ is an external flat if and only if one of the following holds:*

(o) $r = 2$; $\mathcal{E}(\mathcal{C}_2)$ is an external point;
(i) $r = 3$ and $\mathcal{S}_3 \in$ class IIIa.1; $\mathcal{E}(\mathcal{C}_3)$ is a general line;
(ii) $r = 4$ and $\mathcal{S}_4 \in$ class IVb.1; $\mathcal{E}(\mathcal{C}_4)$ is a general plane;
(iii) $r = 5$ and $\mathcal{S}_5 \in$ class Ve.1; $\mathcal{E}(\mathcal{C}_5)$ is a general solid.

Proof. A straightforward check of the 12 regulus-free classes (for $r \geq 3$) described in [5, Table B.2] reveals that $\mathcal{E}(\mathcal{C}_r)$ is an external flat only for the four listed cases (case (i) being that of the non-reguli of lemma 2.3). For example, if $\mathcal{S}_4$ is in class IVa.1, then we find that $\mathcal{E}(\mathcal{C}_4)$ is a plane which is not external, since it is tangent to Rk_2 at the point $m_\Sigma = m_{1234}$. ■

Remark 2.6 *If $\mathcal{E}(\mathcal{C}_5)$ is a general solid, then it follows that $\mathcal{E}(\mathcal{C}_4)$ is a general plane for each 4-subset $\mathcal{C}_4$ of $\mathcal{C}_5$. Note that the properties of partial spreads fit in nicely with cases (ii) and (iii) in the theorem: for of the partial spreads $\mathcal{S}_5$ of size 5, only those of class Ve.1, see [5, lead-in to eq.(3.24)], have the property that $S_5 \setminus \{\mu_i\}$ is of class IVb.1 for each $\mu_i \in S_5$.*

A description of the three classes IIIa.1, IVb.1 and Ve.1 of regulus-free partial spreads, and of their stabilizer groups, is given in appendix A.2.

Remark 2.7 *For $k = 1, 2, 3$ the distinct* $\mathrm{GL}(5,2)$-*orbits of external k-flats in* $\mathrm{PG}(9,2)$ *will be denoted* $\mathrm{orb}(k\alpha)$, $\mathrm{orb}(k\beta)$, ... *Since each of the classes IIIa.1, IVb.1, Ve.1 of partial spreads is a* $\mathrm{GL}(5,2)$-*orbit, the general flats in parts (i), (ii) and (iii) of the theorem form single* $\mathrm{GL}(5,2)$-*orbits. In each case let us agree to label the orbit as the alpha orbit: (i) $\mathcal{E}(\mathcal{C}_3)$* $\in \mathrm{orb}(1\alpha)$; *(ii) $\mathcal{E}(\mathcal{C}_4)$* $\in \mathrm{orb}(2\alpha)$; *(iii) $\mathcal{E}(\mathcal{C}_5)$* $\in \mathrm{orb}(3\alpha)$.

Concerning case (o) in the theorem, every external point arises in this way, since Rk_4 is a single $\mathrm{GL}(5,2)$-orbit. Moreover, see eq. (A.1), for any external point b we have $\mathcal{E}(\mathcal{C}_2) = \langle b \rangle$ for precisely ten choices of 2-cap $\mathcal{C}_2$. Recall also that $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$. For cases (i) and (ii), the corresponding facts are as described in the next two theorems.

Theorem 2.8 *([10]) (i) All general lines belong to the orbit* $\mathrm{orb}(1\alpha)$.

(ii) A general line L can be expressed as $L = \mathcal{E}(\mathcal{C}_3)$ for precisely two choices of non-regulus $\mathcal{S}_3$. If $\mathcal{S}_3$ is one choice, then its twin $\mathcal{S}_3^*$, see appendix A.2(i), is the only other choice.

(iii) The stabilizer $\mathcal{G}_L$ of L is of order 96, and contains, as a subgroup of index 2, the common stabilizer $\mathcal{G}_{\mathcal{S}_3} = \mathcal{G}_{\mathcal{S}_3^*} \cong \mathrm{Sym}(4) \times Z_2$ *of $\mathcal{S}_3$ and $\mathcal{S}_3^*$.*

Proof. (i), (ii) See [10, Theorem 3.6], where use is made of lemma 1.4. (iii) Use lemma A.3. ■

Theorem 2.9 *([10]) (i) A plane $P \in \mathrm{orb}(2\alpha)$ can be expressed in the form $P = \mathcal{E}(\mathcal{C}_4)$ of theorem 2.5(ii) for a uniquely determined 4-cap $\mathcal{C}_4$ of class IVb.1.*

(ii) The stabilizer $\mathcal{G}_P$ of P is isomorphic to $\mathrm{Sym}(3)$. *In its action upon P, the group $\mathcal{G}_P$ fixes a privileged antiflag (b, L) in P; equivalently worded, $\mathcal{G}_P$ fixes a privileged conic whose nucleus is b and whose external line is L.*

Proof. Omitted, see [10]. Concerning $\mathcal{G}_P \cong \mathrm{Sym}(3)$, this follows from $\mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(3)$, see appendix A.2(ii), granted the bijective correspondence in part (i). ■

Next let us look at those $\mathcal{E}(\mathcal{C}_r), r \geq 3$, for which $\mathcal{S}_r$ is of type I, such $\mathcal{E}(\mathcal{C}_r)$ being the only even hyperplanes which are candidates for a special flat. Corresponding to theorem 2.5 we obtain the next theorem, with case (i) being that of the reguli of lemma 2.3.

Theorem 2.10 *Given a partial spread $\mathcal{S}_r, r \geq 3$, in* $\mathrm{PG}(4, 2)$ *of type I, let $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{even}$ be defined as previously. Then $\mathcal{E}(\mathcal{C}_r)$ is an external flat if and only if one of the following holds:*

(i) $r = 3$ and $\mathcal{S}_3 \in$ class IIIb.1; $\mathcal{E}(\mathcal{C}_3)$ is a special line;

(ii) $r = 4$ and $\mathcal{S}_4 \in$ class IVc.1; $\mathcal{E}(\mathcal{C}_4)$ is a special plane;

(iii) $r = 5$ and $\mathcal{S}_5 \in$ class Vg.1; $\mathcal{E}(\mathcal{C}_5)$ is a special solid.

Since each of the classes IIIb.1, IVc.1, Vg.1 of partial spreads is a $\mathrm{GL}(5, 2)$-orbit, the special flats in parts (i), (ii) and (iii) of the theorem form single $\mathrm{GL}(5, 2)$-orbits. *In each case let us agree to label the orbit as the beta orbit:* (i) $\mathcal{E}(\mathcal{C}_3) \in \mathrm{orb}(1\beta)$; (ii) $\mathcal{E}(\mathcal{C}_4) \in \mathrm{orb}(2\beta)$; (iii) $\mathcal{E}(\mathcal{C}_5) \in \mathrm{orb}(3\beta)$.

Theorem 2.11 *If $\mathcal{S}$ is a partial spread in* $\mathrm{PG}(4, 2)$, *and if $\mathcal{C}$ is the corresponding cap on* Rk_2, *then $\mathcal{E}(\mathcal{C})$ is an external flat if and only if $\mathcal{S}$ is in one of the following classes: IIa.1, IIIa.1, IIIb.1, IVb.1, IVc.1, Ve.1, Vg.1.*

Proof. Immediate, from lemmas 2.3, 2.4 and theorems 2.5 and 2.10. ■

3 External 4-flats and their $N(Z_{31})$-construction

Consider a 5-cap $\mathcal{C} = \{m_1, \ldots, m_5\}$ which arises from a partial spread $\mathcal{S}$ of class Ve.1. Then, see eq. (A.10), $\mathcal{G}_{\mathcal{C}} = \langle B\rangle \cong Z_5$. Now $\mathcal{G}_{\mathcal{C}}$ determines (i) an antiflag (u, σ) in $\mathrm{PG}(4, 2)$, the unique antiflag which is fixed by B, and (ii) a special line $L = \{p, p', p''\} \subset \mathcal{W}(\sigma)$, where p, p' and p'' are the 3 fixed points of $T_B = \wedge^2 B$ in $\mathrm{PG}(9, 2)$. Since $m_\Sigma = \sum_{i=1}^{5} m_i$ is clearly fixed by T_B, we may as well take p to be m_Σ.

In the next theorem (see [10] for its proof), we consider the extensions to external 4-flats of the general solid $D = \mathcal{E}(\mathcal{C})$ of theorem 2.5(iii). Recall, remark 2.7, that such solids form the single orbit, $\mathrm{orb}(3\alpha)$.

Theorem 3.1 *([10]) (i) The solid $D = \mathcal{E}(\mathcal{C})$ has precisely two extensions E' and E'' to an external 4-flat, namely*

$$E' = \langle p', D\rangle \qquad \text{and} \qquad E'' = \langle p'', D\rangle. \tag{3.1}$$

(ii) Each $D \in \mathrm{orb}(3\alpha)$ can be expressed as $D = \mathcal{E}(\mathcal{C})$ for a uniquely determined 5-cap $\mathcal{C}$ of class Ve.1.

(iii) (a) $\mathcal{G}_D = \mathcal{G}_{\mathcal{C}} = \langle B\rangle \cong Z_5$; (b) $\mathcal{G}_D \subseteq \mathcal{G}_{E'} \cap \mathcal{G}_{E''}$.

(iv) The 5-flat $F = \langle E', E''\rangle = \langle L, D\rangle$ also has stabilizer $\mathcal{G}_F = \langle B\rangle \cong Z_5$.

Theorem 3.2 *Let E be any external 4-flat and let F be any 5-flat which contains E. Put $\mathcal{C}_F = F \cap \mathrm{Rk}_2$, and let $\mathcal{S}_F$ be the corresponding line-set in* $\mathrm{PG}(4, 2)$. *Then*

(i) $\mathcal{C}_F$ is a cap, and $\mathcal{S}_F$ is a regulus-free partial spread;

(ii) if $F \neq F'$ then $\mathcal{C}_F \cap \mathcal{C}_{F'} = \emptyset$;

(iii) $\{\mathcal{S}_F \,|\, F$ a 5-flat $\supset E\}$ is a partition of the 155 lines of $\mathrm{PG}(4, 2)$*;*

(iv) for each 5-flat $F \supset E$, $|\mathcal{C}_F| = 5$;

(v) $\mathcal{S}_F$ is of class Ve.1 and $\mathcal{G}_{\mathcal{C}_F} \cong Z_5$.

Proof. This is proved in [10]. (In that reference an alternative proof of part (iv) is given which provides the further information that $|F \cap W(\sigma)| = 1, 2, 3$ for respectively 10, 15, 6 solids $\sigma \subset \mathrm{PG}(4, 2)$.) ■

Theorem 3.3 *If E is any external 4-flat, its stabilizer $\mathcal{G}_E$ is the normalizer $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of* $\mathrm{GL}(5, 2)$.

Proof. By theorems 2.5 and 3.2 each of the 31 5-flats $F \supset E$ gives rise to a solid $D_F := \langle \mathcal{C}_F\rangle \cap E = \mathcal{E}(\mathcal{C}_F) \in \mathrm{orb}(3\alpha)$. The 31 5-caps $\mathcal{C}_F$ are distinct, and so by theorem 3.1(ii) the 31 solids $D_F \subset E$ are also distinct. By theorem 3.1(iiia), $\mathcal{G}_{D_F} = \langle B_F\rangle \cong Z_5$; furthermore these 31 stabilizers $\mathcal{G}_{D_F}$ are distinct, since T_{B_F} and $T_{B_{F^*}}$ fix the distinct (if $F \neq F^*$) hyperplanes D_F and D_{F^*} of E. Consequently, by theorem 3.1(iiib), $\mathcal{G}_E$ contains the 31 distinct Z_5-subgroups $\mathcal{G}_{D_F}$.

Now, see [1, p.70], there are just three classes, say $\mathfrak{C}_i$, $i = 1, 2, 3$, of maximal subgroups of $\mathrm{GL}(5, 2)$ which contain elements of order 5. A subgroup $\mathcal{G} \in \mathfrak{C}_1$ stabilizes a point of $\mathrm{PG}(4, 2)$, and a subgroup $\mathcal{G} \in \mathfrak{C}_2$ stabilizes a solid of $\mathrm{PG}(4, 2)$. But no common point or solid is stabilized by all the foregoing thirty-one Z_5-subgroups of $\mathcal{G}_E$. Hence $\mathcal{G}_E$ lies inside a group $\mathcal{G} \in \mathfrak{C}_3$. But such a $\mathcal{G}$ is the normalizer $\mathcal{N} = N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of some Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5, 2)$. But the only subgroup of $\mathcal{N}$ which contains more than one Z_5 is $\mathcal{N}$ itself (which indeed contains 31 distinct Z_5 subgroups). Hence the theorem. ■

Theorem 3.4 *Suppose that the k-flat X, $k = 1, 2, 3$, possesses an extension to an external 4-flat E; then X belongs to the orbit* $\mathrm{orb}(k\alpha)$.

Proof. Let E be an external 4-flat and let $\mathcal{C} = \{m_1, \dots, m_5\}$ be one of the 5-caps $\mathcal{C}_F$ of theorem 3.2. Then, for $k = 1, 2, 3$, E certainly contains a k-flat $X_k \in \mathrm{orb}(k\alpha)$, namely $X_k = \langle m_1, \dots, m_k\rangle_{\mathrm{even}}$. But, by theorem 3.3, $\mathcal{G}_E$ is the normalizer $N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5, 2)$, and such a group acts transitively on (i) the 155 lines of E, (ii) the 155 planes of E, (iii) the 35 solids of E. (The $k = 1$ result is otherwise immediate, since the general lines form the single orbit $\mathrm{orb}(1\alpha)$.) ■

In the rest of this section, where we deal with *the $N(Z_{31})$-construction of external 4-flats,* we will find it convenient to adopt the following notation: if X is a projective r-flat, then we use $\overline{X}$ for the $(r + 1)$-dimensional vector space such that $X = \mathbb{P}\overline{X}$. For example, if the flats X, Y are disjoint, then $\wedge^2 V_5 = \overline{X} \oplus \overline{Y}$ is the same thing as $\mathrm{PG}(9, 2) = \langle X, Y\rangle$.

Given any subgroup $\mathcal{Z} \cong Z_{31}$ of $\mathrm{GL}(5, 2)$, let $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ be its normalizer. Consider the representation T of $\mathcal{N}$ upon $V_{10} = \wedge^2 V_5$ given by $T_C = \wedge^2 C$, $C \in \mathcal{N}$. Since $2 \nmid |\mathcal{N}|$, Maschke's theorem applies, and so the representation T is completely reducible. Now no one-dimensional subspace of V_{10} is $\mathcal{N}$-invariant, and only for $n \geq 5$ does $\mathrm{GL}(n, 2)$ possess a subgroup $\cong Z_{31}$. Hence the 10-dimensional representation T of $\mathcal{N}$ decomposes as the direct sum $T_+ \oplus T_-$ of two 5-dimensional irreducible representations T_+ and T_-, arising from some $\mathcal{N}$-invariant direct sum decomposition $\overline{E_+} \oplus \overline{E_-}$ of V_{10}. Moreover, since Rk_2 is a single $T_{\mathcal{N}}$-orbit, note that both of the 4-flats E_+, E_- are external.

Lemma 3.5 *The external 4-flats E_+, E_- belong to different* $\mathrm{GL}(5, 2)$*-orbits.*

Proof. Suppose there exists $P \in \mathrm{GL}(5, 2)$ such that T_P maps E_+ onto E_-. It follows that $P\mathcal{G}_{E_+}P^{-1} = \mathcal{G}_{E_-}$, that is $P\mathcal{N}P^{-1} = \mathcal{N}$. But $\mathcal{N}$, being maximal, is its own normalizer, whence $P \in \mathcal{N}$; so T_P stabilizes E_+, contradicting the supposition that T_P maps E_+ onto E_-. ■

Theorem 3.6 *The external 4-flats comprise precisely two* $\mathrm{GL}(5, 2)$*-orbits, say* $\mathrm{orb}(4+)$ *and* $\mathrm{orb}(4-)$. *The stabilizer $\mathcal{G}_E$ of any external 4-flat E has order* 155 *and structure* $\cong Z_{31} \rtimes Z_5$.

Proof. We know from lemma 3.5 that there exist *at least* two orbits, say orb(4+) and orb(4−), of external 4-flats. But, theorem 3.4, those solids which extend to an external 4-flat form a single orbit, and so it follows from theorem 3.1(i) that there exist *at most* two orbits. Hence there are precisely two orbits, orb(4+) and orb(4−). The stabilizer $\mathcal{G}_E$ of any external 4-flat E is as described in theorem 3.3. ■

Remark 3.7 *(i) It follows from the theorem that the 5-dimensional irreducible representations* T_+, T_- *of* $\mathcal{N}$ *just obtained are inequivalent. For otherwise the decomposition* $\overline{E_+} \oplus \overline{E_-}$ *of* V_{10} *would not be unique and there would be more than 2 orbits of external 4-flats.*

(ii) It is tempting to say that each subgroup $\mathcal{N} \cong Z_{31} \rtimes Z_5$ *gives rise to* twin *external 4-flats* $\{E_+, E_-\}$*. However it is more accurate to say that external 4-flats which share the same stabilizer occur as* non-identical twins $\{E_+, E_-\}$*. Certainly* E_+, E_- can *be distinguished (unlike twin non-reguli), one way being as follows. In* $\mathcal{N}$, *the 30 elements* A *of order 31 fall into 6 conjugacy classes* K_r *of length 5. Letting* r *take the values* $1, 2, \dots, 6 \pmod 6$ *we may label the classes in such a way that* $K_{r+1} = (K_r)^3$*. Let* f_r *be the characteristic polynomial of elements* $A \in K_r$*. By using the fact that* A *is diagonalizable over the extension field* GF(32) *one easily shows that if* $\chi_A = f_r$ *then* $\chi_{T(A)} = f_{r+1}.f_{r+2}$*. So, for* $A \in K_r$, *we may distinguish between* E_+ *and* E_- *by defining* $\overline{E_+} = \ker f_{r+1}(T_A)$ *and* $\overline{E_-} = \ker f_{r+2}(T_A)$.

4 Construction of another kind of general plane

Let $\mathrm{PG}(4,2) = \mathbb{P}V_5$ be generated by a line $\mu = \{u, v, w\} = \mathbb{P}V_2$ and a plane $\alpha = \mathbb{P}V_3$; so $V_5 = V_2 \oplus V_3$. Choose $A_3 \in \mathrm{GL}(V_3)$ to have order 7 and minimal polynomial $t^3 + t + 1$; then, for any choice of $a_0 \in \alpha$, we have $\alpha = \{a_0, a_1, ..., a_6\}$ where $a_i = (A_3)^i a_0$, the seven lines of α being $\lambda_i = \{a_i, a_{i+1}, a_{i+3}\}$, $i = 0, 1, .., 6 \pmod 7$. Choose $C_3 \in \mathrm{GL}(V_3)$ in the normalizer of A_3 to satisfy $C_3 A_3 (C_3)^{-1} = (A_3)^2$ and $C_3 a_0 = a_0$, and note therefore that $C_3 a_i = a_{2i}$. Define also $C_2 \in \mathrm{GL}(V_2)$ to effect the 3-cycle $u \mapsto w \mapsto v \mapsto u$. Setting $A = I_2 \oplus A_3$ and $C = C_2 \oplus C_3$, observe that A and C are elements of $\mathrm{GL}(5,2)$, of orders 7 and 3, which stabilize both μ and α and which generate a subgroup $F_{21} \cong 7:3$ of $\mathrm{GL}(5,2)$.

Theorem 4.1 *Let* $P = \{b_0, b_1, ..., b_6\}$ *where, in the foregoing notation, the seven points* b_i *of* $\mathrm{PG}(9,2)$ *are defined by*

$$b_i = u \wedge a_i + v \wedge a_{i+1}, \quad (i = 0, 1, .., 6 \pmod 7). \tag{4.1}$$

Then P *is a general plane with stabilizer* $\mathcal{G}_P \cong 2^3{:}(7{:}3)$.

Proof. *(Outline)* Noting that $b_i + b_{i+1} = b_{i+3}$, we see that P is a plane. Since $\mathrm{im}(b_i) = \langle \mu, \lambda_i \rangle \neq \mathrm{im}(b_j)$ for $i \neq j$, it follows that P is a general plane.

Next note that $T_A b_i = b_{i+1}$ and $T_C b_i = b_{2i-1}$, whence $\mathcal{G}_P$ contains F_{21}. But for $f \in V_3^*$ each b_i is seen to be fixed by $T_{J(f)}$, where $J(f) \in \mathrm{GL}(5,2)$ is the involution defined by

$$\begin{aligned} &J(f)u = u, \quad J(f)v = v, \\ &J(f)x = x + f(x)u + f(A_3 x)v, \ x \in V_3. \end{aligned} \tag{4.2}$$

Since $AJ(f)A^{-1} = J(\hat{A}f)$ and $CJ(f)C^{-1} = J(\hat{A}\hat{C}f)$, it follows that $\mathcal{G}_P$ contains a subgroup $\mathcal{J} \rtimes F_{21} \cong 2^3 : (7:3)$, where $\mathcal{J} = \{J(f) | \, f \in V_3^*\} \cong (Z_2)^3$. With a little more effort one sees that $\mathcal{G}_P$ is no larger than $\mathcal{J} \rtimes F_{21}$. ■

Since the stabilizer of a plane $\in \mathrm{orb}(2\alpha)$ is isomorphic to Sym(3), the general plane P in the theorem belongs to a distinct orbit, which we will label $\mathrm{orb}(2\gamma)$.

5 The key counting lemma

So far we have obtained the following ten orbits of k-flats in $\mathrm{PG}(9,2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$:
$k = 0$: Rk_4;
$k = 1$: $\mathrm{orb}(1\alpha)$, $\mathrm{orb}(1\beta)$;
$k = 2$: $\mathrm{orb}(2\alpha)$, $\mathrm{orb}(2\beta)$, $\mathrm{orb}(2\gamma)$;
$k = 3$: $\mathrm{orb}(3\alpha)$, $\mathrm{orb}(3\beta)$;
$k = 4$: $\mathrm{orb}(4+)$, $\mathrm{orb}(4-)$.
Moreover, in the cases $k = 0, 1, 4$, but not in the cases $k = 2, 3$, we have proved that this list is complete. In the final sections of this paper we prove the non-existence of further external planes and solids and hence show that *these ten orbits constitute a complete classification of the flats in* $\mathrm{PG}(9,2)$ *which are external to the Grassmannian* $\mathcal{G}_{1,4,2}$. We also provide further information about the beta planes and solids, and give details of their stabilizer groups.

If X is an external k-flat in PG(9,2), we denote by $\Delta(X)$ the set of all those $(k+1)$-flats Y in PG(9,2) which contain X, and we define

$$\mathcal{C}_Y = Y \cap \mathrm{Rk}_2, \qquad \text{for } Y \in \Delta(X). \tag{5.1}$$

Then $\Delta(X)$ is the disjoint union $\Delta(X) = \cup_r \Delta_r(X)$ where, for $r = 0, 1, 2, \dots$,

$$\Delta_r(X) = \{Y \in \Delta(X) | \, |\mathcal{C}_Y| = r\}. \tag{5.2}$$

Lemma 5.1 *If X is an external flat then, for each $Y \in \Delta_r(X)$, $r > 0$, the r-set $\mathcal{C}_Y$ is a r-cap $\{m_1, \dots, m_r\}$ on* Rk_2 *. Moreover the corresponding r-set of lines $\mathcal{S}_Y = \{\mu_1, \dots, \mu_r\}$ is a partial spread in* $\mathrm{PG}(4,2)$.

Proof. Since the field is GF(2), any r-set of points, $r > 0$, of a projective space Y which lies outside a hyperplane $X \subset Y$ is a r-cap. Since X is external, then for any two distinct points m, m' of $\mathcal{C}_Y$ their sum $m+m' \in X$ has rank 4; so any two distinct lines μ, μ' of $\mathcal{S}_Y$ do not intersect, that is $\mathcal{S}_Y$ is a partial spread. (The case $r=1$ is trivial, with $\mathcal{S}_Y$ a partial spread consisting of a single line.) ■

The next lemma is surprisingly powerful, and should perhaps be viewed as *the key counting lemma.* Certainly its use enables us to avoid some rather tedious deliberations, carried out on a case by case basis. In it we put $N_r = N_r(X) = |\Delta_r(X)|$ and, for $i > 0$, we denote by $M_i = M_i(X)$ the number of i-caps $\mathcal{C}_i$ on Rk_2 which satisfy $\mathcal{E}(\mathcal{C}_i) \subseteq X$. We also put $M_0 = |\Delta(X)|$. If X is an external k-flat, take note that $N_0(X)$ *is the number of extensions of* X *to an external* $(k+1)$*-flat.*

Lemma 5.2 *If X is an external k-flat then the relations*

$$\sum\nolimits_{r=i}^{k+2} \binom{r}{i} N_r = M_i \qquad (R_i)$$

hold for $i = 0, 1, \dots, k+2$.

Proof. The relation (R_0) holds since it reads $\Sigma_{r=0}^{k+2}|\Delta_r(X)| = |\Delta(X)|$. If $\mathcal{C}_r$ is an r-cap on Rk_2 such that $\mathcal{E}(\mathcal{C}_r) \subseteq X$ then it contributes $\binom{r}{i}$ to the number M_i of i-caps $\mathcal{C}_i$ on Rk_2 which satisfy $\mathcal{E}(\mathcal{C}_i) \subseteq X$; so (R_i) holds for $i = 1, 2, \dots k+2$. (The upper limit $k+2$ in the summation suffices, since if $\mathcal{E}(\mathcal{C}_i)$ is an external k-flat then $i = k+2$.) ■

This lemma yields the following results for an external k-flat X:

$$\begin{array}{rll} \text{(o)} & k=0: & N_0 - M_2 = 356; \\ \text{(i)} & k=1: & N_0 + M_3 = 130; \\ \text{(ii)} & k=2: & N_0 - M_4 = 42 - M_3; \\ \text{(iii)} & k=3: & N_0 + M_5 = 58 + M_4 - M_3. \end{array} \qquad (5.3)$$

To derive these results, first note that M_0 $(= |\Delta(X)|) = 2^{9-k} - 1$ and $M_1 = |\,\mathrm{Rk}_2\,| = 155$. (The latter holds since every $m \in \mathrm{Rk}_2$ defines a 1-cap $\mathcal{C}_1 = \{m\}$ on Rk_2 which satisfies $\mathcal{E}(\mathcal{C}_1) \subseteq X$ because $\mathcal{E}(\mathcal{C}_1) = \emptyset$.) If $k=0$ then $(R_0) - (R_1) + (R_2)$ yields result (o). This $k=0$ information is not new; for, see after lemma 1.4, we already know that through an external point there pass $M_2 = 10$ bisecants and $N_0 = 366$ external lines (the latter consisting of 6 special lines and 360 general lines). Concerning the cases $k > 0$, observe that $M_2 = 10(2^{k+1} - 1)$, since 10 bisecants pass through each point of an external k-flat X. From the relations $(R_0), (R_1), \dots, (R_{k+2})$ in the key counting lemma the combination $(R_0) - (R_1) + \dots + (-1)^{k+1}(R_{k+2})$ then yields the results (i), (ii), (iii). These latter will be made use of in succeeding sections.

6 Extensions of external lines to external planes

6.1 Special lines and special planes

Lemma 6.1 *Let* $L = \{b_1, b_2, b_3\} \subset \mathcal{W}(\sigma)$ *be a special line, and let* b' *be an element of* $\mathcal{W}(\sigma')$*, where* $\sigma' \neq \sigma$*. Then at most one of the lines* $\langle b', b_i \rangle$ *is tangent to* Rk_2*.*

Proof. Suppose to the contrary that $m_i := b' + b_i$ lies in Rk_2 for two values of i, say $i = 1$ and $i = 2$. But then $b_3 = b_1 + b_2 = m_1 + m_2$, whence m_1 and m_2 lie in $\mathcal{H}(\sigma)$. It would then follow that b', the sum of elements b_1, m_1 of $\Pi(\sigma)$, also lies in $\Pi(\sigma)$, contradicting b' being in $\mathcal{W}(\sigma')$, where $\sigma' \neq \sigma$. ∎

The next theorem asserts that *special planes* (that is those external planes which contain a special line) *form a single orbit,* $\mathrm{orb}(2\beta)$. The proof, which makes use of the results in lemmas 1.4 and 6.1, is omitted, but can be found in [11].

Theorem 6.2 *([11]) (i) An external line* $L \in \mathrm{orb}(1\beta)$ *possesses precisely 120 extensions to an external plane.*

(ii) External planes containing a special line form a single orbit $\mathrm{orb}(2\beta)$*.*

(iii) If $P \in \mathrm{orb}(2\beta)$ *then* $\mathcal{G}_P$ *is of order* 48 *and has for its centre a subgroup* $\mathcal{G}_P^{\bullet} \cong Z_2$ *which fixes* P *pointwise. The quotient group* $\mathcal{G}_P/\mathcal{G}_P^{\bullet}$ *is isomorphic to* $\mathrm{Sym}(4)$*, and acts faithfully on* P*.*

6.2 Extensions of a line $L \in \mathrm{orb}(1\alpha)$

Lemma 6.3 *If* L *is a general line then it has precisely 112 extensions to a plane* $P \in \mathrm{orb}(2\alpha)$*, precisely 12 extensions to a plane* $P \in \mathrm{orb}(2\beta)$*, and precisely 4 extensions to a plane* $P \in \mathrm{orb}(2\gamma)$*.*

Proof. For $\xi = \alpha, \beta, \gamma$, let n_ξ be the number of extensions of $L \in \mathrm{orb}(1\alpha)$ to a plane $P_\xi \in \mathrm{orb}(2\xi)$, and let h_ξ be the number of lines $L \in \mathrm{orb}(1\alpha)$ contained in a particular plane $P_\xi \in \mathrm{orb}(2\xi)$. By double-counting of such pairs (L, P_ξ) we have $(|\,\mathrm{GL}(5,2)|/|\mathcal{G}_L|) \times n_\xi = (|\,\mathrm{GL}(5,2)|/|\mathcal{G}_{P_\xi}|) \times h_\xi$, whence $n_\xi = (96 \times h_\xi)/|\mathcal{G}_{P_\xi}|$. Now $h_\alpha = 7,\ h_\beta = 6,\ h_\gamma = 7$ and $\mathcal{G}_{P_\xi}$ is of order $6, 48, 168$ according as $\xi = \alpha, \beta, \gamma$. Hence $n_\alpha = 112,\ n_\beta = 12,\ n_\gamma = 4$. ∎

Theorem 6.4 *There are precisely three orbits of external planes, namely* $\mathrm{orb}(2\alpha)$*,* $\mathrm{orb}(2\beta)$ *and* $\mathrm{orb}(2\gamma)$*.*

Proof. For an external line L we have from eq. (5.3)(i) the result $N_0(L) + M_3(L) = 130$. But if L is a general line then it can be expressed as $\mathcal{E}(\mathcal{C}_3)$ for precisely two choices of $\mathcal{C}_3$ (corresponding to a pair of twin non-reguli $\mathcal{S}_3, \mathcal{S}_3^*$); so $M_3(L) = 2$. It follows that $N_0(L) = 128$, that is *L has precisely 128 extensions to an external plane.* But by lemma 6.3 these 128 extensions are completely accounted for by the $112 + 12 + 4$ extensions to planes belonging to the orbits orb(2α), orb(2β) and orb(2γ). The theorem follows from this, since, by lemma 2.4, every external plane arises as the extension of a general line. ■

7 Extensions of external planes to external solids

Of the external planes those belonging to orb(2γ) have the most symmetry, and so we look at this orbit first of all.

7.1 Extensions of a plane $P \in \mathrm{orb}(2\gamma)$

Let the notation be as in section 4 and consider the plane $P \in \mathrm{orb}(2\gamma)$ of theorem 4.1. As an example of a special solid consider the solid $D_0 = \langle c_0, P\rangle$, where $c_0 = a_0 \wedge u + a_3 \wedge a_6 \in \mathrm{Rk}_4$. Then one checks that $c_0 + b_i \in \mathrm{Rk}_4$ for each i, so that D_0 is an external solid. One also finds that D_0 contains a special line, namely the line

$$L_0 = \{b_3, c_0 + b_2, c_0 + b_5\} \subset \mathcal{W}(\sigma_0), \qquad \text{where } \sigma_0 = \langle \lambda_3, \mu\rangle; \tag{7.1}$$

so $D_0 = \langle c_0, P\rangle$ is a special solid. Let us assign D_0 to an orbit orb$(3\beta')$. We will soon show that this orbit coincides with the orbit orb(3β).

We also find that P is the only plane of D_0 which belongs to orb(2γ). To see this, note that the seven kernels, see eq. (1.3), of the points of a plane $\in$ orb(2γ) themselves form a plane in the dual space PG$(4,2)^*$. (For P as in (4.1) the seven kernels $\ker b_i$ form the plane $\mu^{\bigcirc}$, the annihilator of $\mu = \langle u, v\rangle$.) A check on the 13 distinct kernels of the 15 points of $D_0 = \langle c_0, P\rangle$ reveals that no plane other than $\mu^{\bigcirc}$ is present.

Observe that in addition to the unique plane $P \in \mathrm{orb}(2\gamma)$, the solid D_0 contains three planes $\in$ orb(2β), namely those through the unique special line L_0; hence P contains precisely eleven planes of orb(2α).

Finally observe that, under the action of the Z_7-subgroup $\langle A\rangle \subset \mathcal{G}_P$, the external solid D_0 is just one of a septuplet $\{D_0, D_1, \ldots, D_6\}$ of external solids $D_i = (T_A)^i D_0 = \langle c_i, P\rangle$, where $c_i = a_i \wedge u + a_{i+3} \wedge a_{i+6}$. For later use it should be noted that the seven points c_i all lie in $\mathcal{W}(\langle u, \alpha\rangle)$. Of course the unique special line of D_i is $L_i := (T_A)^i(L_0)$, lying inside $\mathcal{W}(\sigma_i)$, where $\sigma_i = \langle \lambda_{i+3}, \mu\rangle$.

Lemma 7.1 *A plane $P \in \mathrm{orb}(2\gamma)$ has precisely 28 extensions to an external solid.*

Proof. Apply the result (5.3)(ii) to P. Recall, see theorem 2.11, that those external planes of the form $\mathcal{E}(\mathcal{C}_4)$ belong either to orb(2α) or to orb(2β). So $M_4(P) = 0$. Moreover $M_3(P) = 2 \times 7 = 14$, since all 7 lines of P are general lines. Hence $N_0(P) = 28$. ■

Consider the plane $P \in \mathrm{orb}(2\gamma)$ as given in (4.1). Under the action of $\mathcal{G}_P \cong 2^3 : (7:3)$ the 31 solids of PG$(4,2)$ fall into two orbits Ω_7 and Ω_{24}, where Ω_7 consists of the 7 solids through the line $\mu = \langle u, v\rangle$ and Ω_{24} consists of the 24 solids which meet μ in a point. (Dually, the 31 points of PG(4,2) fall into the two orbits $\Omega_3 = \mu$ and $\Omega_{28} = \mu^c$). Consider extensions of P to external solids D which belong to the same orbit, $\mathrm{orb}(2\beta')$, as the seven solids $D_i = \langle c_i, P\rangle$ discussed in the preceding section. As noted at the end of the last section, these seven external solids D_i alone make use of seven distinct points c_i of $\mathcal{W}(\sigma)$, where $\sigma = \langle u, \alpha\rangle \in \Omega_{24}$. By the transitivity of $\mathcal{G}_P$ on Ω_{24} it follows that *at least* seven points of each of the 24 $\mathcal{W}(\sigma)$, $\sigma \in \Omega_{24}$ lie on some external solid $D \in \mathrm{orb}(3\beta)$. But for such an external solid D we have $\mathrm{im}\, d \in \Omega_7$ for 9 elements $d \in D$ (namely for $d \in P \cup L_i$, for D_i as in the last section) and $\mathrm{im}\, d \in \Omega_{24}$ for the other 6 elements of D. Consequently there are at least $7 \times 24 \div 6 = 28$ extensions of the plane P to an external solid belonging to $\mathrm{orb}(3\beta')$.

Theorem 7.2 *(i) The 28 extensions of a plane* $P \in \mathrm{orb}(2\gamma)$ *to an external solid form a single* $\mathcal{G}_P$*-orbit.*

(ii) All such extensions belong to the $\mathrm{GL}(5,2)$*-orbit* $\mathrm{orb}(3\beta)$.

(iii) If $D \in \mathrm{orb}(3\beta)$ *then* $\mathcal{G}_D \cong Z_6$.

Proof. (i) By the lead-in to the theorem and lemma 7.1.

(ii) By inspecting the kernels for a solid $D_{3\beta} = \mathcal{E}(\mathcal{C}_5) \in \mathrm{orb}(3\beta)$ we find 7 which form a plane, and so $D_{3\beta}$ contains a plane $\in \mathrm{orb}(2\gamma)$. Hence $\mathrm{orb}(3\beta') = \mathrm{orb}(3\beta)$.

(iii) Since $D \in \mathrm{orb}(3\beta)$ contains a unique plane $P \in \mathrm{orb}(2\gamma)$, we have $\mathcal{G}_D < \mathcal{G}_P$. From part (i), $|\mathcal{G}_D| = |\mathcal{G}_P| \div 28 = 6$. For the solid D_0 we may check that $\mathcal{G}_{D_0}$ contains the element A^2CA^{-2} of order 3 and also the involution $J(f)$, where $f = 0$ is the line $\lambda_2 = \{a_2, a_3, a_5\}$. ■

7.2 Extensions of alpha and beta planes

Lemma 7.3 *A plane* $P \in \mathrm{orb}(2\alpha)$ *has precisely 29 extensions to an external solid, comprising 18 extensions to a solid in* $\mathrm{orb}(3\alpha)$ *and 11 extensions to a solid in* $\mathrm{orb}(3\beta)$.

Proof. Apply the result $N_0(P) - M_4(P) = 42 - M_3(P)$ of (5.3)(ii). Recall that $P = \mathcal{E}(\mathcal{C}_4)$ for a unique 4-cap on Rk$_2$; so $M_4(P) = 1$. Moreover $M_3(P) = 2 \times 7 = 14$, since all 7 lines of P are general lines. Hence $N_0(P) = 29$. Since $|\mathcal{G}_P| = 6$, then by double-counting, as in the proof of lemma 6.3, it follows that P has n_α extensions to a solid $D_{3\alpha}$ in $\mathrm{orb}(3\alpha)$, where $n_\alpha = (6 \times 15)/|\mathcal{G}_{3\alpha}| = 18$. Similarly, since a solid in $\mathrm{orb}(3\beta)$ contains 11 planes of $\mathrm{orb}(2\alpha)$, the plane P has n_β extensions to a solid $D_{3\beta}$ in $\mathrm{orb}(3\beta)$, where $n_\beta = (6 \times 11)/|\mathcal{G}_{3\beta}| = 11$. ■

Lemma 7.4 *If D is an external solid then D contains at least one (in fact at least eleven) planes in* $\mathrm{orb}(2\alpha)$.

Proof. Any extension of a plane $P \in \mathrm{orb}(2\gamma)$ contains, see section 7.1, no further planes in $\mathrm{orb}(2\gamma)$; so D contains at most one plane in $\mathrm{orb}(2\gamma)$. Since D contains at most one special line, D contains at most three planes in $\mathrm{orb}(2\beta)$. Hence D contains at least eleven planes in $\mathrm{orb}(2\alpha)$. ■

Theorem 7.5 *There are just two* $\mathrm{GL}(5,2)$*-orbits of external solids, with the general solids forming a single orbit* $\mathrm{orb}(3\alpha)$, *and the special solids forming a single orbit,* $\mathrm{orb}(3\beta)$.

Proof. By lemma 7.4 an external solid D is an extension of a plane $P \in \mathrm{orb}(2\alpha)$. Hence by lemma 7.3 either $D \in \mathrm{orb}(3\alpha)$ or $D \in \mathrm{orb}(3\beta)$. ■

Lemma 7.6 *A plane $P \in \mathrm{orb}(2\beta)$ has precisely 24 extensions to an external solid, all belonging to* $\mathrm{orb}(3\beta)$.

Proof. By theorem 7.5 all such extensions belong to $\mathrm{orb}(3\beta)$. By double-counting pairs $(P, D_{3\beta})$ with $P \subset D_{3\beta} \in \mathrm{orb}(3\beta)$, we see that P has $(|\mathcal{G}_P| \times 3)/|\mathcal{G}_{3\beta}| = (48 \times 3)/6 = 24$ extensions to a solid in $\mathrm{orb}(3\beta)$. ■

8 Summary and further results

Having now proved theorems 6.4 and 7.5, we have achieved our goal, namely a complete classification of external flats:

Theorem 8.1 *Under the action of* $\mathrm{GL}(5,2)$ *there are precisely ten orbits of flats in* $\mathrm{PG}(9,2)$ *which are external to the Grassmannian* $\mathcal{G}_{1,4,2}$, *namely those listed in Table 1 below.*

Table 1: $|\mathcal{G}_{X_k}|$, $N_0(X_k)$ and $M_{k+2}(X_k)$

k	orbit of X_k	$\lvert\mathcal{G}_{X_k}\rvert$	$N_0(X_k)$	$M_{k+2}(X_k)$
0	Rk_4	11520	366	$M_2 = 10$
1	$\mathrm{orb}(1\alpha)$	96	128	$M_3 = 2$
	$\mathrm{orb}(1\beta)$	5760	120	$M_3 = 10$
2	$\mathrm{orb}(2\alpha)$	6	29	$M_4 = 1$
	$\mathrm{orb}(2\beta)$	48	24	$M_4 = 4$
	$\mathrm{orb}(2\gamma)$	168	28	$M_4 = 0$
3	$\mathrm{orb}(3\alpha)$	5	2	$M_5 = 1$
	$\mathrm{orb}(3\beta)$	6	0	$M_5 = 3$
4	$\mathrm{orb}(4+)$	155	0	$M_6 = 0$
	$\mathrm{orb}(4-)$	155	0	$M_6 = 0$

Concerning column 4, recall that $N_0(X_k)$ is the number of extensions of the k-flat X_k to an external $(k+1)$-flat. Concerning column 5, we have still to obtain the entries for the three orbits $\mathrm{orb}(k\beta)$, $k = 1, 2, 3$. This we will do in theorem 8.2 below. (For the entry $M_2 = 10$, see before eq. (A.1).)

Theorem 8.2 *(i) A line* $L \in \mathrm{orb}(1\beta)$ *can be expressed in the form* $L = \mathcal{E}(\mathcal{C}_3)$ *for precisely ten choices of partial spread* $\mathcal{S}_3$.

(ii) A plane $P \in \mathrm{orb}(2\beta)$ *can be expressed in the form* $P = \mathcal{E}(\mathcal{C}_4)$ *for precisely four choices of partial spread* $\mathcal{S}_4$.

(iii) A solid $D \in \mathrm{orb}(3\beta)$ *can be expressed in the form* $D = \mathcal{E}(\mathcal{C}_5)$ *for precisely three choices of partial spread* $\mathcal{S}_5$.

Proof. (i) From theorem 6.2(i) we have $N_0(L) = 120$, and so from (5.3)(i) it follows that $M_3(L) = 10$, that is that each special line L can be expressed in the form $L = \mathcal{E}(\mathcal{C}_3)$ for precisely ten choices of regulus $\mathcal{S}_3$.

(ii) Apply the result $N_0 - M_4 = 42 - M_3$ of (5.3ii) to P. By lemma 7.6 we have $N_0 = 24$; moreover, since P contains one special line and six general lines, we have $M_3(P) = 10 \times 1 + 2 \times 6 = 22$. Hence $M_4 = 4$.

(iii) Apply the result $N_0 + M_5 = 58 + M_4 - M_3$ of (5.3iii) to D. Since D contains one special line and 34 general lines, we have $M_3(D) = 10 \times 1 + 2 \times 34 = 78$. Since D contains three planes $\in \mathrm{orb}(2\beta)$, one plane $\in \mathrm{orb}(2\gamma)$ and eleven planes $\in \mathrm{orb}(2\alpha)$ we have $M_4(D) = (4 \times 3) + (0 \times 1) + (1 \times 11) = 23$. Moreover $N_0(D) = 0$. Hence $M_5 = 58 + 23 - 78 = 3$. ■

8.1 Further comments

(i) See [11, Appendix] for more details concerning the $M_{k+2}(X_k)$ entries in Table 1 in the cases $X_k \in \mathrm{orb}(k\beta), k = 1, 2, 3$.

(ii) Planes belonging to $\mathrm{orb}(2\gamma)$ are treated further in [12], where they are shown to be in bijective correspondence with *conclaves* (see [15]) of planes in $\mathrm{PG}(4, 2)$. (In $\mathrm{PG}(4, 2)$ a conclave of eight planes is the dual of a regulus-free partial spread of eight lines.)

(iii) (i) See [8] and [10, Section 6.2] for applications of the classification to coding theory, and also for consideration of generalizations to $\mathrm{PG}(n, q)$, as in Cooperstein [2] and Cossidente & Siciliano [3].

A Appendix

A.1 The Grassmannian $\mathcal{G}_{1,3,2}$

In this case we write the decomposition $\mathrm{PG}(5, 2) = \mathrm{Rk}_2(3, 2) \cup \mathrm{Rk}_4(3, 2)$ as $\mathrm{PG}(5, 2) = \mathcal{H} \cup \mathcal{W}$, where $\mathcal{H} = \mathcal{G}_{1,3,2}$, the Klein quadric, is a hyperbolic quadric in $\mathrm{PG}(5, 2)$, and $\mathcal{W}$ consists of the 28 points external to $\mathcal{H}$. It should be noted that *each* $b \in \mathcal{W}$ *can be expressed in the form* $b = m + n$ *for* *precisely ten pairs* $\{\mu, \nu\}$ *of skew lines* μ, ν *of* PG(3, 2). Explicitly, given one expression $b = m + n$, let τ_i, $i = 1, 2, \ldots, 9$, be the nine transversals of the skew pair μ, ν; then the other nine expressions for b are

$$b = m_i + n_i,\ (i = 1, 2, \ldots, 9) \quad \text{where } m_i = m + t_i, \text{ and } n_i = n + t_i. \quad \text{(A.1)}$$

(Here, because τ_i meets μ, it follows that $m + t_i$ is indeed the image of a line μ_i, namely the third line of the pencil determined by μ and τ_i; similarly $n + t_i$ is the image of a line ν_i.) Observe therefore that through each point $b \in \mathcal{W}$ there pass ten bisecants of the Klein quadric $\mathcal{H}$.

In the same area, observe that each $b \in \mathcal{W}$ defines a non-degenerate symplectic form $(u, v)_b$ on V_4, given by $b \wedge u \wedge v = (u, v)_b \omega$, where ω is the (unique!) basis vector for the 1-dimensional space $\wedge^4 V_4$. In projective terms, the 28 different symplectic polarities on $\sigma = \mathrm{PG}(3, 2)$ thus arise from the 28 choices of $b \in \mathcal{W}$. For a subspace $\alpha \subset \sigma$, we will let α^b denote its polar using the b polarity:

$$\alpha^b = \{u \in \sigma \mid b \wedge u \wedge v = 0 \text{ for all } v \in \alpha\}. \quad \text{(A.2)}$$

For a given polarity the ten expressions $b = m + n$ described above correspond to the ten polar pairs $\{\mu, \nu\}$ of lines, with ν being the polar μ^b of μ. The remaining $15 (= 35 - 20)$ lines λ of PG(3, 2) are self-polar, the join $\langle l, b\rangle$ being tangent to $\mathcal{H}$ at the point l. Note that these 15 tangents through b consume 15 further points of $\mathcal{W}$. It follows that, in order to account for the remaining 12 $(= 28 - 1 - 15)$ points of $\mathcal{W}$, through b must pass precisely 6 lines external to $\mathcal{H}$. (See also lemma A.1(i) below.)

Of course the subgroup $\mathcal{G}_b$ of GL(4, 2) which stabilizes $b \in \mathcal{W}$ is one of the 28 subgroups of GL(4, 2) which are isomorphic to $\mathrm{Sp}(4, 2) \cong \mathrm{Sym}(6)$; see also remark A.2 below.

A.1.1 Use of the isomorphism of GL(4, 2) with Alt(8)

Let $ijklmnrs$ be an arbitrary permutation of 12345678. Then we can arrange for the 28 points of $\mathcal{W}$ to be labelled $p_{ij} = p_{ji}$, where the only linear relations amongst the p_{ij} are those of the forms

$$\text{(i) } p_{ij} + p_{ik} + p_{jk} = 0; \qquad \text{(ii) } p_{ij} + p_{kl} + p_{mn} + p_{rs} = 0. \quad \text{(A.3)}$$

The 35 points of $\mathcal{H}$ then have double labellings of the kind $q_{ijkl} = q_{mnrs}$, and are given by $q_{ijkl} = p_{ij} + p_{kl}$. (This labelling can be achieved internally, with p_{ij} being defined by $H_i \cap H_j = \{p_{ij}\}$, where H_i, $i = 1, 2, \ldots, 8$, are the Conwell heptads, *cf.* [6, Lemma 17.5.4]). Any permutation $\pi \in \mathrm{Sym}(8)$ induces, via $p_{ij} \mapsto p_{\pi i\, \pi j}$, a linear transformation of $V_6 = \wedge^2 V_4$ which preserves the Klein quadric; in this way we arrive at the isomorphism of Sym(8) with the full orthogonal group $\mathrm{O}^+(6, 2)$. However elements $A \in \mathrm{GL}(4, 2)$ preserve separately the two systems, Latin and Greek, of internal planes of $\mathcal{H}$, and the elements T_A lie in a subgroup of $\mathrm{O}^+(6,2)$ of index 2, with $A \mapsto T_A$ giving rise to the well-known isomorphism $\mathrm{GL}(4,2) \cong \mathrm{Alt}(8)$. The symplectic scalar product $b.c$ on V_6 determined by the quadric $\mathcal{H}$ satisfies

$$p_{ij}.p_{ik} = 1, \qquad \text{and} \qquad p_{ij}.p_{kl} = 0 \tag{A.4}$$

whenever i, j, k, l are distinct. (In particular the seven elements $p_{ij}, j \neq i$, of the heptad H_i are pairwise non-conjugate — the defining property of a Conwell heptad.) It should be noted that two lines μ, ν of $\mathrm{PG}(3,2)$ meet if and only if the corresponding two points m, n of $\mathcal{H}$ satisfy $m.n = 0$.

The next lemma serves as an indication of the usefulness of the isomorphism $\mathrm{GL}(4,2) \cong \mathrm{Alt}(8)$.

Lemma A.1 *(i) There exist in* $\mathrm{PG}(5,2)$ *precisely fifty-six lines* L *external to* $\mathcal{H}$*, these forming a single* $\mathrm{GL}(4,2)$*-orbit, with six lines passing through each point* $b \in \mathcal{W}$*.*

(ii) The 3-flat $L^\perp$ *polar to an external line* L *meets* $\mathcal{H}$ *in a 5-set* $\mathcal{C} = \{m_i\}_{i=1,2,3,4,5}$ *which is the image of a spread* $\mathcal{S} = \{\mu_i\}_{i=1,2,3,4,5}$ *in* $\mathrm{PG}(3,2)$*.*

(iii) $\mathcal{G}_\mathcal{S} = \mathcal{G}_L \cong (\mathrm{Alt}(5) \times 3).2$*, of order 360.*

(iv) If $\mathcal{S} = \{\mu_i\}_{i=1,2,3,4,5}$ *is a spread in* $\mathrm{PG}(3,2)$ *then* $\Sigma_{i=1}^5 m_i = 0$*.*

Proof. (i) The 56 lines are the $\binom{8}{3}$ lines $L_{ijk} = \{p_{ij}, p_{ik}, p_{jk}\}$, see (A.3). They form a single $\mathrm{GL}(4,2)$-orbit, since the triads ijk form a single $\mathrm{Alt}(8)$-orbit. Through the point $p_{78} \in \mathcal{W}$ pass the 6 external lines L_{i78}, $i = 1, 2, \ldots, 6$.

(ii) If $m \in \mathcal{H}$ satisfies $m.p_{67} = m.p_{68} = m.p_{78} = 0$ then from (A.4) we see that $m \in \mathcal{C}$ where $\mathcal{C} = \{m_i\}_{i=1,2,3,4,5}$ with $m_i = q_{i678}$. Since, by (A.4), $m_i.m_j = (p_{i6} + p_{78}).(p_{j6} + p_{78}) = 1$, for $i \neq j$, the five lines μ_i are pairwise skew and so comprise a spread. ($\mathcal{C}$ is in fact an elliptic quadric $\subset L^\perp$.)

(iii) Since there are 56 spreads $\mathcal{S}$ in $\mathrm{PG}(3,2)$, the construction in part (ii) establishes a bijective correspondence $\mathcal{S} \leftrightarrow L$ with the 56 external lines L, and so the stabilizers $\mathcal{G}_\mathcal{S}$, $\mathcal{G}_L$ are equal. The structure of these stabilizers is as indicated, since the stabilizer in $\mathrm{Alt}(8)$ of the triad 678 is $\mathcal{G}_0 \cup (12)(78)\mathcal{G}_0$, where $\mathcal{G}_0 = \mathrm{Alt}\{1,2,3,4,5\} \times \langle(678)\rangle \cong \mathrm{Alt}(5) \times Z_3$.

(iv) From (A.3) we immediately see that $\Sigma_{i=1}^5 q_{i678} = 0$. ■

Remark A.2 *The isomorphism* $\mathcal{G}_b \cong \mathrm{Sym}(6)$ *at the end of the preceding section can be arrived at in a similar manner to that in part (iii) of the lemma. For if* $b = p_{78}$ *then* $\mathcal{G}_b$ *is isomorphic to the stabilizer in* $\mathrm{Alt}(8)$ *of the duad* 78*, namely to* $\mathcal{A} \cup (12)(78)\mathcal{A}$*, where* $\mathcal{A} = \mathrm{Alt}\{1,2,3,4,5,6\} \cong \mathrm{Alt}(6)$*.*

A.2 Some regulus-free partial spreads

We provide information here about three classes (i) IIIa.1, (ii) IVb.1 and (iii) Ve.1, of partial spreads in $\mathrm{PG}(4,2)$ which are needed in section 2. (For the complete classification of partial spreads in $\mathrm{PG}(4,2)$, consult [5].) At times it will prove convenient to adopt the following shorthand notation. A basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, e_5,\}$ for V_5 will be written $\mathcal{B} = \{1,2,3,4,5\}$, and abbreviations such as $235 = e_2+e_3+e_5$, $u = 12345 = e_1+...+e_5$, $5u = 1234$, $2 \wedge 145 = e_2 \wedge (e_1+e_4+e_5)$, ... will be employed. The $3r$-set of points of $\mathrm{PG}(4,2)$ which support the r lines of a partial spread $\mathcal{S}_r$ will be denoted $\psi(\mathcal{S}_r)$. A line τ which meets each of three lines of $\mathcal{S}_r$ will be referred to not only as a transversal of the three lines, *but also as a transversal of* $\mathcal{S}_r$.

(i) Class IIIa.1. In $\mathrm{PG}(4,2)$ there are only two kinds of partial spreads $\mathcal{S}_3$, reguli (class IIIb.1) and non-reguli (class IIIa.1). With respect to a suitable basis $\mathcal{B}$ any partial spread $\mathcal{S}_3$ which is not a regulus can be taken to be $\mathcal{S}_3 = \{\mu_1, \mu_2, \mu_3\}$ where, in shorthand notation,

$$\mu_1 = \{1,2,12\},\ \mu_2 = \{3,4,34\},\ \mu_3 = \{5,u,5u\}. \tag{A.5}$$

The non-regulus $\mathcal{S}_3$ has a unique transversal $\tau = \{12, 34, 5u\}$, and it also determines the *hyperbasis* $\mathcal{B}_+ = \mathcal{B} \cup \{u\} = \psi(\mathcal{S}_3) \setminus \tau = \{1,2,3,4,5,u\}$. Each permutation $\pi \in \mathrm{Sym}(\mathcal{B}_+)$ gives rise to an element A_π of $\mathrm{GL}(5,2)$ satisfying $A_\pi : b \mapsto \pi(b),\ b \in \mathcal{B}_+$. Observe that the stabilizer $\mathcal{G}_{\mathcal{S}_3}$ of $\mathcal{S}_3$ must consist of those A_π, $\pi \in \mathrm{Sym}(\mathcal{B}_+)$, such that π preserves the syntheme $12\,34\,5u$. It soon follows that

$$\mathcal{G}_{\mathcal{S}_3} = \mathcal{G}^0_{\mathcal{S}_3} \times \langle J \rangle \cong \mathrm{Sym}(4) \times Z_2, \quad \text{where } J = A_{(12)(34)(5u)}. \tag{A.6}$$

See [16], [5] for more details of the group $\mathcal{G}^0_{\mathcal{S}_3} \cong \mathrm{Sym}(4)$.

Of considerable importance to our present concerns is the fact that each non-regulus $\mathcal{S}_3$ comes along with a *twin non-regulus* $\mathcal{S}^*_3 = \{\mu^*_1, \mu^*_2, \mu^*_3\}$, where μ^*_j denotes the third line of the pencil determined by μ_j and τ (and so $\mathcal{S}_3$ and $\mathcal{S}^*_3$ share the same transversal τ). For $\mathcal{S}_3$ as in (A.5), $\mathcal{S}^*_3$ is

$$\mu^*_1 = \{1^*, 2^*, 1^*2^*\},\ \mu^*_2 = \{3^*, 4^*, 3^*4^*\},\ \mu^*_3 = \{5^*, u^*, 5^*u^*\}, \tag{A.7}$$

where $1^* = 134$, $2^* = 234$, $3^* = 124$, $4^* = 123$, $5^* = 125$, $u^* = 1^*2^*3^*4^*5^* = 345$. Note that $\tau^* = \{1^*2^*, 3^*4^*, 5^*u^*\}$ indeed coincides with $\tau = \{12, 34, 5u\}$.

Lemma A.3 *If* $\mathcal{S}_3$ *and* $\mathcal{S}^*_3$ *are twin non-reguli in* $\mathrm{PG}(4,2)$ *then*
(i) The stabilizer $\mathcal{G}_{\mathcal{S}^*_3}$ *of* $\mathcal{S}^*_3$ *coincides with the stabilizer* $\mathcal{G}_{\mathcal{S}_3}$ *of* $\mathcal{S}_3$.
(ii) The stabilizer $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}^*_3\}}$ *of* $\{\mathcal{S}_3, \mathcal{S}^*_3\}$ *is a group* $\mathcal{G}_{\mathcal{S}_3}.2$, *of order 96.*

Proof. (i) An element $A \in \mathcal{G}_{\mathcal{S}_3}$ permutes the lines μ_i but fixes $\tau (= \tau^*)$, and hence permutes the lines μ^*_i.

(ii) The element $K \in \mathrm{GL}(5,2)$ which maps $i \in \mathcal{B}$ onto $i^* \in \mathcal{B}^* := \{1^*, 2^*, 3^*, 4^*, 5^*\}$ (and so u onto u^*) is an involution ($i \rightleftarrows i^*$) which effects the three interchanges $\mu_j \rightleftarrows \mu^*_j, j = 1,2,3$, whence $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}^*_3\}} = \mathcal{G}_{\mathcal{S}_3}.\langle K \rangle$. ■

(In fact $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}^*_3\}}$ is the normalizer in $\mathrm{GL}(5,2)$ of $\mathcal{G}_{\mathcal{S}_3}$.)

(ii) Class IVb.1. This class is represented by $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ where

$$\mu_1 = \{1, 2, 12\},\ \mu_2 = \{3, 4, 34\},\ \mu_3 = \{5, u, 5u\},\ \mu_4 = \{13, 15, 35\}. \quad (A.8)$$

So $\mathcal{S}_4 = \mathcal{S}_3 \cup \{\mu_4\}$, where $\mathcal{S}_3 = \{\mu_1, \mu_2, \mu_3\}$ is as in (A.5). Observe that the line μ_4 is distinguished: each point of μ_4 lies on precisely one transversal of $\mathcal{S}_4$, but in contrast the points $1, 3, 5$ lie on precisely two transversals, and the points $2, 4, u$ lie on no transversals. It follows that $\mathcal{G}_{\mathcal{S}_4}$ is a subgroup of $\mathcal{G}_{\mathcal{S}_3}$, and that any $T \in \mathcal{G}_{\mathcal{S}_4}$ must permute amongst themselves the three points $1, 3, 5$, and also three points $2, 4, u$. It quickly follows that

$$\mathcal{G}_{\mathcal{S}_4} = \langle A, C \rangle \cong \mathrm{Sym}(3),$$

where A and C in $\mathrm{GL}(5,2)$ effect the permutations $(135)(24u)$ and $(13)(24)$ of the hyperbasis $\mathcal{B}_+$. Incidentally, observe that $\mathcal{G}_{\mathcal{S}_4}$ fixes the point $135 = 24u$ (and no other).

(iii) Class Ve.1. This class is represented by $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ where the lines $\mu_1, \ldots, \mu_5$ are, in shorthand notation,

$$\{1u, 2, 345\},\ \{2u, 3, 451\},\ \{3u, 4, 512\},\ \{4u, 5, 123\},\ \{5u, 1, 234\}. \quad (A.9)$$

Note that $\mathcal{S}_5$ is cyclic, since if $B \in \mathrm{GL}(5,2)$ is defined by $B : 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 5 \mapsto 1$, then B effects the permutation $(\mu_1\mu_2\mu_3\mu_4\mu_5)$. In fact, since any $T \in \mathcal{G}_{\mathcal{S}_5}$ must permute amongst themselves those points which lie on only one transversal, namely the points $1, 2, 3, 4, 5$, it then quickly follows from (A.9) that $\mathcal{G}_{\mathcal{S}_5}$ is precisely $\langle B \rangle$:

$$\text{class Ve.1:} \qquad \mathcal{G}_{\mathcal{S}_5} = \langle B \rangle \cong Z_5. \quad (A.10)$$

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