Grassmann and Segre varieties over GF(2): some graph theory links

R. SHAW r.shaw@hull.ac.uk
Department of Mathematics, University of Hull, Hull HU6 7RX, United Kingdom

Abstract

Some recent results (and a conjecture) concerning the polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$ of the lines of $\mathrm{PG}(n,2)$ are shown to be equivalent to results (and a conjecture) concerning certain kinds of subgraphs of any (simple) graph $\Gamma = (\mathcal{V}, \mathcal{E})$ which is of order $|\mathcal{V}| = n+1$. It turns out that those graphs Γ of size $|\mathcal{E}| = n = |\mathcal{V}| - 1$ are of particular significance.

Somewhat similarly, results concerning the polynomial degree of the Segre variety $\mathcal{S}_{m,n,2}$ are translated into equivalent assertions concerning certain subgraphs of any graph Γ which is a subgraph of the complete bipartite graph $K_{m+1,n+1}$.

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1 The polynomial degree of a subset ψ of $\mathrm{PG}(N,2)$

In succeeding sections we will be interested in the polynomial degrees of the following varieties over the finite field GF(2):
(i) the Grassmann variety $\mathcal{G}_{1,n,2}$ of the lines of $\mathrm{PG}(n,2)$, considered as a subset of points of the finite projective space $\mathrm{PG}(\binom{n+1}{2}-1,2) = \mathbb{P}(\wedge^2 V_{n+1,2})$;
(ii) the Segre variety $\mathcal{S}_{m,n,2}$, considered as a subset of points of the finite projective space $\mathrm{PG}(mn+m+n,2) = \mathbb{P}(V_{m+1,2} \otimes V_{n+1,2})$.
However it will help to first consider material concerned with the polynomial degree of a general subset ψ of points of a general finite projective space $\mathrm{PG}(N,2) = \mathbb{P}(V)$, where $V = V_{N+1} = V(N+1,2)$.

For the most part the notation will be as in [7]. In particular $S = \mathrm{PG}^{(0)}(N,2)$ denotes the set of points (0-flats) of $\mathrm{PG}(N,2) = \mathbb{P}(V)$, and we identify S with the nonzero vectors $V \setminus \{0\}$ of the vector space V. The set $F(V)$ of all functions $V \to \mathrm{GF}(2)$ is a vector space over GF(2) of dimension

$|V| = 2^{N+1}$, and its elements are the characteristic functions $\chi(\psi)$, also denoted χ_ψ, of the subsets $\psi \subseteq V$. In the case when ψ is a singleton set $\{a\}$, $a \in V$, we put $\chi_a := \psi_{\{a\}}$.In fact, rather than $F(V)$, *our main focus is on the vector space* $F(S)$, *of dimension* $|S| = 2^{N+1} - 1$ *over* GF(2), *consisting of all functions* $S \to$ GF(2). *Moreover we will consider* $F(S)$ *to be a subspace of* $F(V)$, by identifying an element $f \in F(S)$ with that element $f_0 \in F(V)$ such that $f_0(0) = 0$ and $f_0(a) = f(a)$ for $a \in S$.

Upon choosing a basis $\mathcal{B} = \{e_1, e_2, \ ..., e_{N+1}\}$ for V an element $x \in V$ may be viewed as an $(N+1)$-tuple $(x_1, x_2, \ ..., x_{N+1}) \in \mathrm{GF}(2)^{N+1}$, where the coordinates x_i are elements of the dual $\tilde{V} = \tilde{V}_{N+1}$ of V. The basis $\mathcal{B}$ for V gives rise to an associated *monomial basis* $\mathcal{M}$ for $F(S)$, namely

$$\mathcal{M} = \Xi_1 \cup \Xi_2 \cup \ ... \ \cup \Xi_{N+1}, \quad \text{where } \Xi_r = \{x_{i_1} x_{i_2} ... x_{i_s}\}_{1 \le i_1 < i_2 < \ ... \ < i_r \le N+1}, \tag{1.1}$$

and by adding the constant function 1 to $\mathcal{M}$ we obtain a basis $\mathcal{M}' := \{1\} \cup \mathcal{M}$ for $F(V) = \prec 1 \succ \oplus F(S) = \prec \chi_0 \succ \oplus F(S)$.

If ψ^c denotes the complement *within the set* S of ψ then $\chi(\psi) + \chi(\psi^c) = I$, where I denotes that element of $F(S)$ such that $I(x) = 1$ for all $x \in S$. The characteristic functions $\chi_a, a \in S$, have the coordinate expression:

$$\chi_a(x) = \chi_0(a + x), \qquad \text{where } \chi_0(x) = \Pi_{i=1}^{N+1}(1 + x_i), \tag{1.2}$$

and $I = \chi(S)$ has the coordinate expression

$$I(x) = 1 + \Pi_{i=1}^{N+1}(1 + x_i) = \Sigma_i x_i + \Sigma_{i<j} x_i x_j + \ ... \ + x_1 x_2 ... x_{N+1}. \tag{1.3}$$

This last expression (1.3) may be viewed as the special case $r = N+1$, $X^c = S$ of the following easily verified result:
if X is an $(N-r)$-flat in $\mathrm{PG}(N, 2)$ which is the intersection of the r hyperplanes $f_1(x) = 0, \ ... \ , f_r(x) = 0$, $(f_i \in \tilde{V}_{N+1} \setminus \{0\})$, then

$$\begin{aligned} \chi(X^c) &= 1 + \Pi_{i=1}^{r}(1 + f_i) \\ &= \sum\nolimits_i f_i + \sum\nolimits_{i<j} f_i f_j + \sum\nolimits_{i<j<k} f_i f_j f_k + \ ... \ + f_1 f_2 ... f_r. \end{aligned} \tag{1.4}$$

For $r > 0$, let $F_r = F_r(S)$ denote the subspace of $F(S)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \ ..., x_{N+1})$ with $\deg f \le r$ and $f(0) = 0$; we put $F_0 := \{0\}$. The subspaces F_r are thus nested:

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \ ... \ \subset F_N \subset F_{N+1} = F(S), \tag{1.5}$$

with F_r, $r \ge 1$, possessing the monomial basis $\mathcal{M}_r$ where

$$\mathcal{M}_r = \Xi_1 \cup \Xi_2 \cup \ ... \ \cup \Xi_r, \qquad 1 \le r \le N+1. \tag{1.6}$$

The subspace F_r thus has dimension $|\mathcal{M}_r| = \sum_{s=1}^{r} |\Xi_s| = \sum_{s=1}^{r} \binom{N+1}{s}$. Observing that Ξ_{N+1} consists of the single monomial $m_{N+1} := x_1 x_2 ... x_{N+1}$, it follows from (1.2) that F_N consists of the characteristic functions of all even subsets of S.

The subspace F_r of $F(S)$ has just been given an algebraic definition, but *there exists an equivalent geometric definition*, namely as that subspace of $F(S)$ which is generated by the characteristic functions $\chi(X^c)$ of the complements X^c of the $(N-r)$-flats X of $\mathrm{PG}(N,2)$. For if we define subspaces C_r, $0 \le r < N$, of $F(S)$ by

$$C_r = \prec \chi(X^c) \succ_{X \in \mathrm{PG}^{(r)}(N,2)}, \tag{1.7}$$

then it can be shown, see [7, Theorem 1.5], *cf.* [1, Section 5.3], that

$$C_{N-r} = F_r, \quad r = 1, 2, \ \ldots \ , N. \tag{1.8}$$

Setting $Q_\psi := \chi(\psi^c)$, a subset ψ of S has equation $Q_\psi(x) = 0$. If $Q_\psi \in F_r \diagdown F_{r-1}$ we will say that ψ has *polynomial degree* r, and we write $\deg Q_\psi = r$ for the degree of Q_ψ. (Here $\deg Q_\psi$ is the *reduced* degree of Q_ψ; if $\deg Q_\psi = r$ then of course, see (1.5), $Q_\psi \in F_s$ for each $s \ge r$.) Recall that the subspace $C_0 = F_N$ consists of the characteristic functions of all the even subsets of S. Consequently if ψ is an odd subset of S (and so ψ^c is an even subset) then ψ has polynomial degree $\le N$. On the other hand, since $\chi(\psi) + \chi(\psi^c) = I$, and, see (1.3), $\deg I = N+1$, *an even subset always has polynomial degree* $N+1$.

In general the determination of the polynomial degree of a subset $\psi \subset S$ is a formidable problem. This is certainly usually the case if a direct algebraic approach is attempted, based for example upon (1.2). But quite often progress can be made by using a geometrical approach based upon the next theorem.

Theorem 1.1 *Let ψ be an odd subset of $S = \mathrm{PG}^{(0)}(N,2)$, and consider the following two conditions:*
(A) $|X \cap \psi|$ is odd for all r-flats X of $\mathrm{PG}(N,2)$;
(B) there exists an $(r-1)$-flat X of $\mathrm{PG}(N,2)$ for which $|X \cap \psi|$ is even.
Then

(i) $Q_\psi \in F_r$ if and only if ψ satisfies condition (A);

(ii) ψ has polynomial degree r if and only if ψ satisfies both (A) and (B).

Proof. (i) See [7, Theorem 1.7], or see [5].
(ii) From (B) it follows from (i) that $Q_\psi \notin F_{r-1}$. So both (A) and (B) will hold if and only if $Q_\psi \in F_r \diagdown F_{r-1}$. ∎

In the present paper we adopt this geometric approach in the cases (i) $\psi = \mathcal{G}_{1,n,2}$ and (ii) $\psi = \mathcal{S}_{m,n,2}$, our chief aim being to demonstrate a tie-in with problems of enumeration of certain kinds of subgraphs of (i) (simple) graphs of order $n+1$, and (ii) graphs which are subgraphs of the complete bipartite graph $K_{m+1,n+1}$.

To this end, with respect to a choice of basis $\{e_1, e_2, \ \dots\ , e_{N+1}\}$ for V_{N+1}, let $X_{i_1 i_2 \dots i_s}$ denote that $(N-s)$-flat with coordinate equation $x_{i_1} = x_{i_2} = \dots = x_{i_s} = 0$, and let $Y(j_1, j_2, \ \dots\ j_{s+1})$ denote the s-flat $\langle e_{j_1}, e_{j_2}, \ \dots\ e_{j_{s+1}} \rangle$. Observe that if $\{i_1, i_2, \dots\ i_{N-s}\} = \{1, 2, \ \dots\ , N+1\} \setminus \{j_1, j_2, \ \dots\ , j_{s+1}\}$ then

$$X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \ \dots\ j_{s+1}). \tag{1.9}$$

Upon writing $\chi_{i_1 \dots i_s} := \chi(X^c_{i_1 \dots i_s})$ observe that elements of the set $\mathcal{F}_s := \{\chi_{i_1 \dots is}\}_{1 \le i_1 < i_2 < \ \dots\ < i_s \le N+1}$ are in bijective correspondence with the faces of the simplex of reference of projective dimension $N-s$.

Theorem 1.2 (Simplex Basis) *For* $0 \le r < N$

$$\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ \dots\ \cup \mathcal{F}_{N-r} \tag{1.10}$$

is a basis for $C_r = F_{N-r}$.

Proof. This follows from (1.6) upon noting from (1.4) that the element $\chi_{i_1 i_2 \dots i_s}$ of C_{N-s} differs from $x_{i_1} x_{i_2} \dots x_{i_s}$ by elements of $F_{s-1} = C_{N-s+1}$. ∎

Remark 1.3 *One presumes that other researchers must have made use of this simplex basis (1.10); the present author came across it in 1989 — albeit, see [6, Section 2], in the course of some researches into Clifford algebras!*

By appeal to the basis (1.10) for C_r it follows that in theorem 1.1 we can replace "for all r-flats X of $\mathrm{PG}(N, 2)$" in condition (A) by "for all s-flats $X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \ \dots\ j_{s+1}), \ s \ge r$". In this manner we obtain from theorem 1.1 the following theorem. In this theorem a flat X of $\mathrm{PG}(N, 2)$ is said to be ψ-odd or ψ-even according as X meets ψ in an odd or even number of points.

Theorem 1.4 *Let* ψ *be an odd subset of* $S = \mathrm{PG}^{(0)}(N, 2)$ *and consider the following two conditions:*
(A) for* $s \ge r$ *each of the s-flats* $X_{i_1 i_2 \dots i_{N-s}} = Y(j_1, j_2, \ \dots\ j_{s+1})$ *is* ψ*-odd;*
(B) at least one of the* $(r-1)$*-flats* $Y(j_1, j_2, \ \dots\ j_r)$ *is* ψ*-even.*
Then
(i) $Q_\psi \in F_r$ *if and only if* ψ *satisfies condition (A*);*
(ii) ψ *has polynomial degree* r *if and only if* ψ *satisfies (A*) and (B*).*

2 The polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$

2.1 Some results, and the main conjecture

For $V_{n+1} = V(n+1, 2)$ the bivector space $\wedge^2 V_{n+1}$ has vector space dimension $\binom{n+1}{2}$. We will be dealing with the associated projective space $\mathbb{P}(\wedge^2 V_{n+1}) =$

PG$(N,2)$, where $N := N_n = \binom{n+1}{2} - 1$, and, for $n \geq 3$, we will be interested in the Grassmannian $\mathcal{G}_{1,n,2} \subset S := \text{PG}^{(0)}(N,2)$, which consists of the Grassmann images $m = a \wedge b$ of the lines $\mu = \langle a, b \rangle$ of PG$(n,2) = \mathbb{P}V_{n+1}$. Observe that $|\mathcal{G}_{1,n,2}| = \frac{1}{3}(2^{n+1}-1)(2^n-1)$ is odd. The natural action of $A \in \text{GL}(n+1,2)$ upon $V_{N+1} := \wedge^2 V_{n+1}$ is by $T_A = \wedge^2 A : a \wedge b \mapsto Aa \wedge Ab$. Now, for $n > 3$, the subgroup $G(\mathcal{G}_{1,n,2})$ of GL$(N+1,2)$ which stabilizes $\mathcal{G}_{1,n,2}$ is the isomorphic image under T of GL$(n+1,2)$. Consequently the space $F(S)$ will be viewed as a GL$(n+1,2)$-space under the action L defined by

$$(L_A f)(x) = f(T_A^{-1}x), \quad A \in \text{GL}(n+1,2),\ x \in S. \tag{2.1}$$

In the following we set $Q_{1,n,2} = \chi((\mathcal{G}_{1,n,2})^c)$, and we also write

$$N_n = n + d_n, \qquad \text{where } d_n := N_n - n = \binom{n}{2} - 1. \tag{2.2}$$

Note therefore, from equation 1.8, that

$$F_{d_n} = C_n, \qquad F_n = C_{d_n}. \tag{2.3}$$

Theorem 2.1 *If δ_n denotes the polynomial degree of $\mathcal{G}_{1,n,2}$ then*
(i) $\delta_n \leq d_n + 1$ for all $n \geq 3$;
(ii) $\delta_n \geq d_n$ for all $n \geq 3$;
(iii) $\delta_n = d_n$ for n in the range $3 \leq n \leq 7$.

Proof. See [4] and [7, Theorems 3.1, 3.2, 3.5]. Important ingredients in the proof are: (i) the existence of (Latin) $(n-1)$-flats internal to $\mathcal{G}_{1,n,2}$; (ii) the existence, see [3], of $(d_n - 1)$-flats external to $\mathcal{G}_{1,n,2}$. Appeal is also made to theorem 1.1. Further, for (iii), the computer was needed in [7] to handle the cases $n = 5, 6, 7$. ∎

The results in theorem 2.1(iii) provide good evidence for:

Conjecture 2.2 (Main Conjecture) *For all $n \geq 3$ the Grassmannian $\mathcal{G}_{1,n,2}$ has polynomial degree $d_n = \binom{n}{2} - 1$. That is $Q_{1,n,2} \in F_{d_n} \setminus F_{d_n - 1}$* $(= C_n \setminus C_{n+1})$.

As explained in [7, Section 3.1] this *main conjecture* is just the $q = 2$ special case of a $\mathcal{G}_{1,n,q}$ conjecture put forward (with almost no supporting evidence!) in [4].

Theorem 2.3 *Let $\Omega(i), i = 1, 2, \ ...$ denote the* GL$(n+1,2)$*-orbits of d_n-flats of* PG$(N,2)$, *and let $X(i)$ be a representative of $\Omega(i)$. Then the main conjecture holds if and only if each of the d_n-flats $X(i), i = 1, 2, \ ...$ has odd intersection with $\mathcal{G}_{1,n,2}$.*

Proof. We know from theorem 2.1 that $d_n \leq \delta_n \leq d_n + 1$. So from theorem 1.1 it follows that $\delta_n = d_n$ if and only if every d_n-flat of PG$(N,2)$ is $\mathcal{G}_{1,n,2}$-odd. ∎

2.2 Using simple graphs on $n+1$ vertices

The verification in [7] that the main conjecture at least holds up if n is in the range $3 \le n \le 7$ was helped by making use of certain simple graphs. Let $\Gamma = (\mathcal{V}_n, \mathcal{E})$ denote a (simple) graph having vertex set $\mathcal{V}_n := \{1, 2, \ldots, n+1\}$ and edge set $\mathcal{E}$. Along with Γ we also need its complement $\bar{\Gamma} = (\mathcal{V}_n, \bar{\mathcal{E}})$. In section 1 a point $x \in S = \mathrm{PG}^{(0)}(N, 2)$ had coordinates $(x_1, x_2, \ldots, x_{N+1})$ relative to a choice of basis $\{e_1, e_2, \ldots, e_{N+1}\}$ for V_{N+1}. In our present area of concern a basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for V_{n+1} gives rise to a product basis $\{e_i \wedge e_j\}_{1 \le i < j \le n+1}$ for $V_{N+1} = \wedge^2 V_{n+1}$, and a point $x = \Sigma_{1 \le i < j \le n+1} x_{ij} e_i \wedge e_j \in S$ has coordinates $(x_{ij})_{1 \le i < j \le n+1}$.

Given such a graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ to each edge $E = ij := \{i, j\} \in \mathcal{E}$ let us, relative to our choice of basis, associate:
(i) the coordinate $x_E := x_{ij} (= x_{ji})$, and hence the hyperplane $x_E = 0$;
(ii) the basis element $e_E := e_i \wedge e_j$ for V_{N+1}.
Further let $X_{\mathcal{E}}$ denote the flat having coordinate equations $x_E = 0$, each $E \in \mathcal{E}$, and let $Y(\mathcal{E})$ denote the flat $\langle \{e_E\}_{E \in \mathcal{E}} \rangle$. So $x \in Y(\bar{\mathcal{E}})$ if and only if $x = \sum_{E \in \bar{\mathcal{E}}} x_E e_E$, that is if and only if $x_E = 0$, for each $E \in \mathcal{E}$. Hence, *cf.* eq. (1.9), we have result (i) of:

$$\text{(i) } X_{\mathcal{E}} = Y(\bar{\mathcal{E}}); \qquad \text{(ii) } \chi((X_{\mathcal{E}})^c) = 1 + \Pi_{E \in \mathcal{E}} (1 + x_E), \tag{2.4}$$

with the result (ii) being an instance of the result (1.4). If we define

$$\chi_{\mathcal{E}} := \chi((X_{\mathcal{E}})^c) = \chi(Y(\bar{\mathcal{E}})^c) \qquad \text{and} \quad \mathcal{F}_s = \{\chi_{\mathcal{E}}\}_{|\mathcal{E}| = s} \tag{2.5}$$

then it follows from theorem 1.2 that, for $0 \le r < N$, the set

$$\{\chi_{\mathcal{E}}\}_{|\mathcal{E}| \le N-r} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_{N-r} \tag{2.6}$$

is a basis for $C_r = F_{N-r}$. Of relevance to the next theorem, take note of the following result which, for s in the range $-n \le s \le d_n$, is seen to hold because $|\mathcal{E}| + |\overline{\mathcal{E}}| = \binom{n+1}{2} = N + 1$:

$$Y(\bar{\mathcal{E}}) \text{ is a } (d_n - s)\text{-flat} \iff |\mathcal{E}| = n + s. \tag{2.7}$$

Theorem 2.4 *(i) For each graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ of size $|\mathcal{E}| < n$ the flat $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ has odd intersection with $\mathcal{G}_{1,n,2}$.*

(ii) For at least one of the graphs $\Gamma = (\mathcal{V}_n, \mathcal{E})$ of size $|\mathcal{E}| = n+1$ the $(d_n - 1)$-flat $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ has even intersection with $\mathcal{G}_{1,n,2}$.

Proof. Recall that in the present context the flats $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ take over from those in equation (1.9) and theorem 2.1. Consequently part (i) of the present theorem follows from theorem 2.1(i) in view of theorem 1.4(i), and part (ii) similarly follows from theorem 2.1(ii) in view of theorem 1.4(ii). ■

Remark 2.5 *As in [7, P.6 of Theorem 3.4], we need only to check the intersection properties in the preceding theorem for one graph belonging to each isomorphism class of graphs* $(\mathcal{V}_n, \mathcal{E})$ *of the relevant sizes, since the stabilizer* $G(\mathcal{G}_{1,n,2})$ *contains that subgroup* G_0 *of* $\mathrm{GL}(n+1,2)$ *which effects all permutations of the basis* $\{e_1, e_2, \ ..., e_{n+1}\}$ *for* V_{n+1}.

In the preceding theorem we have translated the results (i) and (ii) of theorem 2.1 in terms of properties of the flats $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$. Concerning the main conjecture, theorem 2.3 translates similarly into the following theorem.

Theorem 2.6 *The Grassmannian* $\mathcal{G}_{1,n,2}$ *has polynomial degree* d_n *if and only if for each isomorphism class of graph* $\Gamma = (\mathcal{V}_n, \mathcal{E})$ *of size* $|\mathcal{E}| = n$ *the* d_n*-flat* $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ *has odd intersection with* $\mathcal{G}_{1,n,2}$.

Remark 2.7 *Direct application of theorem 2.3 would require a knowledge of the* $\mathrm{GL}(n+1,2)$*-orbits of the* d_n*-flats of* $\mathrm{PG}(N,2)$*. But for* $n > 4$ *this knowledge will probably never be gained. For even in the case* $n = 4$ *of* $\mathcal{G}_{1,4,2}$ *much effort was expended in [8] to classify just one kind of flat in* $\mathrm{PG}(9,2)$*, namely those external to* $\mathcal{G}_{1,4,2}$*. Possibly this work could be built upon to determine the* $\mathrm{GL}(5,2)$*-orbits of all of the 53,743,987 5-flats in* $\mathrm{PG}(9,2)$*, but it would appear to be a daunting task.*

In contrast, as we will see below, application of theorem 2.6 is quite feasible, at least with computer help, for small values of n*.*

2.3 Computation of $|Y(\overline{\mathcal{E}}) \cap \mathcal{G}_{1,n,2}|$

In order to apply theorem 2.6 we need to consider, for each isomorphism class of graph $\Gamma = (\mathcal{V}_n, \mathcal{E})$ of size $|\mathcal{E}| = n$, the intersection of the d_n-flat $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ with $\mathcal{G}_{1,n,2}$. Setting $h(\mathcal{E}) := |Y(\overline{\mathcal{E}}) \cap \mathcal{G}_{1,n,2}|$, we would like to show that $h(\mathcal{E})$ is odd for all such graphs Γ.

In the earlier paper [7] the points of $\mathbb{P}(\wedge^2 V_{n+1}) = \mathrm{PG}(N,2)$ were viewed as alternating matrices of size $n+1$, and, for $n \leq 7$, Magma [2] was used to compute the ranks of all the elements of a d_n-flat $Y(\overline{\mathcal{E}})$. For example in the $n = 7$ case there are 115 isomorphism classes of graphs $(\mathcal{V}_7, \mathcal{E})$ of order 8 and size $|\mathcal{E}| = 7$, and for one of these 115 edge sets $\mathcal{E}$ Magma reported that of the $2^{21} - 1 = 2,097,151$ points of the 20-flat $Y(\bar{\mathcal{E}})$ there were 747, 84,308, 1,233,856 and 778,240 points which had rank 2, 4, 6 and 8, respectively. Of course not all of this detail was needed! — we just needed to know that $h(\mathcal{E})$, the number of points of $Y(\overline{\mathcal{E}})$ having rank 2, was odd.

Rather than computing the ranks of the bivectors $x \in X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$*, we now explain how the intersection numbers* $h(\mathcal{E}) := |X_{\mathcal{E}} \cap \mathcal{G}_{1,n,2}|$ *can be computed using simple combinatorial considerations.* To this end we will need some further notation.

In the following α,β,γ will denote subsets of $\mathcal{V}_n := \{1,2,\ \dots\ ,n+1\}$ *which are pairwise disjoint and non-empty.* Given the graph $\Gamma = (\mathcal{V}_n,\mathcal{E})$, if two such subsets α,β satisfy

$$ij \notin \mathcal{E} \text{ (that is } ij \in \overline{\mathcal{E}}\text{) for all } i \in \alpha \text{ and for all } j \in \beta, \tag{2.8}$$

then we write $\alpha \perp_{\mathcal{E}} \beta$ (or equally $\beta \perp_{\mathcal{E}} \alpha$). Such a non-ordered pair $\{\alpha,\beta\}$ will be termed a *dyad* for the graph $(\mathcal{V}_n,\mathcal{E})$. Also if three (pairwise disjoint and non-empty) subsets α,β,γ of $\mathcal{V}_n$ satisfy $\alpha \perp_{\mathcal{E}} \beta,\ \alpha \perp_{\mathcal{E}} \gamma$ and $\beta \perp_{\mathcal{E}} \gamma$, then we write $\alpha \perp_{\mathcal{E}} \beta \perp_{\mathcal{E}} \gamma$, and we will term the non-ordered triple $\{\alpha,\beta,\gamma\}$ a *triad* for the graph $(\mathcal{V}_n,\mathcal{E})$. For an agreed edge set $\mathcal{E}$ we abbreviate $\perp_{\mathcal{E}}$ by $\perp$. We also put $e_\alpha := \Sigma_{i\in\alpha} e_i$, and (for $\alpha\cap\beta=\emptyset$) we put $e_{\alpha\beta} := e_{\alpha\cup\beta}$.

Lemma 2.8 *The flat $X_{\mathcal{E}} = Y(\overline{\mathcal{E}})$ meets $\mathcal{G}_{1,n,2}$ in $h(\mathcal{E}) = p(\mathcal{E}) + q(\mathcal{E})$ points, where*
$p(\mathcal{E})$ is the number of dyads $\{\alpha,\beta\}$ for the graph $(\mathcal{V}_n,\mathcal{E})$;
$q(\mathcal{E})$ is the number of triads $\{\alpha,\beta,\gamma\}$ for the graph $(\mathcal{V}_n,\mathcal{E})$.

Proof. Every line λ in $\mathrm{PG}(n,2)$ is (with respect to the agreed choice of basis $\{e_1,e_2,\ \dots, e_{n+1}\}$ for V_{n+1}) of one of the following two kinds:
(1) $\lambda_{\{\alpha,\beta\}} = \{e_\alpha, e_\beta, e_{\alpha\beta}\}$ for two disjoint non-empty subsets α,β of $\mathcal{V}_n$;
(2) $\lambda_{\{\alpha,\beta,\gamma\}} = \{e_{\alpha\gamma}, e_{\beta\gamma}, e_{\alpha\beta}\}$ for three pairwise disjoint non-empty subsets α,β,γ of $\mathcal{V}_n$.
Now the Grassmann image $l_{\{\alpha,\beta\}} = e_\alpha \wedge e_\beta = \Sigma_{i\in\alpha}\Sigma_{j\in\beta} e_i\wedge e_j$ of $\lambda_{\{\alpha,\beta\}}$ lies in $Y(\overline{\mathcal{E}})$ if and only if $ij \in \overline{\mathcal{E}}$ for all $i\in\alpha$ and for all $j\in\beta$, that is if and only if $\alpha\perp\beta$. Similarly the Grassmann image

$$\begin{aligned} l_{\{\alpha,\beta,\gamma\}} &= e_{\alpha\gamma}\wedge e_{\beta\gamma} = e_\alpha\wedge e_\beta + e_\alpha\wedge e_\gamma + e_\beta\wedge e_\gamma \\ &= \Sigma_{i\in\alpha}\Sigma_{j\in\beta} e_i\wedge e_j + \Sigma_{i\in\alpha}\Sigma_{j\in\gamma} e_i\wedge e_j + \Sigma_{i\in\beta}\Sigma_{j\in\gamma} e_i\wedge e_j \end{aligned}$$

of $\lambda_{\{\alpha,\beta,\gamma\}}$ lies in $Y(\overline{\mathcal{E}})$ if and only if $\alpha\perp\beta\perp\gamma$. ■

Given the graph $\Gamma = (\mathcal{V}_n,\mathcal{E})$, the following considerations help in the computation of $p(\mathcal{E})$ and $q(\mathcal{E})$. For each non-empty subset α of $\mathcal{V}_n$ we define the subset $\alpha^\perp \subseteq \mathcal{V}_n$ by

$$\alpha^\perp = \{j\in\mathcal{V}_n \mid ij\in\overline{\mathcal{E}} \text{ for all } i\in\alpha\}. \tag{2.9}$$

Denote by Γ_α the subgraph of Γ which is induced on $\alpha^\perp$; so $\Gamma_\alpha = (\alpha^\perp,\mathcal{E}_\alpha)$ where $\mathcal{E}_\alpha \subset \mathcal{E}$ consists of those edges of Γ which lie in $\alpha^\perp$. For given α, observe that $\alpha\perp\beta$ holds if and only if $\beta(\neq\emptyset)\subseteq\alpha^\perp$; so we have $\alpha\perp\beta$ for $2^{|\alpha^\perp|}-1$ choices of β. Hence we have

$$p(\mathcal{E}) = \frac{1}{2}\sum\nolimits_{\alpha(\neq\emptyset)\subseteq\mathcal{V}_n} N_\alpha, \quad \text{where } N_\alpha := 2^{|\alpha^\perp|}-1. \tag{2.10}$$

Here the $\frac{1}{2}$ corrects for the double counting due to $\{\alpha, \beta\} = \{\beta, \alpha\}$.

For given α, observe that $\alpha \perp \beta \perp \gamma$ holds if and only if $\{\beta, \gamma\}$ is a dyad for the induced graph $(\alpha^\perp, \mathcal{E}_\alpha)$; so we have $\alpha \perp \beta \perp \gamma$ for $p(\mathcal{E}_\alpha)$ choices of $\{\beta, \gamma\}$. Hence

$$q(\mathcal{E}) = \frac{1}{3} \sum\nolimits_{\alpha(\neq\emptyset)\subseteq\mathcal{V}_n} p(\mathcal{E}_\alpha), \tag{2.11}$$

where the $\frac{1}{3}$ corrects for a triple counting of α. So

$$h(\mathcal{E}) = \sum\nolimits_{\alpha(\neq\emptyset)\subseteq\mathcal{V}_n} (\tfrac{1}{2} N_\alpha + \tfrac{1}{3} p(\mathcal{E}_\alpha)). \tag{2.12}$$

Of course $p(\mathcal{E}_\alpha)$ can be computed using (2.10), but with $\mathcal{E}_\alpha$ replacing $\mathcal{E}$:

$$p(\mathcal{E}_\alpha) = \frac{1}{2} \sum\nolimits_{\beta(\neq\emptyset)\subseteq\alpha^\perp} N_{\alpha,\beta}, \quad \text{where } N_{\alpha,\beta} := 2^{|\beta^\alpha|} - 1, \tag{2.13}$$

and where $\beta^\alpha := \{j \in \alpha^\perp | \ ij \in \overline{\mathcal{E}_\alpha} \text{ for all } i \in \beta\}$.

2.3.1 A simple example

Consider the following graph $\Gamma_A = (\mathcal{V}_5, \mathcal{E}_A)$, of order 6 and size 5:

$$\mathcal{E}_A = \{12, 23, 34, 45, 36\} : \tag{2.14}$$

which is of relevance for the case of $\mathcal{G}_{1,5,2}$. The following table summarizes the computation of the number $p(\mathcal{E}_A)$ of dyads. For example the seventh entry conveys the information that if $\alpha = \{1, 2\}$ then $\alpha^\perp = \{4, 5, 6\}$, whence $N_\alpha = 2^3 - 1 = 7$. Observe that $\mathcal{E}_A$ is stable under the graph isomorphism J which fixes the vertices 3 and 6 and effects the interchanges $1 \leftrightarrow 5$ and $2 \leftrightarrow 4$. In cases where $J(\alpha) \neq \alpha$, the entries for α and $J(\alpha)$ are given in adjacent rows. Thus the seventh entry which has $\alpha = \{1, 2\}$ is paired with the eighth entry which has $J(\alpha) = \{5, 4\}$.

α	$\alpha^\perp$	N_α	α	$\alpha^\perp$	N_α	α	$\alpha^\perp$	N_α	α	$\alpha^\perp$	N_α
1	3456	15	14	6	1	36	15	3	236	5	1
5	3216	15	52	6	1	123	5	1	436	1	1
2	456	7	15	36	3	543	1	1	136	5	1
4	216	7	16	45	3	124	6	1	536	1	1
3	15	3	56	21	3	542	6	1	1236	5	1
6	1245	15	23	5	1	125	6	1	5436	1	1
12	456	7	43	1	1	541	6	1	1245	6	1
54	216	7	24	6	1	126	45	3			
13	5	1	26	45	3	546	21	3			
53	1	1	46	21	3						

From the table we see that $\Sigma_\alpha N_\alpha = 120$, whence, from (2.10), $p(\mathcal{E}_A) = 60$.

Concerning triads for the graph $\Gamma_A = (\mathcal{V}_5, \mathcal{E}_A)$ they are either of the kind $\alpha \perp \beta \perp \{6\}$ with $\alpha \subseteq \{1,2\}$ and $\beta \subseteq \{4,5\}$, or of the kind $\{1\} \perp \{5\} \perp \gamma$ with $\gamma \subseteq \{3,6\}$. There are thus $3 \times 3 = 9$ triads of the first kind and 3 of the second kind. However just one triad, namely $\{1\} \perp \{5\} \perp \{3\}$, is of both kinds, and so the number $q(\mathcal{E}_A)$ of distinct triads is 11.

It thus follows from lemma 2.8 that the 9-flat $X_{\mathcal{E}_A} = Y(\overline{\mathcal{E}_A})$ of PG(14, 2) meets $\mathcal{G}_{1,5,2}$ in $h(\mathcal{E}_A) = 60+11 = 71$ points. This result is in agreement with the computer result in [7, Section 3.3.3] that $|Y(\overline{\mathcal{E}_{11}}) \cap \mathcal{G}_{1,5,2}| = 71$, since the graph $(\mathcal{V}_5, \mathcal{E}_{11})$ in [7, equation (3.10)] is isomorphic to the above graph $(\mathcal{V}_5, \mathcal{E}_A)$.

Remark 2.9 *One may similarly compute by hand the value of* $h(\mathcal{E})$ *for each of the* 15 *distinct isomorphism classes of graphs* $(\mathcal{V}_5, \mathcal{E})$, *of order 6 and size* $|\mathcal{E}| = 5$, *and hence, by theorem 2.6, deduce from all the* $h(\mathcal{E})$ *being odd that the Grassmannian* $\mathcal{G}_{1,5,2}$ *has polynomial degree* $d_5 = 9$. *However even in this* $n = 5$ *case calculations by hand are prone to error, and certainly for* $n > 5$ *it is advisable to resort to computer results.*

2.4 Spin-off results in graph theory

For a simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ observe that two non-empty subsets α, β of $\mathcal{V}$, of sizes $|\alpha| = a$ and $|\beta| = b$, satisfy $\alpha \perp \beta$ if and only if $\bar{\Gamma} = (\mathcal{V}, \bar{\mathcal{E}})$ contains as a subgraph (but not necessarily an induced subgraph) the complete bipartite graph $K_{a,b}$ with parts the a-set α and the b-set β. Similarly if $\gamma \subset \mathcal{V}$ is of size $|\gamma| = c \neq 0$, then $\alpha \perp \beta \perp \gamma$ holds if and only if $\bar{\Gamma}$ contains as a subgraph (but not necessarily an induced subgraph) the complete tripartite graph $K_{a,b,c}$ with parts the a-set α, the b-set β and the c-set γ. For any simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ let us define $p(\Gamma)$ to be the total number of subgraphs of $\bar{\Gamma}$ which are isomorphic to $K_{a,b}$ for some a, b satisfying $a \geq b > 0$. Similarly we define $q(\Gamma)$ to be the total number of subgraphs of $\bar{\Gamma}$ which are isomorphic to $K_{a,b,c}$ for some a, b, c satisfying $a \geq b \geq c > 0$.

The finite geometry results of theorem 2.4 may then be translated into the following theorem in graph theory.

Theorem 2.10 *Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be any finite simple graph.*
(i) If $|\mathcal{E}| < |\mathcal{V}| - 1$ *then* $p(\Gamma) + q(\Gamma)$ *is odd.*
(ii) If $|\mathcal{E}| > |\mathcal{V}| - 1$ *then* $p(\Gamma) + q(\Gamma)$ *is even for some edge sets* $\mathcal{E}$.

Similarly theorem 2.6 translates as follows:

Theorem 2.11 *Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be any finite simple graph such that* $|\mathcal{E}| = |\mathcal{V}| - 1 = n$. *Then* $p(\Gamma) + q(\Gamma)$ *is odd for all such edge sets* $\mathcal{E}$ *if and only if the Grassmannian* $\mathcal{G}_{1,n,2}$ *has polynomial degree* $d_n = \binom{n}{2} - 1$.

So conceivably(?) subgraph enumeration techniques may enable us to settle the main conjecture 2.2 concerning the polynomial degree of the Grassmannian $\mathcal{G}_{1,n,2}$! Freeing ourselves from reference to Grassmannians take note that the following graph theory conjecture at least holds good for graphs of order ≤ 8.

Conjecture 2.12 *If $\Gamma = (\mathcal{V}, \mathcal{E})$ is any finite simple graph such that $|\mathcal{E}| < |\mathcal{V}|$ then $p(\Gamma) + q(\Gamma)$ is odd.*

3 The polynomial degree of the Segre variety $\mathcal{S}_{m,n,2}$

In this section $V = V_{N+1}$ is the tensor product space $V_{m+1} \otimes V_{n+1}$ of two vector spaces V_{m+1} and V_{n+1} over GF(2); so $N = mn + m + n$. We deal with the Segre variety $\mathcal{S}_{m,n,2} \subset S := \mathrm{PG}^{(0)}(N, 2)$ which consists of all the decomposable elements $u \otimes v$, $u(\neq 0) \in V_{m+1}$, $v(\neq 0) \in V_{n+1}$, of V_{N+1}. Given a basis $\{e_1, \ldots, e_{m+1}\}$ for V_{m+1} and a basis $\{f_1, \ldots, f_{n+1}\}$ for V_{n+1} then for non-empty subsets α of $\mathcal{U} := \{1, 2, \ldots, m+1\}$ and β of $\mathcal{V} := \{1, 2, \ldots, n+1\}$ we set $e_\alpha := \Sigma_{i\in\alpha} e_i$ and $f_\beta := \Sigma_{j\in\beta} e_j$. So in this notation we have

$$\mathcal{S}_{m,n,2} = \{e_\alpha \otimes f_\beta | \ (\emptyset \neq)\alpha \subseteq \mathcal{U}, \ (\emptyset \neq)\beta \subseteq \mathcal{V}\}. \tag{3.1}$$

With respect to the product basis $\{e_i \otimes f_j\}_{1\leq i\leq m+1, 1\leq j\leq n+1}$ for V_{N+1} a point $x = \Sigma_{i=1}^{m+1} \Sigma_{j=1}^{n+1} x_{ij} e_i \otimes f_j \in S$ has coordinates $(x_{ij})_{1\leq i\leq m+1, 1\leq j\leq n+1}$, and the polynomial spaces $F_r = F_r(S)$ are accordingly defined.

Observe that the set $\mathcal{S}_{m,n,2}$ is the disjoint union of the $(2^{m+1} - 1)$ n-flats $Y_\alpha := \mathbb{P}(e_\alpha \otimes V_{n+1})$, $(\emptyset \neq)\alpha \subseteq \mathcal{U}$, and is also the disjoint union of the $(2^{n+1} - 1)$ m-flats $Z_\beta := \mathbb{P}(V_{m+1} \otimes f_\beta)$, $(\emptyset \neq)\beta \subseteq \mathcal{V}$. Without loss of generality *we will assume that* $m \leq n$; in which case we set $d_{m,n} := mn + m$ and aim to show that the polynomial degree of $\mathcal{S}_{m,n,2}$ is $d_{m,n}$.

Lemma 3.1 *There exists a $(d_{m,n} - 1)$-flat X of $\mathrm{PG}(N, 2)$ which is skew to the flat $Y_\mathcal{U}$ but which meets each of the $2^{m+1} - 2$ flats Y_α, $\alpha \subset \mathcal{U}$.*

Proof. We give an explicit example of such a $(d_{m,n} - 1)$-flat X. Let X be that flat which is spanned by basis vectors of two kinds:

$$\text{(a) } \{e_i \otimes f_j\}_{1\leq i\leq m, \ j\neq i} \qquad \text{(b) } \{e_{m+1} \otimes f_j\}_{j\leq m}. \tag{3.2}$$

Since mn basis vectors are of the kind (a) and m are of the kind (b) it follows that X is a $(mn + m - 1)$-flat as required. Alternatively X is that flat of $\mathrm{PG}(N, 2)$ which satisfies the coordinate conditions

$$\text{(a) } x_{ii} = 0, \ 1 \leq i \leq m \qquad \text{(b) } x_{(m+1)j} = 0, \ j \geq m + 1. \tag{3.3}$$

(The total number of independent conditions is thus $m+(n+1-m)=n+1$, so that X is a flat of projective dimension $N-n-1=d_{m,n}-1$, as required.) Suppose α is strictly a subset of $\mathcal{U}$, and so $i \notin \alpha$ for some $i \in \mathcal{U}$; then, from (3.2), $e_\alpha \otimes f_i \in X$, and so X meets Y_α. On the other hand every point $(\Sigma_{i=1}^{m+1} e_i) \otimes v$ of $Y_{\mathcal{U}}$ violates the coordinate conditions (3.3). ■

Theorem 3.2 *For $m \leq n$ the polynomial degree $\delta_{m,n}$ of the Segre variety $\mathcal{S}_{m,n,2} \subset \mathrm{PG}(mn+m+n,2)$ is $d_{m,n}=mn+m$.*

Proof. Since $N-n=d_{m,n}$, every $d_{m,n}$-flat X in $\mathrm{PG}(N,2)$ meets each of the n-flats Y_α in an odd number of points, and so meets $\mathcal{S}_{m,n,2}$ in an odd number of points. Hence, by theorem 1.1(i), $\delta_{m,n} \leq mn+m$. On the other hand in $\mathrm{PG}(N,2)$ there exists, lemma 3.1, a $(d_{m,n}-1)$-flat X which is skew to $Y_{\mathcal{U}}$ and which meets (in an odd number of points) the remaining even number of n-flats Y_α, $\alpha \subset \mathcal{U}$; so X meets $\mathcal{S}_{m,n,2}$ in an even number of points. Hence, by theorem 1.1(ii), $\delta_{m,n}=d_{m,n}$. ■

Upon recalling the use of (simple) graphs in section 2.2 which were subgraphs of the complete graph Γ_{m+1}, let us in the present context consider subgraphs Γ of the complete bipartite graph $\Gamma_{m+1,n+1}$ whose parts are $\mathcal{U}=\{1,2,\ \ldots\ ,m+1\}$ and $\mathcal{V}'=\{m+2,m+3,\ \ldots\ ,m+n+2\}$, of sizes $|\mathcal{U}|=m+1$ and $|\mathcal{V}'|=n+1$ While we can proceed on quite a similar path to that followed in section 2, our present considerations are simpler because, see theorem 3.2, we have a precise result for the polynomial degree of $\mathcal{S}_{m,n,2}$, in contrast to the conjecture 2.2 for the polynomial degree of $\mathcal{G}_{1,n,2}$. Consequently we can derive, see theorem 3.5 below, a corresponding precise result in graph theory.

For such a bipartite graph $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$, each edge $E \in \mathcal{E}$ is of the form $E=ij' := \{i,j'\}$ for some $i \in \mathcal{U}$ and some $j' \in \mathcal{V}'$. Along with Γ we also consider its "bipartite complement" $\Gamma^*=(\mathcal{V}\cup\mathcal{V}',\mathcal{E}^*)$, where $\mathcal{E}^*=\{ij' |\ i \in \mathcal{U}, j' \in \mathcal{V}', ij' \notin \mathcal{E}\}$. For $j \in \mathcal{V}$ define $j' := j+m+1$ and note that $j \longleftrightarrow j'$ establishes a bijective correspondence between the sets $\mathcal{V}$ and $\mathcal{V}'$. To each edge $E=ij' \in \mathcal{E}$, let us associate:
(i) the coordinate $x_E := x_{ij}$, and hence the hyperplane $x_E=0$;
(ii) the basis element $e_E := e_i \otimes f_j$ for V_{N+1}.
Further let $X_{\mathcal{E}}$ denote the flat having coordinate equations $x_E=0$, each $E \in \mathcal{E}$, and let $Y(\mathcal{E})$ denote the flat $\langle\{e_E\}_{E\in\mathcal{E}}\rangle$. Take note that $X_{\mathcal{E}}=Y(\mathcal{E}^*)$, *cf.* eq. (2.4)(i).

Then, just as theorem 2.1 led to theorem 2.4, our current theorem 3.2 leads to the results in the next theorem.

Theorem 3.3 *(i) For each of the foregoing bipartite graphs $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$ of size $|\mathcal{E}| \leq n$ the flat $X_{\mathcal{E}}=Y(\mathcal{E}^*)$ has odd intersection with $\mathcal{S}_{m,n,2}$.*

(ii) For at least one of the graphs $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$ of size $|\mathcal{E}|=n+1$ the $(d_{m,n}-1)$-flat $X_{\mathcal{E}}=Y(\mathcal{E}^*)$ has even intersection with $\mathcal{S}_{m,n,2}$.

Observe that the $(d_{m,n}-1)$-flat X in lemma 3.1 is of the kind $X_{\mathcal{E}}$ in part (ii) of the theorem, with $\mathcal{E}$ consisting of the m edges $\{j,j'\},\ 1\le j\le m$ together with the $n+1-m$ edges $\{m+1,j'\},\ 2m+2\le j'\le m+n+2$.

Next let us look at the analogue for $\mathcal{S}_{m,n,2}$ of lemma 2.8 and of equation (2.10). Things are simpler in the present context: since every point of $S_{m,n,2}$ is of the form $e_\alpha\otimes f_\beta$ we will have no need of "triads". Given a bipartite graph $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$ then, for (non-empty) subsets $\alpha\subseteq\mathcal{U},\ \beta'\subseteq\mathcal{V}'$, we write $\alpha\perp\beta'$ whenever $ij'\notin\mathcal{E}$ (that is $ij'\in\mathcal{E}^*$) for all $i\in\alpha$ and for all $j'\in\beta'$. Such an *ordered* pair (α,β') will be termed a *dyad* for the graph $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$. Also for each non-empty subset α of $\mathcal{U}$ we define the subset $\alpha^\perp$ of $\mathcal{V}'$ by

$$\alpha^\perp=\{j'\in\mathcal{V}'|\ ij'\in\mathcal{E}^*\text{ for all } i\in\alpha\}. \tag{3.4}$$

The following lemma now follows in a straightforward way.

Lemma 3.4 *The flat $X_{\mathcal{E}}=Y(\mathcal{E}^*)$ meets $\mathcal{S}_{m,n,2}$ in $p(\mathcal{E})$ points, where $p(\mathcal{E})$ is the number of dyads for the bipartite graph $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$.*

Further the analogue of equation (2.10) is easily seen to be

$$p(\mathcal{E})=\sum\nolimits_{\alpha(\neq\emptyset)\subseteq\mathcal{U}}N_\alpha,\quad\text{where } N_\alpha:=2^{|\alpha^\perp|}-1. \tag{3.5}$$

Note that (α,β') is a dyad for the graph $\Gamma=(\mathcal{U}\cup\mathcal{V}',\mathcal{E})$ if and only if the graph $\Gamma^*=(\mathcal{V}\cup\mathcal{V}',\mathcal{E}^*)$ contains as a subgraph the complete bipartite graph having parts α and β'. Bearing this in mind we may now translate the finite geometry results of theorems 3.2 and 2.4 into the following bipartite graph theorem. (In this theorem a,b play the roles of the preceding $m+1,n+1$, and $(\mathcal{A}\cup\mathcal{B},\mathcal{E})$ plays the role of $(\mathcal{V}\cup\mathcal{V}',\mathcal{E}^*)$.)

Theorem 3.5 *Let $\Gamma=(\mathcal{A}\cup\mathcal{B},\mathcal{E})$ be any finite bipartite graph which is a subgraph of the complete bipartite graph $K_{a,b}$ whose parts $\mathcal{A}$, $\mathcal{B}$ have sizes $|\mathcal{A}|=a$, $|\mathcal{B}|=b$ with $a\le b$. Let $N(\Gamma)$ denote the total number of subgraphs of Γ which are isomorphic to $\Gamma_{a',b'}$ for some $a'\le a,\ b'\le b$. Then $N(\Gamma)$ is odd for all such graphs Γ of size $|\mathcal{E}|>ab-b$. Moreover for each $s\le ab-b$ there exists at least one such graph Γ of size s for which $N(\Gamma)$ is even.*

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