Partial spreads in $\mathrm{PG}(4, 2)$ and flats in $\mathrm{PG}(9, 2)$ external to the Grassmannian $G_{1,4,2}$

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Abstract

We consider the following *'even hyperplane construction'* of flats in the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2)$. Let the Plücker image in $\mathcal{G}_{1,4,2}$ of a partial spread $\mathcal{S}_r = \{\mu_1, \dots, \mu_r\}$. in $\mathrm{PG}(4,2)$ be $\mathcal{C}_r = \{m_1, \dots, m_r\}$. Then $\mathcal{C}_r$ is a r-cap on $\mathcal{G}_{1,4,2}$. Using the recent classification [3] of partial spreads in $\mathrm{PG}(4,2)$, we determine those partial spreads $\mathcal{S}_r$ such that the projective space $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{\text{even}}$ generated by the $\binom{r}{2}$ points $m_{ij} := m_i + m_j$ is an external flat. We show that, in this simple manner, we may construct seven out of the ten $\mathrm{GL}(5,2)$-orbits of external flats.

1 Introduction

We will be concerned with flats in $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ arising from the 155 lines of $\mathrm{PG}(4,2) = \mathbb{P}(V(5,2))$. Since we work over $\mathrm{GF}(2)$, we will identify the nonzero elements of a vector space $V_{n+1} = V(n+1,2)$ with the points of the associated projective space $\mathbb{P}V_{n+1} = \mathrm{PG}(n,2)$. We use $\prec u, v, \dots \succ$ for the vector subspace spanned by $u, v, \dots$, and $\langle u, v, \dots \rangle$ for the projective flat generated by points $u, v, \dots$. The Plücker map $\prec u, v \succ \mapsto \prec u \wedge v \succ$ sends the 2-spaces of V_5 to those 1-spaces of $\wedge^2 V_5$ which are spanned by decomposable bivectors. Projectively the lines of $\mathrm{PG}(4,2)$ are mapped onto the points of $\mathcal{G}_{1,4,2}$. *Throughout this paper the images in* $\mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$ *of lines* λ, μ, ν, τ *in* $\mathrm{PG}(n,2)$ *will be denoted* l, m, n, t.

An element $A \in \mathrm{GL}(5,2)$ gives rise to the induced mapping $T_A := \wedge^2 A$ on $V_{10} = \wedge^2 V_5$. Under this action T of $\mathrm{GL}(5,2)$ the projective space $\mathrm{PG}(9,2)$ ($= V_{10} \diagdown \{0\}$) is the disjoint union $\mathrm{Rk}_2 \cup \mathrm{Rk}_4$ of two $\mathrm{GL}(5,2)$-orbits, where the orbit $\mathrm{Rk}_2 = \mathcal{G}_{1,4,2}$ is of length 155 and where Rk_4, of length $1023 - 155 = 868$, consists of the bivectors b of rank 4; such a bivector is expressible in the form $b = u \wedge v + x \wedge y$ for linearly independent x, y, u, v, and defines a solid $\operatorname{im} b := \langle u, v, x, y \rangle \subset \mathrm{PG}(4,2)$. Each solid $\sigma = \mathbb{P}(V_4)$ in $\mathrm{PG}(4,2)$ defines a 5-flat $\Pi(\sigma) = \mathbb{P}(\wedge^2 V_4)$ in $\mathrm{PG}(9,2)$ which is the disjoint union $\mathcal{H}(\sigma) \cup \mathcal{W}(\sigma)$ of a hyperbolic quadric $\mathcal{H}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_2$, a 35-point set, and of a 28-set

$\mathcal{W}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_4$. Since $b \in \mathcal{W}(\sigma)$ if and only if $\mathrm{im}\, b = \sigma$, take note that *the subsets* $W(\sigma)$, σ *a solid in* $\mathrm{PG}(4, 2)$, *yield a partition of the 868 points of* Rk_4 *into 31 subsets of size 28.*

Each $b \in \mathcal{W}(\sigma), \sigma = \mathbb{P}(V_4)$, defines a non-degenerate symplectic form $(u, v)_b$ on V_4, given by $b \wedge u \wedge v = (u, v)_b \omega$, where ω is the (unique!) basis vector for the 1-dimensional space $\wedge^4 V_4$. In projective terms, the 28 different symplectic polarities on σ thus arise from the 28 choices of $b \in \mathcal{W}(\sigma)$. For a subspace $\alpha \subset \sigma$, we will let $\alpha^b \subset \sigma$ denote its polar using the b polarity:

$$\alpha^b = \{u \in \sigma \,|\, b \wedge u \wedge v = 0 \text{ for all } v \in \alpha\}. \tag{1.1}$$

An external line which lies inside one of the 28-sets $\mathcal{W}(\sigma)$ is termed a *special line;* the other external lines, which meet three distinct $\mathcal{W}(\sigma)$, will be referred to as *general lines.* An external flat which contains a special line is a *special flat;* the other external flats are termed *general flats.*

Lemma 1.1 *An external flat* X *contains at most one special line.*

Proof. Suppose to the contrary that X contains two distinct special lines, $L \subset \mathcal{W}(\sigma)$ and $L' \subset \mathcal{W}(\sigma')$. Then $\sigma \neq \sigma'$, since $\mathcal{W}(\sigma)$ contains no planes. Consider the plane $\alpha = \sigma \cap \sigma' \subset \mathrm{PG}(4, 2)$. Recall (1.1) and note that the three polar points α^b, $b \in L$, form a line $\lambda \subset \alpha$, and the three polar points $\alpha^{b'}$, $b' \in L'$, form a line $\lambda' \subset \alpha$. Let λ and λ' meet in the point $p \in \alpha$. So there exists $b \in L$ and $b' \in L'$ such that $\alpha^b = p = \alpha^{b'}$. Let μ be any line in α such that $p \notin \mu$; then μ^b is a line $\nu \subset \sigma$ which meets α in p, and $\mu^{b'}$ is a line $\nu' \subset \sigma'$ which also meets α in p. Moreover $b = m + n$ and $b' = m + n'$, and so $b + b' = n + n'$. Since the lines ν and ν' meet, $b + b'$ has rank 2 — contradicting the assumption that X is external. ■

2 The even hyperplane construction

Given a partial spread $\mathcal{S}_r = \{\mu_1, \dots, \mu_r\}$ in $\mathrm{PG}(4, 2)$, let the corresponding r-set of points of $\mathrm{Rk}_2 \subset \mathrm{PG}(9, 2)$ be $\mathcal{C}_r = \{m_1, \dots, m_r\}$. For $1 \leq i < j < \dots \leq r$ we define $m_{ij} = m_i + m_j$, $m_{ijk} = m_i + m_j + m_j$, ... , $m_\Sigma = m_{12\dots r} = \sum_{i=1}^{r} m_i$. One should be aware that some of these vectors may be zero, and that coincidences may therefore occur amongst the points (= nonzero vectors). For example, if $\mathcal{S}_5 = \{\mu_1, \dots, \mu_5\}$ is a spread for a solid $\sigma \subset \mathrm{PG}(4, 2)$ then $m_\Sigma = 0$, whence $m_{123} = m_{45}$, $m_{1234} = m_5$, etc.

Theorem 2.1 *(i)* $\mathcal{C}_r$ *is a* r*-cap: that is, no three points of* $\mathcal{C}_r$ *are collinear.*
(ii) The $\binom{r}{2}$ *points* m_{ij} *are distinct and are external.*
(iii) The $\binom{r}{3}$ *points* m_{ijk} *are distinct and are external.*

Proof. (i) No three points of $\mathcal{C}_r$ can be collinear, since lines internal to Rk_2 arise only from three intersecting lines of a pencil in $\mathrm{PG}(4, 2)$.

(ii) Since μ_i does not meet μ_j, it follows that $m_i + m_j \in \mathrm{Rk}_4$. Concerning distinctness, suppose to the contrary that, for example, $m_{12} = m_{34}$. Then all four lines μ_i, $i = 1, 2, 3, 4$, must lie in the solid $\sigma = \mathrm{im}(m_{12}) = \mathrm{im}(m_{34})$, and so must be members of a spread $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ for σ. But then, as noted prior to the theorem, $m_{12} + m_{34} = m_5$, contradicting $m_{12} = m_{34}$.

(iii) Whether or not $\{\mu_i, \mu_j, \mu_k\}$ is a regulus we have $m_i = e_1 \wedge e_3$, $m_j = e_2 \wedge e_4$ and $m_k = (e_1 + e_2) \wedge a$, for suitably chosen independent points e_1, e_2, e_3, e_4 generating the solid $\sigma_{ij} = \langle \mu_i, \mu_j \rangle$. It follows that $m_{ijk} = e_1 \wedge (e_3 + a) + e_2 \wedge (e_4 + a)$, whence $m_{ijk} \in \mathrm{Rk}_4$, since the lines $\langle e_1, e_3 + a \rangle$ and $\langle e_2, e_4 + a \rangle$ are mutually skew. Concerning distinctness, $m_{125} \neq m_{345}$, since $m_{1234} \neq 0$ for any $\mathcal{S}_4$. Also $m_{123} \neq m_{456}$, since a check shows that $m_\Sigma = m_{123456}$ is nonzero for all $\mathcal{S}_6$. ■

Since we are interested in external flats, part (ii) of the theorem suggests that we should investigate the projective space $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{\mathrm{even}}$ which is generated by the $\binom{r}{2}$ points m_{ij}. In cases where $m_1, \ldots, m_r$ are linearly independent, note that $\langle \mathcal{C}_r \rangle_{\mathrm{even}}$ is the *even hyperplane* of the generating set $\{m_1, \ldots, m_r\}$ for $\langle \mathcal{C}_r \rangle$; it is the unique hyperplane in $\langle \mathcal{C}_r \rangle$ which contains no element of $\mathcal{C}_r$, and so, amongst the hyperplanes in $\langle \mathcal{C}_r \rangle$, *only* $\mathcal{E}(\mathcal{C}_r)$ *is a candidate for being an external flat.* Now in general we can not expect $\mathcal{E}(\mathcal{C}_r)$ to be external, since, for example, some of the points m_{ijkl} may be internal. However, because all of the points m_{ij} are external, at least $\mathcal{E}(\mathcal{C}_r)$ is external if $r = 2$, in which case $\mathcal{E}(\mathcal{C}_2)$ is the external point $\{m_{12}\}$, and also if $r = 3$, in which case $\mathcal{E}(\mathcal{C}_3)$ is the external line $\{m_{12}, m_{13}, m_{23}\}$, as in lemma 3.2.

In the case $r = 2$ each external point b can in fact be expressed in the form $\{b\} = \mathcal{E}(\mathcal{C}_2)$ for precisely ten choices of partial spread $\mathcal{S}_2$. For if $b \in \mathcal{W}(\sigma)$ has the expression $b = m_0 + n_0$ for one pair $\mathcal{S}_2 = \{\mu_0, \nu_0\}$ of skew lines μ_0, ν_0 of σ, then it also has the nine expressions $b = m_i + n_i$, $i = 1, 2, \ldots, 9$, where $m_i = m_0 + t_i$ and $n_i = n_0 + t_i$, and where τ_i, $i = 1, 2, \ldots, 9$, are the nine transversals of the skew pair μ_0, ν_0. (Here, because τ_i meets μ_0, it follows that $m_0 + t_i$ is indeed the image of a line μ_i, namely the third line of the pencil determined by μ_0 and τ_i; similarly $n_0 + t_i$ is the image of a line ν_i.) The ten expressions $b = m_i + n_i$, $i = 0, 1, \ldots, 9$, correspond to the existence, for the polarity in σ determined by b, of precisely ten polar pairs $\{\mu_i, \nu_i\}$ of lines of σ, with ν_i being the polar $(\mu_i)^b$ of μ_i as in (1.1).

To follow this line of investigation further, that is to look at $\mathcal{E}(\mathcal{C}_r)$ for all $r > 2$, we need to know more about the partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4, 2)$.

3 Partial spreads in PG(4, 2)

The partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4, 2)$ have recently received a complete classification: see [3]. (The corresponding classification in $\mathrm{PG}(4, q)$ is not known for any other value of q.) For convenience we reproduce in Appendix A two tables from [3] which summarize the classification. In these tables roman numerals are used to indicate the size of the partial spread. Thus the six GL$(5,2)$-orbits for the underlying point-set of a partial spread $\mathcal{S}_7$ of size 7 are labelled VIIa, VIIb, ... , VIIf, and the five equivalence classes (GL$(5,2)$-orbits) of partial spreads $\mathcal{S}_7$ whose point-sets lie on the orbit VIIe are labelled VIIe.1, VIIe.2, ... ,VIIe.5. *[Note: we also use the same labels for the* GL$(5,2)$*-orbits (classes) of the corresponding r-caps $\mathcal{C}_r$.]*

Up to the action of GL$(5,2)$ there are, see Tables 1a, 1b in the Appendix, 64 distinct classes of partial spreads, the number s_r of equivalence classes of partial spreads $\mathcal{S}_r$ of size r being

$$s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 4,\ s_5 = 10,\ s_6 = 14,\ s_7 = 19,\ s_8 = 9,\ s_9 = 4.$$

For $r = 3$ the class IIIb.1 consists of the *reguli* (any 3 mutually skew lines lying in some solid), and the other class IIIa.1 consists of the *non-reguli* (any 3 mutually skew lines which generate the whole space PG$(4,2)$). Of the 64 classes, the eight classes Vj.1; VIIf.1, VIIf.2, VIIf.3; IXa.1, IXa.2, IXa.3, IXa.4 consist of maximal partial spreads; of these, class Vj.1 consists of spreads in some solid $\sigma \subset$ PG$(4,2)$ and the others are as described in [4]. A partial spread in PG$(4,2)$ is said to be of type I if it contains precisely one regulus, and to be of type O if it is regulus-free. See Table 2 in Appendix A for the other reguli patterns that occur. For $r \geq 3$ there are there are fourteen classes of partial spreads of type I:

$$\text{IIIb.1; IVc.1; Vf.1, g.1; VIc.1, c.2, d.1, d.2, e.1, e.2, f.1; VIIb.1, b.2,b.3,} \quad (3.1)$$

and twelve classes of regulus-free partial spreads:

$$\text{IIIa.1; IVa.1, b.1; Va.1, b.1, c.1, d.1, e.1; VIa.1, b.1; VIIa.1; VIIIa.1.} \quad (3.2)$$

Of the regulus-free partial spreads, three classes, namely Va.1, VIb.1 and VIIIa.1, have the maximality property of admitting no extension to a larger regulus-free partial spread, with Va.1 and VIIIa.1 being noteworthy in having relatively large symmetry groups. The stabilizer for class VIb.1 is only of order 2, but a partial spread $\mathcal{S}_5$ of class Va.1 (which is a spread on some parabolic quadric $\mathcal{P}_4 \subset$ PG$(4,2)$) has stabilizer $\mathcal{G}(\mathcal{S}_5) \cong$ Sym(5), and a partial spread $\mathcal{S}_8$ of class VIIIa.1 has stabilizer $\mathcal{G}(\mathcal{S}_8) \cong 2^3 : (7:3)$. The next lemma describes one aspect of this high symmetry of the regulus-free partial spreads of size 8.

Lemma 3.1 *If $\mathcal{S}_5 \subset \mathcal{S}_6 \subset \mathcal{S}_7 \subset \mathcal{S}_8$ is any chain of partial spreads with $\mathcal{S}_8$ of class VIIIa.1, then $\mathcal{S}_7$ is of class VIIa.1, $\mathcal{S}_6$ is of class VIa.1 and $\mathcal{S}_5$ is of class Vd.1.*

Proof. First note that $\mathcal{S}_5$, $\mathcal{S}_6$ and $\mathcal{S}_7$ are regulus-free. The assertions concerning $\mathcal{S}_6$ and $\mathcal{S}_7$ thus follow from the list (3.2), since class VIb.1 for $\mathcal{S}_6$

is ruled out because of the maximality property mentioned after (3.2). See [3, Theorem 6.4(ii)] for the assertion concerning $\mathcal{S}_5$. ∎

Let $\mathcal{E}(\mathcal{C}_r)$ be the projective flat, defined as in section 2, which arises from a partial spread $\mathcal{S}_r$ of size r. The following lemmas will enable us to determine all those partial spreads $\mathcal{S}_r$ for which $\mathcal{E}(\mathcal{C}_r)$ is external.

Lemma 3.2 *If $\mathcal{S}_3$ is any partial spread of size 3 then $\mathcal{E}(\mathcal{C}_3)$ is an external line. Moreover*
(a) if $\mathcal{S}_3$ is a non-regulus (of class IIIa.1) then $\mathcal{E}(\mathcal{C}_3)$ is a general line;
(b) if $\mathcal{S}_3$ is a regulus (of class IIIb.1) then $\mathcal{E}(\mathcal{C}_3)$ is a special line.

Proof. By theorem 2.1(ii), $\mathcal{E}(\mathcal{C}_3) = \{m_{12}, m_{13}, m_{23}\}$ is an external line. If $\mathcal{S}_3$ is a regulus in a solid σ, each of $\mathrm{im}(m_{ij}) = \sigma$, so each $m_{ij} \in \mathcal{W}(\sigma)$. On the other hand, for a non-regulus, the $\mathrm{im}(m_{ij})$ are three distinct solids. ∎

Lemma 3.3 *If $\mathcal{E}(\mathcal{C}_r)$ is an external flat then the partial spread $\mathcal{S}_r$ is either of type O or of type I.*

Proof. This follows immediately from lemmas 1.1 and 3.2. ∎

Lemma 3.4 *If $\mathcal{S}_4$ is any partial spread of size 4 then $\mathcal{E}(\mathcal{C}_4)$ is a plane. Moreover the plane $\mathcal{E}(\mathcal{C}_4)$ is external for just two of the four classes of $\mathcal{S}_4$:*
(a) if $\mathcal{S}_4$ is of class IVb.1 then $\mathcal{E}(\mathcal{C}_4)$ is a general plane;
(b) if $\mathcal{S}_4$ is of class IVc.1 then $\mathcal{E}(\mathcal{C}_4)$ is a special plane.

Proof. A straightforward check shows that m_{1234} is an internal point in the case of classes IVa.1 and IVd.1, but is an external point for the other two classes IVb.1 and IVc.1. Case (a) gives rise to a general plane, because $\mathcal{S}_4$ is regulus-free, while case (b) gives rise to a special plane, because $\mathcal{S}_4$ is of type I. ∎

Lemma 3.5 *If $\mathcal{S}_5$ is a partial spread of size 5 then $\langle\mathcal{C}_5\rangle$ is a 4-flat — except if $\mathcal{S}_5$ is of class Va.1 or of class Vj.1, when $\langle\mathcal{C}_5\rangle$ is a 3-flat. Without exception $\mathcal{E}(\mathcal{C}_5)$ is a 3-flat. Moreover the solid $\mathcal{E}(\mathcal{C}_5)$ is external for precisely two of the ten classes of $\mathcal{S}_5$:*
(a) if $\mathcal{S}_5$ is of class Ve.1 then $\mathcal{E}(\mathcal{C}_5)$ is a general solid;
(b) if $\mathcal{S}_5$ is of class Vg.1 then $\mathcal{E}(\mathcal{C}_5)$ is a special solid.

Proof. Representatives for the ten classes of partial spread $\mathcal{S}_5$ of size 5 are given in [3]. Using these, a straightforward check shows that the five elements $m_i \in \mathcal{C}_5$ are linearly independent — except for the two classes Va.1 and Vj.1, for which the single linear relation $m_\Sigma := m_{12345} = 0$ holds. So $\langle\mathcal{C}_5\rangle$ is a 4-flat, except for $\mathcal{S}_5$ of class Va.1 or Vj.1, but for all ten classes $\mathcal{E}(\mathcal{C}_5)$ is a 3-flat. For the solid $\mathcal{E}(\mathcal{C}_5)$ to be an external general solid we must, by the preceding lemma, have $\mathcal{C}_4$ of class IVb.1 for each $\mathcal{C}_4 \subset \mathcal{C}_5$.

This requirement rules out classes Va.1-Vd.1, since of the partial spreads $\mathcal{S}_5$ of type O only those of class Ve.1, see [3, lead-in to eq.(3.24)], have the property that $\mathcal{S}_5 \setminus \{\mu\}$ is of class IVb.1 for each $\mu \in \mathcal{S}_5$. One now checks that $\mathcal{E}(\mathcal{C}_5)$ is indeed external if $\mathcal{S}_5$ is of class Ve.1. Finally, there are just two classes, Vf.1 and Vg.1, of partial spread $\mathcal{S}_5$ of type I, and a straightforward check shows that only for class Vg.1 is $\mathcal{E}(\mathcal{C}_5)$ external. ■

Lemma 3.6 *If $\mathcal{S}_6$ is a partial spread of size 6 then $\langle\mathcal{C}_6\rangle$ is a 5-flat — except if $\mathcal{S}_6$ is of class VIc.2, when $\langle\mathcal{C}_6\rangle$ is a 4-flat. Without exception $\mathcal{E}(\mathcal{C}_6)$ is a 4-flat, and moreover $\mathcal{E}(\mathcal{C}_6)$ is external for no choice of $\mathcal{S}_6$.*

Proof. Using representatives for the fourteen classes of partial spread $\mathcal{S}_6$ of size 6 as given in [3] one finds that $m_\Sigma := m_{123456}$ is nonzero in all cases. Consequently the only possible linear relations amongst the six elements $m_i \in \mathcal{C}_6$ are 5-term ones present in some $\mathcal{C}_5 \subset \mathcal{C}_6$. Since a $\mathcal{C}_5$ of class Vj.1 has no extensions, by the preceding lemma a $\mathcal{C}_6$ for which a linear relation holds must be an extension of a $\mathcal{C}_5$ of class Va.1. By [3, Lemma A.10(ii)], all such extensions are of class VIc.2.

Consider the two regulus-free classes VIa.1 and VIb.1. If $\mathcal{S}_6$ is of class VIa.1 then $\mathcal{E}(\mathcal{C}_6)$ is not external because, by lemmas 3.1 and 3.5, $\mathcal{C}_6$ contains a $\mathcal{C}_5$ of class Vd.1, for which $\mathcal{E}(\mathcal{C}_5)$ is not external. Similarly, if $\mathcal{S}_6$ is of class VIb.1 then $\mathcal{E}(\mathcal{C}_6)$ is not external because, see [3, Theorem 6.5], it contains 3-flats $\mathcal{E}(\mathcal{C}_5)$ which are not external. By lemma 3.3, the only other possibilities for the 4-flat $\mathcal{E}(\mathcal{C}_6)$ to be external arise from the seven classes of $\mathcal{S}_6$ of type I. But on consulting [3, Section 7.3.1] we see that such a $\mathcal{S}_6$ is an extension of a $\mathcal{S}_5$ of class Vf.1, and so by lemma 3.5 contains a 3-flat $\mathcal{E}(\mathcal{C}_5)$ which is not external. ■

Remark 3.7 *If $\mathcal{S}_r, r > 5$, contains a $\mathcal{S}_5$ of class Va.1, then accordingly there exist five elements of $\mathcal{C}_r$ which satisfy a linear relation. In fact,* with one exception, *linear relations for a $\mathcal{C}_r$ with $r > 5$ exist only if $\mathcal{S}_r$ is an extension of a $\mathcal{S}_5$ of class Va.1 (that is of a spread on a parabolic quadric). The exception occurs for $r = 8$ with $\mathcal{S}_8$ of class VIIIa.1, where the eight elements of $\mathcal{C}_8$ are found to satisfy the single linear relation $m_\Sigma = 0$.*

4 Conclusion and discussion

Using the even hyperplane construction we have seen in lemmas 3.2, 3.4, 3.5 that, for each $k = 1, 2, 3$, we can construct an orbit of general k-flats and also an orbit of special k-flats. Let us label the former three orbits orb($k\alpha$), $k = 1, 2, 3$, and the latter three orb($k\beta$), $k = 1, 2, 3$. The former three orbits consist of external k-flats of the form $\mathcal{E}(\mathcal{C}_{k+2})$ with $\mathcal{S}_{k+2}$ of class IIIa.1, IVb.1, Ve.1 for $k = 1, 2, 3$, respectively; similarly for the latter three orbits, but with $\mathcal{S}_{k+2}$ of class IIIb.1, IVc.1, Vg.1 for $k = 1, 2, 3$, respectively.

In section 2 we also noted that the external 0-flats, which form the orbit Rk_4, are of the form $\mathcal{E}(\mathcal{C}_2)$. Recently, see [5], all k-flats in $\mathrm{PG}(9,2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ have been classified. There are precisely ten orbits:

$$\begin{array}{llll} k=0: & \mathrm{Rk}_4; & k=1: & \mathrm{orb}(1\alpha),\ \mathrm{orb}(1\beta); \\ k=2: & \mathrm{orb}(2\alpha),\ \mathrm{orb}(2\beta),\ \mathrm{orb}(2\gamma); & & \\ k=3: & \mathrm{orb}(3\alpha),\ \mathrm{orb}(3\beta); & k=4: & \mathrm{orb}(4+),\ \mathrm{orb}(4-). \end{array} \quad (4.1)$$

Theorem 4.1 *Of the ten orbits (4.1), all but* $\mathrm{orb}(2\gamma)$, $\mathrm{orb}(4+)$ *and* $\mathrm{orb}(4-)$ *can be obtained using the even hyperplane construction.*

Proof. By using the even hyperplane construction we have already obtained seven of the orbits; however, as a consequence of lemma 3.6, no further external k-flats can be thus obtained. ■

Although our construction has, very simply and efficiently, yielded seven orbits of external flats, it has to be said that it offers no help in proving that there exist just the three further orbits listed in (4.1). In [5] can be found two different proofs that external 5-flats do not exist, see [5, Theorems 1.1, 1.3]. See also [1], [2] for some more general results. As shown in [5] an external 4-flat belonging to either of the two orbits $\mathrm{orb}(4\pm)$ has stabilizer the normalizer $N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5,2)$. Incidentally, since the external 4-flats are general flats, see [5, Theorem 1.3], note that the special solids of $\mathrm{orb}(3\beta)$ are, along with the 4-flats of $\mathrm{orb}(4\pm)$, maximal external flats.

A representative P of the orbit $\mathrm{orb}(2\gamma)$ is displayed in [5, Theorem 4.1], and its stabilizer $\mathcal{G}_P$ is shown to have the structure $2^3:(7:3)$. Thus $\mathcal{G}_P$ is isomorphic to the stabilizer $\mathcal{G}_{\mathcal{S}_8}$ of a regulus-free partial spread $\mathcal{S}_8$ (that is one of class VIIIa.1). Naturally one wonders whether there exist constructions of an external plane $P \in \mathrm{orb}(2\gamma)$ which start out form a regulus-free partial spread $\mathcal{S}_8$. For the (affirmative) answer, see [6].

A Appendix: partial spreads in $\mathrm{PG}(4,2)$

The following tables, reproduced from [3], provide details of the 64 $\mathrm{GL}(5,2)$-orbits of partial spreads which exist in $\mathrm{PG}(4,2)$. In the second column N_r is the number of reguli contained in the partial spread $\mathcal{S}_r$. In the final column it should be noted that *the references are to sections, lemmas and theorems of [3], and **not** to ones in the present paper.*

Profiles are as described in.[3, Section 1.1]. In the tables a profile $(1,2,3)^3(2,2,2)$ is written $(123)^3(222)$. We use `t` as an abbreviation for 10. Dots are used to separate groups of lines of the same valency, with the valencies occurring in descending order. Thus for an $\mathcal{S}_6$ of type `L` a profile $(557).(456)^4.(355)$ conveys the information that the line of valency 2 has profile $(5,5,7)$, each of the four lines of valency 1 has profile $(4,5,6)$ and the line of valency 0 has profile $(3,5,5)$. The entry $2\times(456)^2(447)$ for the profile of class VIg.1, of type II, indicates that both reguli contribute $(4,5,6)^2(4,4,7)$ to the overall profile. In the cases VIIIc.5, c.6 of an $\mathcal{S}_8$ of type IL the 7 lines which belong to a single regulus have a natural $4+3$ split, with the 3 forming the stand-alone regulus. A vertical line | is used to separate the profile of the 4 from that of the 3.

Table 1a: the classes of partial spreads $\mathcal{S}_r$, $1\le r\le 6$

Class($\mathcal{S}_r$)	N_r	Type	$\mathcal{G}(\mathcal{S}_r)$	profile($\mathcal{S}_r$)	Notes
Ia.1	0	O	64512	(111)	2^6:($L_2(2)\times L_3(2)$)
IIa.1	0	O	1152	$(111)^2$	$\{2^4$:($L_2(2)\times L_2(2)$)$\}$.2
IIIa.1	0	O	$\mathrm{Sym}(4)\times Z_2$	$(112)^3$	§3.2, non-regulus, cyclic
IIIb.1	1	I	576	$(222)^3$	§3.1, regulus, cyclic
IVa.1	0	O	Sym(4)	$(222)^4$	§3.3.1, cyclic
IVb.1	0	O	Sym(3)	$(123)^3(222)$	§3.3.2
IVc.1	1	I	$\mathrm{Sym}(3)\times Z_2$	$(233)^3(114)$	§3.1, §3.5, eq. (3.28)
IVd.1	4	$\binom{4}{3}$	1152	$(444)^4$	§3.1, cyclic
Va.1	0	O	Sym(5)	$(333)^5$	§3.4, §A.3.2, cyclic
Vb.1	0	O	D_8	$(333)^4(135)$	§3.4
Vc.1	0	O	$\mathrm{Sym}(3)\times Z_2$	$(333)^5$	§3.4
Vd.1	0	O	Z_3	$(234)^3(333)^2$	§3.4
Ve.1	0	O	Z_5	$(234)^5$	§3.4, cyclic
Vf.1	1	I	Z_2	$(344)^2(335).(234)^2$	§3.5
Vg.1	1	I	Z_2	$(344)^3.(135)(225)$	§3.5
Vh.1	2	L	D_8	$(355).(335)^4$	§3.5, lemma 4.1
Vi.1	4	$\binom{4}{3}$	$\mathrm{Sym}(4)\times Z_2$	$(555)^4.(117)$	§3.1
Vj.1	10	$\binom{5}{3}$	$2^4{:}\Gamma\,\mathrm{L}(2,4)$	$(777)^5$	§3.1, **maximal**, cyclic
VIa.1	0	O	Z_6	$(445)^6$	§6.2, cyclic
VIb.1	0	O	Z_2	$(355)^2(445)^4$	§6.2
VIc.1	1	I	Sym(3)	$(555)^3.(355)^3$	§7.3.1
VIc.2	1	I	Sym(3)	$(555)^3.(355)^3$	§7.3.1
VId.1	1	I	Sym(3)	$(555)^3.(445)^3$	§7.3.1
VId.2	1	I	Sym(3)	$(555)^3.(445)^3$	§7.3.1
VIe.1	1	I	1	$(456)^2(555).(346)^2(445)$	§7.3.1
VIe.2	1	I	1	$(456)^2(555).(346)^2(445)$	§7.3.1
VIf.1	1	I	Z_2	$(555)^3.(355)^2(337)$	§7.3.1
VIg.1	2	II	Z_4	$2\times(456)^2(447)$	§7.2.1
VIh.1	2	L	1	$(566).(456)^2(447)(555).(355)$	§4.2, lemma 4.2
VIh.2	2	L	Z_2	$(557).(456)^4.(355)$	§4.2, lemma 4.2
VIi.1	3	Δ	Sym(3)	$(557)^3.(555)^3$	§4.2, lemma 4.3
VIj.1	4	$\binom{4}{3}$	D_8	$(667)^4.(337)^2$	§7.1, lemma 7.2

Table 1b: the classes of partial spreads $\mathcal{S}_r$, $r > 6$

Class($\mathcal{S}_r$)	N_r	Type	$\mathcal{G}(\mathcal{S}_r)$	profile($\mathcal{S}_r$)	Notes
VIIa.1	0	O	7:3	$(666)^7$	§6.1, cyclic
VIIb.1	1	I	Z_2	$(668)(677)^2.(567)^4$	§7.3.2
VIIb.2	1	I	Z_2	$(677)^3.(558)(567)^2(666)$	§7.3.2
VIIb.3	1	I	Z_2	$(677)^3.(558)(567)^2(666)$	§7.3.2
VIIc.1	2	II	Z_4	$\{2 \times (668)(677)^2\}.(558)$	§7.2.2, lemma 7.3
VIIc.2	2	II	Z_4	$\{2 \times (668)(677)^2\}.(558)$	§7.2.2, lemma 7.3
VIIc.3	2	L	Z_2	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.4	2	L	Z_2	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.5	2	L	1	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.6	2	L	1	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIId.1	2	II	1	$(668)^6.(666)$	§7.2.2, lemma 7.3
VIIe.1	3	Δ	Sym(3)	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.2	3	Δ	Z_3	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.3	3	Δ	Z_2	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.4	3	Y	Z_6	$(888).(677)^6$	§4.4
VIIe.5	3	F	Z_2	$(778)^2.(668)(677)^4$	§4.4
VIIf.1	4	$\binom{4}{3}_{(4,0)}$	Sym(4)	$(888)^4.(558)^3$	§7.1, **maximal**
VIIf.2	4	$\binom{4}{3}_{(0,4)}$	Sym(4)	$(888)^4.(558)^3$	§7.1, **maximal**
VIIf.3	4	$\binom{4}{3}_{(2,2)}$	Sym(3) × Z_2	$(888)^4.(558)^3$	§7.1, **maximal**
VIIIa.1	0	O	$2^3{:}\mathrm{F}_{21}$	$(888)^8$	§6.1, transitive
VIIIb.1	2	II	Z_2	$(88\mathtt{t})^6.(888)^2$	§5.2
VIIIc.1	3	Δ	Sym(3)	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.2	3	Δ	Z_3	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.3	3	Δ	Z_2	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.4	3	Y	Z_3	$(\mathtt{ttt}).(899)^6.(888)$	§5.2
VIIIc.5	3	$\mathsf{I}^\rho\mathsf{L}$	Z_2	$(99\mathtt{t}).(899)^2(88\mathtt{t})^2\vert(899)^3$	§5.2
VIIIc.6	3	$\mathsf{I}^\kappa\mathsf{L}$	Z_2	$(99\mathtt{t}).(899)^2(88\mathtt{t})^2\vert(899)^3$	§5.2
VIIIc.7	3	F	1	$(99\mathtt{t})^2.(888)(899)^4.(888)$	§5.2
IXa.1	4	X	Alt(4) × Z_2	$(13,13,13).(11,11,11)^8$	§5.1, thms 5.1, 6.2 **maximal**
IXa.2	4	$\mathsf{I}^\rho\Delta$	Sym(3)	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**
IXa.3	4	$\mathsf{I}^\kappa\Delta$	Sym(3)	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**
IXa.4	4	E	Z_6	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**

Reguli patterns and type. Given a partial spread $\mathcal{S}_r$ in PG(4, 2), let R_{ijk}, for distinct i, j, k, carry the meaning that the triple of lines $\{\mu_i, \mu_j, \mu_k\} \subset \mathcal{S}_r$ is a regulus. The only *reguli patterns* which arise are those listed in the second column of the following Table 2, which is reproduced from [3]. The regulus *type,* see the third column of Table 2 (and also of Tables 1a, 1b), informs us of the pattern in a conveniently abbreviated form.

Table 2 The possible reguli patterns

N	Regulus Pattern	Type	Types for $\mathcal{S}_r \setminus \{\mu\}$
0	regulus-free	O	O
1	R_{123}	I	I (if $r > 3$), O
2	R_{123}, R_{456}	II	II (if $r > 6$), I
2	R_{123}, R_{345}	L	L (if $r > 5$), I, O
3	$R_{123}, R_{456}, R_{678}$	IL	L, II, I
3	$R_{123}, R_{145}, R_{167}$	Y	Y (if $r > 7$), L, O
3	$R_{123}, R_{145}, R_{267}$	F	F (if $r > 7$), II, L, I
3	$R_{123}, R_{345}, R_{561}$	Δ	Δ (if $r > 6$), L, I
4	$R_{123}, R_{345}, R_{561}, R_{789}$	IΔ	Δ, IL, II
4	$R_{123}, R_{145}, R_{167}, R_{189}$	X	Y, O
4	$R_{123}, R_{145}, R_{267}, R_{389}$	E	F, II
4	$R_{123}, R_{124}, R_{134}, R_{234}$	$\binom{4}{3}$	$\binom{4}{3}$ (if $r > 4$), I
10	$R_{ijk},\ 1 \leq i < j < k \leq 5$	$\binom{5}{3}$	$\binom{4}{3}$ ($r = 5$)

References

[1] B. N. Cooperstein, External flats to varieties in PG($\Lambda^2 V$) over finite fields. *Geom. Dedicata* **69** (1998), 223-235.

[2] A. Cossidente and A. Siciliano, On tangent spaces and external flats to Grassmannians of lines over finite fields, *Linear Algebra Appl.,* **347** (2002), 81-89.

[3] N. A. Gordon, R. Shaw and L. H. Soicher, Classification of Partial Spreads in PG(4, 2), submitted to *Electronic J. Combinatorics,* (63 pp., available from: http://www.hull.ac.uk/maths/people/rs/staffdetails.html .)

[4] R. Shaw, Subsets of PG(n, 2) and maximal partial spreads in PG(4, 2), *Designs, Codes and Cryptography,* **21** (2000), 209-222.

[5] R. Shaw, J.G. Maks and N.A. Gordon, The classification of flats in PG(9,2) which are external to the Grassmannian $\mathcal{G}_{1,4,2}$, to appear in *Des. Codes Cryptogr.* (24 pp., available from: http://www.hull.ac.uk/maths/people/rs/staffdetails.html .)

[6] R. Shaw and J.G. Maks, Conclaves of planes in PG(4, 2) and certain planes external to the Grassmannian $\mathcal{G}_{1,4,2} \subset$ PG(9, 2), *J. Geom.* **78** (2003), 168-180.

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