# The quintic Grassmannian $\mathcal{G}_{1,4,2}$ in PG(9, 2)

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**Abstract**

The 155 points of the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2) = \mathbb{P}V(5,2)$ are those points $x \in \mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which satisfy a certain quintic equation $Q(x) = 0$. (The quintic polynomial $Q$ is given explicitly in [3].) A projective flat $X \subset \mathrm{PG}(9,2)$ will be termed *odd* or *even* according as $X$ intersects $\mathcal{G}_{1,4,2}$ in an odd or even number of points. Let $Q^{\ddagger}(x_1, \ ... \ x_5)$ denote the alternating quinquelinear form obtained by completely polarizing $Q$. We define the *associate* $Y = X^{\#}$ of a $r$-flat $X \subset \mathrm{PG}(9,2)$ by
$Y = \{y \in \mathrm{PG}(n,2) \,|\, Q^{\ddagger}(x_1, x_2, x_3, x_4, y) = 0 \text{ for all } x_1, x_2, x_3, x_4 \in X\}$.
Because $Q^{\ddagger}$ is quinquelinear, the associate $X^{\#}$ of an $r$-flat $X$ is an $s$-flat for some $s$. The cases where $r = 4$ are of particular interest: if $X$ is an odd 4-flat then $X \subseteq X^{\#}$ while if $X$ is an even 4-flat then $X^{\#}$ is necessarily also a 4-flat which is moreover disjoint from $X$. We give an example of an odd 4-flat $X$ which is self-associate: $X^{\#} = X$. An example of an even 4-flat $X$ such that $(X^{\#})^{\#} = X$ is provided by any 4-flat $X$ which is external to $\mathcal{G}_{1,4,2}$. However it appears that the two possibilities just illustrated, namely $X^{\#} = X$ for an odd 4-flat and $(X^{\#})^{\#} = X$ for an even 4-flat, are the exception rather than the rule. Indeed, we provide examples of odd 4-flats for which $X^{\#} = \mathrm{PG}(9,2)$ and of even 4-flats for which $X^{\#\#\#} = X$.

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## 1 Introduction

### 1.1 Preliminaries and aim

Throughout we work over GF(2), and so *we may identify the nonzero elements of a vector space* $V(n+1,2) = V_{n+1}$ *with the points* $S_0$ *of the associated projective space* $\mathbb{P}V_{n+1} = \mathrm{PG}(n,2)$. Consequently we identify

$\mathrm{GL}(V_{n+1}) = \mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. Similarly nonzero elements of the dual vector space $V_{n+1}^*$ will be identified with the points of the dual projective space $\mathbb{P}V_{n+1}^* = \mathrm{PG}(n,2)^*$. We use $\prec u, v, \ldots \succ$ for the vector subspace spanned by vectors $u, v, \ldots$, and $\langle u, v, \ldots \rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$ . Recall that the annihilator $U^{\mathrm{O}} := \{f \in V_{n+1}^* | f(u) = 0, \text{ for all } u \in U\}$ of any subset $U \subset V_{n+1}$ is always a subspace of $V_{n+1}^*$; moreover if $\dim \prec U \succ = r+1$ then $\dim U^{\mathrm{O}} = n - r$. *We use the same notation also for projective subspaces.* Thus if $\alpha$ is a plane in $\mathrm{PG}(4,2)$ then $\alpha^{\mathrm{O}}$ is a line in $\mathrm{PG}(4,2)^*$.

The vector space $F(S_0)$ of all functions $S_0 \to \mathrm{GF}(2)$ is of dimension $|S_0| = 2^{n+1} - 1$, one basis being $\{\chi_a\}_{a \in S_0}$, where $\chi_a$ is the characteristic function of the singleton subset $\{a\} \subset S_0$. Given a choice of coordinates $x_1, x_2, \ldots, x_{n+1}$ in $V_{n+1}$, there are $\binom{n+1}{r}$ monomials $\{x_{i_1} x_{i_2} \ldots x_{i_r}\}_{1 \le i_1 < \ldots < i_r \le n+1}$ of (reduced) degree $r$. So altogether we have $\Sigma_{r=1}^{n+1} \binom{n+1}{r} = |S_0|$ linearly independent monomials, and these form another basis for $F(S_0)$. Given a point-set $\psi \subset \mathrm{PG}(n,2)$ it follows that it has equation $Q(x) = 0$ for some *uniquely determined* polynomial function $Q$ on $V_{n+1}$ (of reduced degree $\le n+1$ and satisfying $Q(0) = 0$).

In this paper we deal throughout with the case $\psi = \mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$ of the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2)$, for which, see [3], $Q$ has reduced degree 5. Our aim is to study the *associate* $X^{\#}$ of flats $X \subset \mathrm{PG}(9,2)$, as defined in section 1.3 in terms of the completely polarized form $Q^{\ddagger}$ of the quintic $Q$. The associate $X^{\#}$ is a generalization to $\deg Q > 2$ of the orthogonal complement $X^{\perp}$ in the case $\deg Q = 2$. In order to bypass the complexity of dealing with the quinquelinear function $Q^{\ddagger}$ our chief weapon is the rephrasing, see lemma 1.5, of the definition of $X^{\#}$ in terms of incidence properties.

It should be noted that *many of our considerations can be generalized so as to apply to any point-set $\psi$ of any projective space* $\mathrm{PG}(n,2)$ *over* $\mathrm{GF}(2)$; *see [4].* However our considerations do *not* generalize to projective spaces $\mathrm{PG}(n,q)$ for $q \neq 2$. In addition to the lack of uniqueness of $Q$, there is no $q > 2$ analogue of the crucial lemma 1.5. Also in most cases the degree of $Q$ is uncomfortably high; for example $\deg Q = 10$ in the case of the $q = 3$ Grassmannian $\mathcal{G}_{1,4,3}$.

In the following, points, lines, ... , of $\mathrm{PG}(9,2)$ which lie on the Grassmannian $\mathcal{G}_{1,4,2}$ will be called *internal* points, lines, ... , and those which lie off $\mathcal{G}_{1,4,2}$ will be termed *external.* A subset $X \subset \mathrm{PG}(9,2)$ is termed *odd* or *even* according as it contains an odd or even number of internal points.

**Theorem 1.1** *For $r \ge 5$ every $r$-flat $X$ in* $\mathrm{PG}(9,2)$ *is odd.*

*Proof.* This follows from $\deg Q = 5$; see the theorem in [9]. ■

We are starting out from a vector space $V_5 = V(5,2)$, and its projective space $\mathrm{PG}(4,2) = \mathbb{P}V_5$, along with the concomitant space $V_{10} := \wedge^2 V_5$ of bivectors. Whilst the dual $(V_{10})^*$ of $V_{10}$ may be viewed as the space $\wedge^2 V_5^*$ of dual bivectors, we will usually view it as the space $V_{10}^* := \wedge^3 V_5$ of trivectors, by means of the natural nondegenerate bilinear pairing $\lceil \cdot, \cdot \rfloor$ of $\wedge^3 V_5$ with $\wedge^2 V_5$ defined by

$$t \wedge b = \lceil t, b \rfloor\, e, \quad t \in \wedge^3 V_5,\ b \in \wedge^2 V_5. \tag{1.1}$$

Here $e$ is the (unique!) basis vector for the 1-dimensional space $\wedge^5 V_5$. The (projective) annihilator of a subset $U$ of $\mathrm{PG}(9,2)^* = \mathbb{P}V_{10}^*$ is thus that flat $U^{\mathrm{O}}$ of $\mathrm{PG}(9,2) = \mathbb{P}V_{10}$ defined by

$$U^{\mathrm{O}} = \{b \in \mathrm{PG}(9,2) |\ u \wedge b = 0,\ \text{for all } u \in U\}. \tag{1.2}$$

If $U$ is a single point $t$ then $U^{\mathrm{O}}$ is a hyperplane, to be denoted $H(t) := \langle t \rangle^{\mathrm{O}}$.

For more material surrounding this choice of $\wedge^3 V_5$ as dual of $V_{10}$ see [4, Appendix A], where details may also be found concerning:
(i) the fact, [8, Section 9.6.2], that $\wedge^3 V_5$ and $\wedge^2 V_5^*$ are images of each other under the (unique!) Poincaré isomorphisms $\perp : \wedge^2 V_5^* \to \wedge^3 V_5$, $\perp' = (\perp)^{-1} : \wedge^3 V_5 \to \wedge^2 V_5^*$;
(ii) the alternating bilinear form $t(\cdot, \cdot)$ determined by $t \in \wedge^3 V_5$:

$$t \wedge v_1 \wedge v_2 = t(v_1, v_2) e; \tag{1.3}$$

(iii) the 'polar' $\alpha^t$ of a flat $\alpha \subset \mathrm{PG}(9,2)$ with respect to a particular $t \in \wedge^3 V_5$:

$$\alpha^t = \{v \in \mathrm{PG}(4,2) | t(x, v) = 0 \quad \text{for all } x \in \alpha\}. \tag{1.4}$$

A flat $\alpha \subset \mathrm{PG}(4,2)$ is termed *isotropic* if $\alpha \subset \alpha^t$, and *selfpolar* if $\alpha = \alpha^t$. In the case when $t \in \mathrm{Rk}_2^*$ is the Grassmann image of a plane $\pi \subset \mathrm{PG}(4,2)$ *a line* $\lambda$ *is isotropic if and only if* $\lambda$ *meets* $\pi$.

## 1.2 The action of GL(5, 2) upon PG(9, 2) and PG(9, 2)*

Each $A \in \mathrm{GL}(5,2)$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\mathrm{GL}(V_{10})$ whose effect on the decomposable bivectors $u \wedge v \in V_{10}$ is $T_A(u \wedge v) = Au \wedge Av$, $A \in \mathrm{GL}(5,2)$. Similarly we put $\hat{T}_A = \wedge^3 A \in \mathrm{GL}(V_{10}^*)$. Since $(\wedge^5 A)e = e$ for all $A \in \mathrm{GL}(5,2)$, note the invariance property

$$\hat{T}_A t \wedge T_A b = t \wedge b, \quad t \in \wedge^3 V_5,\ b \in \wedge^2 V_5,\ \text{for all } A \in \mathrm{GL}(5,2). \tag{1.5}$$

Thus $\hat{T}_A$ is the contragredient of $T_A$ : $\lceil \hat{T}_A t, T_A b \rfloor = \lceil t, b \rfloor$.

If $X$ is an object of some GL(5, 2)-space then $\mathcal{G}_X < \mathrm{GL}(5,2)$ denotes its stabilizer group. In particular if $X$ is an object in $\wedge^2 V_5$ its stabilizer is $\mathcal{G}_X = \{A \in \mathrm{GL}(5,2) | T_A(X) = X\}$. Under the action $T$ of GL(5, 2) the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$ is the union $\mathrm{Rk}_2 \cup \mathrm{Rk}_4$ of two GL(5, 2)-orbits, consisting of those bivectors having rank 2 and rank 4, respectively.

The Grassmann map $\prec u,v\succ\mapsto\prec u\wedge v\succ$ sends the 2-spaces of $V_5$ to those 1-spaces of $\wedge^2 V_5$ which are spanned by decomposable bivectors. Projectively, the lines of $\mathrm{PG}(4,2)$ are mapped onto the points of the orbit $\mathrm{Rk}_2$, the latter, being the Grassmannian $\mathcal{G}_{1,4,2}\subset\mathrm{PG}(9,2)$ of lines of $\mathrm{PG}(4,2)$, having length 155. Consequently $|\,\mathrm{Rk}_4\,| = 1023-155 = 868$. Throughout this paper the images in $G_{1,4,2}\subset\mathrm{PG}(9,2)$ of lines $\lambda,\mu$ in $\mathrm{PG}(4,2)$ will be denoted $l,m$. Similarly under the action $\hat{T}$ the projective space $\mathrm{PG}(9,2)^* = V_{10}^*\setminus\{0\}$ splits into two $\mathrm{GL}(5,2)$-orbits, say $\mathrm{Rk}_2^*$ and $\mathrm{Rk}_4^*$, of lengths 155 and 868. Here the orbit $\mathrm{Rk}_2^*$ of length 155 is the Grassmannian $\mathcal{G}_{2,4,2}$ consisting of the 155 decomposable trivectors, the Grassmann images of the 155 planes of $\mathrm{PG}(4,2)$, with $a\wedge b\wedge c$ being the image of the plane $\langle a,b,c\rangle$. Of course the decomposition $\mathrm{PG}(9,2)^* = \mathrm{Rk}_2^*\cup\mathrm{Rk}_4^*$ is the image under the Poincaré isomorphism $\perp$ of the decomposition, say $\mathbb{P}(\wedge^2 V_5^*) = \mathrm{Rk}_2(V_5^*)\cup\mathrm{Rk}_4(V_5^*)$, of $\mathbb{P}(\wedge^2 V_5^*)$ into dual bivectors of ranks 2 and 4.

A bivector $b\in\mathrm{Rk}_4$ can be expressed $b = u\wedge v + x\wedge y$, for linearly independent $x,y,u,v\in V_5$ and defines a solid $\mathrm{im}\, b := \langle u,v,x,y\rangle\subset\mathrm{PG}(4,2)$, and also a point $k^* := \ker b\in\mathrm{PG}(4,2)^*$, where $\mathrm{im}\, b$ has equation $k^*(x)=0$. Similarly a dual bivector $b^*\in\mathrm{Rk}_4(V_5^*)$ can be expressed $b^* = u^*\wedge v^* + x^*\wedge y^*$, for linearly independent $x^*,y^*,u^*,v^*\in V_5^*$ and defines a solid $\mathrm{im}\, b^* := \langle u^*,v^*,x^*,y^*\rangle\subset\mathrm{PG}(4,2)^*$, and also a point $k := \ker b^*\in\mathrm{PG}(4,2)$, namely $k = (\mathrm{im}\, b^*)^{\mathrm{O}}$. If $b^* = e^1\wedge e^2 + e^3\wedge e^4\in\mathrm{Rk}_4(V_5^*)$, the corresponding trivector $t = \perp b^*\in\mathrm{Rk}_4^*$ is

$$t = \perp(e^1\wedge e^2 + e^3\wedge e^4) = (e_3\wedge e_4 + e_1\wedge e_2)\wedge e_5. \tag{1.6}$$

Observe that $t$ is of the form $b\wedge k$ where $k$ is the kernel $e_5$ of $b^*$. For a trivector $t=\perp b^*\in\mathrm{Rk}_4^*$ we define the point $\ker t\in\mathrm{PG}(4,2)$ to be $\ker b^*$; Alternatively, for $t\in\mathrm{Rk}_4^*$, we may define $\ker t$ to be that unique point $k\in\mathrm{PG}(4,2)$ which satisfies $t\wedge k = 0$.

### 1.3 The associate $X^{\#}$ of a flat $X\subset\mathrm{PG}(9,2)$

Let $Q^{(r)}$ be the $r$th polarization of $Q$. Thus

$$\begin{aligned} Q^{(1)}(x_1,x_2) &= Q(x_1+x_2)+Q(x_1)+Q(x_2) = Q(x_2,x_1),\\ Q^{(2)}(x_1,x_2,x_3) &= Q^{(1)}(x_1,x_2+x_3)+Q^{(1)}(x_1,x_2)+Q(x_1,x_3)\\ &= \textstyle\sum_{i=1}^{3} Q(x_i)+\sum_{1\le i<j\le 3} Q(x_i+x_j)+Q(x_1+x_2+x_3), \end{aligned}$$

and so on, with $Q^{(r)}(x_1,\ldots,x_r,x_{r+1})$ being equal to

$$\begin{aligned} &Q^{(r-1)}(x_1,\ldots,x_r+x_{r+1})+Q^{(r-1)}(x_1,\ldots,x_r)+Q^{(r-1)}(x_1,\ldots,x_{r+1})\\ &= \textstyle\sum_{i=1}^{r+1} Q_i+\sum_{1\le i<j\le r+1} Q_{ij}+\sum_{1\le i<j\le r+1} Q_{ijk}+\ldots+Q_{12\ldots r(r+1)}\,. \end{aligned} \tag{1.7}$$

(Here we have abbreviated $Q(x_i), Q(x_i+x_j), Q(x_i+x_j+x_k),\ldots$ as $Q_i$, $Q_{ij}$, $Q_{ijk}$, ...etc.) Consequently the following lemma holds.

**Lemma 1.2** *If the points* $x_1, \ldots, x_{r+1}$ *generate a projective r-flat* $U$ *then*

$$Q^{(r)}(x_1, \ldots, x_{r+1}) = \textstyle\sum_{x \in U} Q(x), \tag{1.8}$$

*while if* $x_1, \ldots, x_{r+1}$ *are dependent then* $Q^{(r)}(x_1, \ldots, x_{r+1}) = 0$.

Granted that the (reduced) degree of $Q$ is 5, observe that $Q^{(4)}$, the completely polarized form of $Q$, is a multilinear function of its five arguments. In the following we will adopt the abbreviation $Q^{\ddagger} := Q^{(4)}$. Note therefore that $Q^{\ddagger}(x_1, \ldots, x_5)$ *is an alternating multilinear form.*

**Definition 1.3** *If* $X \subset \mathrm{PG}(9, 2)$ *is an r-flat* $(r > 0)$, *then (with respect to the Grassmannian* $\mathcal{G}_{1,4,2}$*) its* associate $X^{\#}$ *is the following subset of* PG(9, 2)*:*

$$X^{\#} = \{y \in \mathrm{PG}\,(9, 2) \,|\, Q^{\ddagger}(x_1, x_2, x_3, x_4, y) = 0 \text{ for all } x_1, x_2, x_3, x_4 \in X\}. \tag{1.9}$$

The next lemma lists some immediate consequences of this definition

**Lemma 1.4** *Let* $X$ *be an r-flat in* PG(9, 2). *Then*
*(i)* $X^{\#}$ *is always a flat;*
*(ii) if* $r$ *is* $< 3$ *then* $X^{\#} = \mathrm{PG}(9, 2)$; *(iii) if* $r = 3$ *then* $X \subset X^{\#}$;
*(iv) if* $W$ *is a flat then* $W \subset X \Longrightarrow X^{\#} \subseteq W^{\#}$; *(v)* $\mathcal{G}_X \leq \mathcal{G}_{X^{\#}}$.

In attempting to determine the associate of flats in PG(9, 2) it appears that we are embarking upon a journey through previously unexplored terrain. Moreover, since the explicit coordinate form of the multilinear form $Q^{\ddagger}$ is, see eqs. (4.1), (4.2), quite complicated, the determination of $X^{\#}$ directly from definition 1.3 will usually be an extremely daunting task. Fortunately the definition of $X^{\#}$ may be rephrased, as in the next lemma, in terms of certain incidence properties. So, in most cases, it would appear sensible to attempt to determine $X^{\#}$ by appeal to lemma 1.5 and its off-shoots.

**Lemma 1.5** *Let* $X \subset \mathrm{PG}(9, 2)$ *be an r-flat,* $r \geq 3$. *Then*

*(i) a point* $y \in X^c$ *is in the associate* $X^{\#}$ *of* $X$ *if and only if for each 3-flat* $D$ *of* $X$ *the join* $\langle y, D\rangle$ *(a 4-flat) is odd;*

*(ii) a point* $y \in X$ *is in the associate* $X^{\#}$ *of* $X$ *if and only if the following holds: if* $K$ *is any 4-flat of* $X$ *which contains* $y$ *then* $K$ *is odd.*

*Proof.* This follows from the following two observations. Firstly, if $x_1, x_2, x_3, x_4, y$ are independent, generating a 4-flat $U = \langle y, H\rangle$, where $H = \langle x_1, x_2, x_3, x_4\rangle$, then, see (1.8), $Q^{\ddagger}(x_1, x_2, x_3, x_4, y) = \sum_{u \in U} Q(u)$, which last $= 0$ if and only if $Q(u) = 0$ for an odd number of points $u \in U$. Secondly, if $x_1, x_2, x_3, x_4, y$ are dependent, then, lemma 1.2, $Q^{\ddagger}(x_1, x_2, x_3, x_4, y) = 0$. ■

**Corollary 1.6** *(i) If* $X$ *is a 3-flat then a point* $y \in X^c$ *is in the associate* $X^{\#}$ *of* $X$ *if and only if* $\langle y, X\rangle$ *is odd.*

*(ii) If* $X$ *is an even 4-flat then* $X^{\#}$ *is disjoint from* $X$.

*(iii) If* $X$ *is an odd 4-flat then* $X^{\#} \supseteq X$; *moreover* $X^{\#}$ *is odd.*

### 1.4 Flats in $\mathrm{PG}(9,2)$ from partial spreads in $\mathrm{PG}(4,2)$

With which flats in $\mathrm{PG}(9,2)$ should we start our investigation? Certainly one good source of flats in $\mathrm{PG}(9,2)$ arises from the partial spreads in $\mathrm{PG}(4,2)$. A *partial spread* $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ of size $r(>0)$ is a set $\{\mu_1, \ldots, \mu_r\}$ of $r$ pairwise disjoint lines. Such partial spreads have recently been completely classified: see [2], where, under the action of $\mathrm{GL}(5,2)$, it is shown that they fall into 64 distinct classes ($\mathrm{GL}(5,2)$-orbits). Given a partial spread $\mathcal{S}_r = \{\mu_1, \ldots, \mu_r\}$ in $\mathrm{PG}(4,2)$, let the corresponding $r$-set of points of $\mathrm{Rk}_2 \subset \mathrm{PG}(9,2)$ be $\mathcal{C}_r = \{m_1, \ldots., m_r\}$. For $1 \leq i < j < \ \ldots \ \leq r$ we define $m_{ij} = m_i + m_j$, $m_{ijk} = m_i + m_j + m_j$, ...., $m_\Sigma = m_{12\ldots r} = \sum_{i=1}^{r} m_i$. Note that some of these vectors may be zero, and that coincidences may therefore occur amongst the points (= nonzero vectors). For example, if $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ is a spread for a solid $\sigma \subset \mathrm{PG}(4,2)$ then $m_\Sigma = 0$, whence $m_{123} = m_{45}$, $m_{1234} = m_5$, etc.

**Theorem 1.7** *([5]) (i) $\mathcal{C}_r$ is a $r$-cap: that is, no three points of $\mathcal{C}_r$ are collinear.*

*(ii) The $\binom{r}{2}$ points $m_{ij}$ are distinct and are external.*

*(iii) The $\binom{r}{3}$ points $m_{ijk}$ are distinct and are external.*

If $m_1, \ldots., m_r$ are independent then in addition to the $(r-1)$-flat $\langle \mathcal{C}_r \rangle = \langle m_1, \ldots., m_r \rangle$ we may also consider the $(r-2)$-flat $\mathcal{E}(\mathcal{C}_r)$ which is generated by the $\binom{r}{2}$ points $m_{ij}$. The flats in $\mathrm{PG}(9,2)$ of the kind $\mathcal{E}(\mathcal{C}_r)$ were used in [6] to construct representatives for seven out of the ten orbits of external flats which exist in $\mathrm{PG}(9,2)$.

If for a flat $X \subset \mathrm{PG}(9,2)$ we have $|X \cap \mathrm{Rk}_2| = n_1$ and $|X \cap \mathrm{Rk}_4| = n_2$ then we will say that $X$ is a flat of type $(n_1, n_2)$. For flats of the form $X = \langle \mathcal{C}_5 \rangle$ or $X = \mathcal{E}(\mathcal{C}_5)$, where $\mathcal{C}_5$ is the Grassmann image of a partial spread $\mathcal{S}_5$, the value of $(n_1, n_2)$ can be read off from Table 1 below. See Table 2 for flats of the form $X = \langle \mathcal{C}_6 \rangle$ or $X = \mathcal{E}(\mathcal{C}_6)$. The labelling, Vc.1, VId.2, etc., of the different classes of partial spreads is as in [2, Table B.2].

Table 1: values of $n_1(X) = |X \cap \mathrm{Rk}_2|$ for $X$ arising from an $\mathcal{S}_5$

| Class Vx | x = | a.1 | b.1 | c.1 | d.1 | e.1 | f.1 | g.1 | h.1 | i.1 | j.1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_5 \rangle$ | $n_1(X) =$ | — | 7 | 9 | 6 | 5 | 6 | 5 | 7 | 7 | — |
| $X = \mathcal{E}(\mathcal{C}_5)$ | $n_1(X) =$ | 5 | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 1 | 5 |

Table 2: values of $n_1(X) = |X \cap \mathrm{Rk}_2|$ for $X$ arising from an $\mathcal{S}_6$

| Class VIx | x = | a.1 | b.1 | c.1 | c.2 | d.1 | d.2 | e.1 | e.2 |
|---|---|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_6 \rangle$ | $n_1(X) =$ | 9 | 11 | 13 | — | 11 | 15 | 9 | 7 |
| $X = \mathcal{E}(\mathcal{C}_6)$ | $n_1(X) =$ | 3 | 3 | 4 | 6 | 4 | 6 | 2 | 1 |

Table 2 (continued)

| Class VIx | x = | f.1 | g.1 | h.1 | h.2 | i.1 | j.1 |
|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_6 \rangle$ | $n_1(X) =$ | 9 | 7 | 9 | 11 | 13 | 11 |
| $X = \mathcal{E}(\mathcal{C}_6)$ | $n_1(X) =$ | 2 | 1 | 2 | 3 | 4 | 3 |

*Notes.* (i) In Table 1, $\mathcal{E}(\mathcal{C}_5)$ is a 3-flat, and so it is of type $(n_1, 15 - n_1)$, where $n_1$ is as given in the last row. With the exception of an $\mathcal{S}_5$ of class Va.1 or of class Vj.1, $\langle \mathcal{C}_5 \rangle$ is a 4-flat, and so it is of type $(n_1, 31 - n_1)$, where $n_1$ is as given in the penultimate row. But for an $\mathcal{S}_5$ of class Va.1 or of class Vj.1 we have $m_\Sigma = \Sigma_{i=1}^5 m_i = 0$, whence $\langle \mathcal{C}_5 \rangle$ coincides with $\mathcal{E}(\mathcal{C}_5)$, and, for both of these classes, $\langle \mathcal{C}_5 \rangle$ is of type $(5, 10)$.
(ii) In Table 2, $\mathcal{E}(\mathcal{C}_6)$ is a 4-flat, and so it is of type $(n_1, 31 - n_1)$, where $n_1$ is as given in the last row. With the exception of an $\mathcal{S}_6$ of class VIc.2, $\langle \mathcal{C}_6 \rangle$ is a 5-flat, and so it is of type $(n_1, 63 - n_1)$, where $n_1$ is as given in the penultimate row. But for an $\mathcal{S}_6$ of class VIc.2, $\langle \mathcal{C}_6 \rangle$ coincides with $\mathcal{E}(\mathcal{C}_6)$, and is of type $(6, 25)$.

## 2 The associate $D^\#$ of a 3-flat $D \subset \mathrm{PG}(9, 2)$

Bearing in mind lemma 1.4(ii), (iii), let us first concentrate upon finding $D^\#$ for certain 3-flats $D$.

### 2.1 The linear form $f_D$ and trivector $t_D$ of a 3-flat $D$

Let $D = \langle a_1, a_2, a_3, a_4 \rangle$ be a 3-flat of $\mathrm{PG}(9, 2) = \mathbb{P}V_{10}$. Then we define its *associated linear form* $f_D \in (V_{10})^*$ by

$$f_D(x) = Q^\ddagger(a_1, a_2, a_3, a_4, x), \quad x \in \wedge^2 V_5. \tag{2.1}$$

This is a good definition: because $Q^\ddagger$ is alternating and quinquelinear, any choice of independent points $a_1, a_2, a_3, a_4 \in D$ yield the same linear form. This form $f_D$ determines, via $f_D(x) = \lceil t_D, x \rfloor$, a trivector $t_D \in V_{10}^* := \wedge^3 V_5$, the *associated trivector* $t_D$ of the 3-flat $D$: $t_D \wedge x = f_D(x)e, \quad x \in \wedge^2 V_5$.

In this notation the next lemma is now immediate. Note that part (i) justifies the strict inclusion $D \subset D^\#$ which was asserted in lemma 1.4(iii).

**Lemma 2.1** *Let $D$ be a 3-flat in* PG(9, 2)*.*

*(i) If $f_D$ is nonzero, then $D^\#$ is the hyperplane $H(t_D) := \langle t_D \rangle^O$; if $f_D$ is the zero form then $D^\# = \mathrm{PG}(9, 2)$.*

*(ii) The trivector $t_D$ is invariant under the action $\hat{T}$ of $\mathcal{G}_D$.*

*(iii) If $t_D \in \mathrm{Rk}_2^*$, and so is the Grassmann image of a plane $\alpha_D \subset$ PG(4, 2), then $\alpha_D$ is stabilized by $\mathcal{G}_D$.*

*(iv) If $t_D \in \mathrm{Rk}_4^*$ let $k_D = \ker t_D$ (see after eq. (1.6)); then the point $k_D$ of* PG(4, 2) *is a fixed point of $\mathcal{G}_D$.*

**Remark 2.2** *Cases where* $t_D = 0$*, that is where* $f_D$ *is the zero form, do arise, see for example theorems 2.9, 4.1 and 4.3 below. In such cases it is sometimes convenient to interpret* $\mathrm{PG}(9,2)$ *as the "hyperplane"* $H(0)$*.*

**Remark 2.3** *Lemma 1.5(i) asserts that a point* $y \in X^c$ *is in the associate* $X^\#$ *of the* $r$*-flat* $X$ *if and only if* $f_D(y) = 0$ *for each* 3*-flat* $D$ *of* $X$*.*

On account of lemma 2.1(i) we need to know about the hyperplanes in $\mathrm{PG}(9,2)$. Since $\mathrm{PG}(9,2)^*$ consists of just two $\mathrm{GL}(5,2)$-orbits, $\mathrm{Rk}_2^*$ and $\mathrm{Rk}_4^*$, it follows that there are just two $\mathrm{GL}(5,2)$-orbits of hyperplanes in $\mathrm{PG}(9,2)$:

**Theorem 2.4** *[4, Theorem 3.4] Under the natural action of* $\mathrm{GL}(5,2)$ *there are two orbits of hyperplanes in* $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$*, say* $\mathcal{H}_{155}$ *and* $\mathcal{H}_{868}$*, of lengths 155 and 868. A hyperplane* $H \in \mathcal{H}_{155}$ *intersects* $\mathcal{G}_{1,4,2}$ *in 91 points while a hyperplane* $H \in \mathcal{H}_{868}$ *intersects* $\mathcal{G}_{1,4,2}$ *in 75 points. The hyperplanes in* $\mathcal{H}_{155}$ *are in bijective correspondence* $\pi \leftrightarrow H(\pi)$ *with the planes* $\pi \subset \mathrm{PG}(4,2)$*, the elements of the 91-set* $H(\pi) \cap \mathrm{Rk}_2$ *being the Grassmann images of the 91 lines of* $\mathrm{PG}(4,2)$ *which meet the plane* $\pi$*. The hyperplanes in* $\mathcal{H}_{868}$ *are in bijective correspondence* $t \leftrightarrow H(t)$ *with the trivectors* $t \in \mathrm{Rk}_4^*$*, the elements of the 75-set* $H(t) \cap \mathrm{Rk}_2$ *being the Grassmann images of the 75 lines of* $\mathrm{PG}(4,2)$ *which are isotropic, see after eq. (1.4), for the alternating bilinear form* $t(\cdot,\cdot)$*.*

## 2.2 The associate $D^\#$ of $D = \langle \mathcal{C}_4 \rangle$

First we look at certain 3-flats in $\mathrm{PG}(9,2)$ arising from partial spreads $\mathcal{S}_4$ of size 4 in $\mathrm{PG}(4,2)$. There exist, see [2, Table B.2], four distinct classes of $\mathcal{S}_4$, and for an $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ in each of these classes we consider the solid $D = \langle \mathcal{C}_4 \rangle$ in $\mathrm{PG}(9,2)$, where $\mathcal{C}_4 = \{m_1, m_2, m_3, m_4\} \subset \mathrm{Rk}_2$ is the Grassmann image of $\mathcal{S}_4$. We aim to find the associate $D^\# = \langle t_D \rangle^{\mathrm{O}}$ of $D$.

**Lemma 2.5** *Consider the associate* $D^\# = \langle t_D \rangle^O$ *of a solid in* $\mathrm{PG}(9,2)$ *which is of the form* $D = \langle \mathcal{C}_4 \rangle$ *just described. Then each line* $\mu_i \in \mathcal{S}_4$ *is isotropic for the alternating bilinear form* $t_D(\cdot,\cdot)$*, see eq. (1.3). If* $t_D \in \mathrm{Rk}_2^*$*, and so is the Grassmann image of a plane* $\alpha_D \subset \mathrm{PG}(4,2)$*, then* $\alpha_D$ *meets each line* $\mu_i \in \mathcal{S}_4$*.*

*Proof.* Since $D \subset D^\#$ these results follow from theorem 2.4. ■

Let us give a brief description of the four classes of $\mathcal{S}_4$. If $\mathcal{S}_4$ is regulus-free then, see [2, Table B.2], it is either of class IVa.1 or of class IVb.1. If of class IVa.1, with stabilizer $\mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(4)$, then it possesses a unique extension to a spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\mu_5\}$ on a parabolic quadric $\mathcal{P}_4$. Since $m_5 = \Sigma_{i=1}^4 m_i$, note that $D = \langle \mathcal{C}_4 \rangle$ can be expressed $D = \langle \mathcal{C}_5 \rangle = \mathcal{E}(\mathcal{C}_5)$. So $D$ is of type

(5, 10), and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_5} \cong \mathrm{Sym}(5)$. The stabilizer $\mathcal{G}_D$ possesses a unique fixed point $p_0 \in \mathrm{PG}(4,2)$, namely the nucleus of $\mathcal{P}_4$.

If $\mathcal{S}_4$ is of class IVb.1, then $D = \langle \mathcal{C}_4 \rangle$ is of type (4, 11), and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(3)$. The partial spread $\mathcal{S}_4$ has a unique decomposition $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3\} \cup \{\mu_4\}$ such that $\mathcal{G}_{\mathcal{S}_4}$ effects all permutations of $\mu_1, \mu_2, \mu_3$ but leaves $\mu_4$ invariant. If the transversal of $\{\mu_1, \mu_2, \mu_3\}$ meets $\mu_i$ in $c_i$, $i = 1, 2, 3$ then we have $\mu_i = \langle a_i, c_i \rangle$ where the points $a_i \in \mu_i$, $i = 1, 2, 3$, are uniquely determined by the requirement that the invariant line $\mu_4$ is of the form $\{a_2 + a_3, a_3 + a_1, a_1 + a_2\}$. Then $\mathcal{G}_D = \langle A, J \rangle$ where $A$ effects the permutation $(a_1a_2a_3)(c_1c_2c_3)$ and $J$ effects $(a_1a_2)(c_1c_2)(a_3)(c_3)$. Observe that $\mathcal{G}_D$ possesses a unique fixed point $p_0$, namely $p_0 = a_1 + a_2 + a_3$.

There are just two other classes, IVc.1 and IVd.1, of partial spread $\mathcal{S}_4$. A partial spread $\mathcal{S}_4$ of class IVc.1 consists of a regulus $\rho = \{\mu_1, \mu_2, \mu_3\}$ together with a line $\mu_4$ not lying in the ambient solid $\sigma$ of $\rho$. In this case $D = \langle \mathcal{C}_4 \rangle$ is of type (4, 11), and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(3) \times Z_2$. The stabilizer $\mathcal{G}_D$ has a unique fixed point $p_0 \in \mathrm{PG}(4,2)$, namely the point where $\mu_4$ meets $\sigma$.

Finally a partial spread $\mathcal{S}_4$ of class IVd.1 is of the form $\mathcal{S}_5 \setminus \{\mu_5\}$ where $\mathcal{S}_5$ is a spread for some solid $\sigma \subset \mathrm{PG}(4,2)$. Since $m_5 = \Sigma_{i=1}^4 m_i$, note that $D := \langle \mathcal{C}_4 \rangle$ can be expressed $D = \langle \mathcal{C}_5 \rangle = \mathcal{E}(\mathcal{C}_5)$. So $D$ is of type (5, 10), and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_5} \cong 2^4 : \Gamma\mathrm{L}(2,4)$.

In [4] we used of lemmas 2.1 and 2.5 to obtain the following theorems.

**Theorem 2.6** *Let $\mathcal{S}_4$ be of class IVa.1, and let $\mathcal{P}_4$ be that parabolic quadric in* PG(4, 2) *such that $\mathcal{S}_4$ extends to a spread $\mathcal{S}_5$ on $\mathcal{P}_4$. Then $\langle \mathcal{C}_4 \rangle^{\#}$ is the hyperplane $H(t)$, where $t(\cdot,\cdot)$ is that alternating bilinear form (whose kernel is the nucleus of $\mathcal{P}_4$) obtained by polarizing the quadratic form of $\mathcal{P}_4$.*

**Theorem 2.7** *Let $\mathcal{S}_4$, of class IVb.1, be $\{\mu_1, \mu_2, \mu_3\} \cup \{\mu_4\}$ as in the preamble. Then $D = \langle \mathcal{C}_4 \rangle$ has for its associate $D^{\#}$ the hyperplane $H(t_D)$, $t_D = m_{123} \wedge p_0$, where $p_0$ is the fixed point of $\mathcal{G}_{\mathcal{S}_4}$ and $m_{123} = m_1 + m_2 + m_3$.*

**Theorem 2.8** *Let $\mathcal{S}_4 = \rho \cup \{\mu_4\}$ be of class IVc.1. Let $\Sigma_5 = \rho \cup \{\xi, \eta\}$ be the extension of the regulus $\rho$ to a spread $\Sigma_5$ for the ambient solid $\sigma$ of $\rho$, and suppose that $\mu_4$ meets $\sigma$ in the point $p_0 \in \xi$. Then $D = \langle \mathcal{C}_4 \rangle$ has for its associate $D^{\#}$ the hyperplane $H(t)$ where $t$ is the Grassmann image of the plane $\langle p_0, \eta \rangle$.*

**Theorem 2.9** *If $\mathcal{S}_4$ is of class IVd.1 then $D = \langle \mathcal{C}_4 \rangle$ has for its associate $D^{\#}$ the whole of* PG(9, 2).

Actually the last theorem is a special case of theorem 4.4 below.

## 2.3 The associate $D^{\#}$ of a 3-flat of the form $D = \mathcal{E}(\mathcal{C}_5)$

We may also construct 3-flats in PG(9, 2) from partial spreads of size 5 in PG(4, 2). Consider a solid $D$ of the form $D = \mathcal{E}(\mathcal{C}_5)$, where $\mathcal{C}_5 = \{m_i\}_{1 \le i \le 5} \subset$

$\mathrm{Rk}_2$ is the Grassmann image of a partial spread $\mathcal{S}_5 = \{\mu_i\}_{1\le i\le 5}$. Since $D = \langle m_{15}, m_{25}, m_{35}, m_{45}\rangle$ its associated linear form is $f_D$, where

$$f_D(x) = Q^{\ddagger}(m_{15}, m_{25}, m_{35}, m_{45}, x). \tag{2.2}$$

Setting $\mathcal{C}(i) := \mathcal{C}_5 \diagdown \{m_i\}$ and $D_i = \langle \mathcal{C}(i)\rangle$, let $t_i := t_{D_i}$ be the associated trivector of the solid $D_i$.

**Theorem 2.10** *(i) The trivector of the solid $D = \mathcal{E}(\mathcal{C}_5)$ is $t_D = \Sigma_{i=1}^5 t_i$.*

*(ii) With respect to the bilinear form $t_D(\cdot,\cdot)$, see eq. (1.3), there are just two possibilities for the five lines $\mu_i$ of $\mathcal{S}_5$. Either*
*(a) each line $\mu_i$ is isotropic: $t_D \wedge m_i = 0$, $i = 1, \dots, 5$, or*
*(b) each line $\mu_i$ is non-isotropic: $t_D \wedge m_i = e$, $i = 1, \dots, 5$.*

*Proof.* (i) Upon expanding $f_D(x)$ in (2.2) we obtain $f_D = \Sigma_{i=1}^5 f_{D_i}$. (ii) By lemma 1.4(iii), $m_{ij} \in D^{\#} = \langle t_D\rangle^{\mathrm{O}}$; so $t_D \wedge (m_i + m_j) = 0$. ■

By appeal to the last theorem we are able to find $D^{\#}$ for a solid of the form $D = \mathcal{E}(\mathcal{C}_5)$ in the two cases (i) $\mathcal{S}_5$ of class Ve.1 (ii) $\mathcal{S}_5$ of class Vd.1.

**Theorem 2.11** *Let $\mathcal{C}_5 = \{m_1, m_2, m_3, m_4, m_5\}$ be of class Ve.1, and so, see [6, Theorem 2.5], $D = \mathcal{E}(\mathcal{C}_5)$ is an external solid $\in \mathrm{orb}(3\alpha)$. Let $u \in \mathrm{PG}(4,2)$ be the unique fixed point of the stabilizer $\mathcal{G}_D \cong Z_5$. Then the associate of the solid $D$ is the hyperplane*

$$D^{\#} = H(t_D) \quad \textit{where } t_D = u \wedge m_{\Sigma}. \tag{2.3}$$

Concerning a partial spread $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ of class Vd.1, let $\langle C\rangle \cong Z_3$ be its stabilizer $\mathcal{G}_{\mathcal{S}_5}$ and let $p_0$ be the unique fixed point of $C$. Let the lines of $\mathcal{S}_5$ be labelled so that $C$ stabilizes $\mu_4$ and $\mu_5$ and effects the permutation $(\mu_1\mu_2\mu_3)$. Let $\mathcal{S}_8 = \{\mu_1, \dots, \mu_8\}$ be the unique extension of $\mathcal{S}_5$ to a regulus-free partial spread $\mathcal{S}_8$, *cf.* [7, Theorem 1.1(v)]. Then $C$ cyclically permutes the lines $\mu_6, \mu_7, \mu_8$. The Grassmann images $t_{123}, t_{678} \in \mathcal{G}_{1,4,2}$ of the transversals $\tau_{123}$, $\tau_{678}$ are then fixed points of $T_C$. Also the image $t_\pi$ of the invariant plane $\pi := (\mu_1 \cup \dots \cup \mu_8)^c$ of $\mathcal{S}_8$ is a fixed point of $\hat{T}_C$.

**Theorem 2.12** *If $D = \mathcal{E}(\mathcal{C}_5)$, where $\mathcal{S}_5$ is of class Vd.1, then*

$$D^{\#} = H(t_D) \quad \textit{where } t_D = t_\pi + p_0 \wedge (t_{123} + t_{678}). \tag{2.4}$$

## 3 The associate $X^{\#}$ of a 4-flat $X \subset \mathrm{PG}(9,2)$

First, some general results for 4-flats. Given a 4-flat $X \subset \mathrm{PG}(9,2)$ let $\mathcal{B} = \{a_1, a_2, a_3, a_4, a_5\}$ be *any* choice of points which generate $X$, and consider the five independent solids $D_i = \langle \mathcal{B}_i\rangle \subset X$, where $\mathcal{B}_i := \mathcal{B}\diagdown\{a_i\}$, $i = 1, 2, 3, 4, 5$. Let $t_i := t_{D_i}$ be the associated trivector of $D_i$ and consider the five "hyperplanes" $H_i := H(t_i) = \langle t_i\rangle^{\mathrm{O}}$. (Recall, see remark 2.2, that $H_i = \mathrm{PG}(9,2)$ if $t_i = 0$.)

**Theorem 3.1** *Given a 4-flat* $X = \langle a_1, a_2, a_3, a_4, a_5\rangle$ *in* PG(9,2), *let the notation be as in the preamble. Then*

$$X^{\#} = T^{O}, \qquad \text{where } T := \langle t_1, t_2, t_3, t_4, t_5\rangle \subset \mathrm{PG}(9,2)^{*}. \tag{3.1}$$

*Proof.* Since $Q^{\ddagger}$ is alternating and quinquelinear the 4-flat $X^{\#}$ is the intersection $\cap_{i=1}^{5}\langle t_i\rangle^{\mathrm{O}} = \langle t_1, t_2, t_3, t_4, t_5\rangle^{\mathrm{O}}$ of the five "hyperplanes" $H_i$. ■

**Corollary 3.2** *If* $X$ *is an even 4-flat then* $X^{\#}$ *is a disjoint 4-flat.*

*Proof.* By the theorem, $X^{\#}$ is the intersection of at most five independent hyperplanes; hence $X^{\#}$ is an $s$-flat for some $s \geq 4$. But if $X$ is even then, see corollary 1.6(ii), $X \cap X^{\#} = \emptyset$, and so $s \leq 4$. Hence $s = 4$. ■

Next we consider 4-flats $X$ of the form $X = \mathcal{E}(\mathcal{C}_6)$ where $\mathcal{C}_6 = \{m_i\}_{1\leq i\leq 6} \subset \mathrm{Rk}_2$ is the Grassmann image of a partial spread $\mathcal{S}_6 = \{\mu_i\}_{1\leq i\leq 6}$. Setting $\mathcal{C}_5(i) = \mathcal{C}_6\diagdown\{m_i\}$ and $\mathcal{C}_4(ij) = \mathcal{C}_6\diagdown\{m_i, m_j\}$ then we may define, for $1 \leq i \neq j \leq 6$, solids $D_i := \mathcal{E}(\mathcal{C}_5(i))$ and $D_{ij} := \langle \mathcal{C}_4(ij)\rangle$. Let the associated trivectors of these solids be $t_i := t_{D_i}$ and $t_{ij} := t_{D_{ij}}$. The next theorem follows immediately from theorems 3.1, 2.10(i); in it we choose $m_6$ as preferred element of $\mathcal{C}_6$.

**Theorem 3.3** *The associate of the 4-flat* $X = \mathcal{E}(\mathcal{C}_6)$ *in the preamble is*

$$X^{\#} = T^{O}, \qquad \text{where } T := \langle t_1, t_2, t_3, t_4, t_5\rangle \subset \mathrm{PG}(9,2)^{*}. \tag{3.2}$$

*Moreover* $t_1 = \Sigma_{j\neq 1} t_{1j}$, ... , $t_5 = \Sigma_{j\neq 5} t_{5j}$.

Next let us look at some particular 4-flats. If a 4-flat $X$ in PG(9,2) is of the form $X = \langle \mathcal{C}_5\rangle$, or of the form $X = \mathcal{E}(\mathcal{C}_6)$, then we can hope to determining its associate $X^{\#}$ by making use of the general results of theorems 2.10 and 3.1 - 3.3 in conjunction with the particular results in section 2.2. In the cases when $X$ is an even 4-flat we would also like to determine $X^{\#\#} := (X^{\#})^{\#}$, $X^{\#\#\#} := (X^{\#\#})^{\#}$, ... . However in these even cases it may prove difficult to pin down the 4-flat $X^{\#}$; also $X^{\#}$ may not be of the form $\langle \mathcal{C}_5\rangle$ or $\mathcal{E}(\mathcal{C}_6)$, and so theorems 2.10 and 3.3 may not be of help in determining $X^{\#\#}$. Consequently arriving at a reasonable understanding of such a sequence $X, X^{\#}, X^{\#\#}, X^{\#\#\#}, \ldots$ is in general quite a tough undertaking.

We list briefly below some cases where we have succeeded in determining $X^{\#}$; for full details consult [4].

1. If $X = \langle \mathcal{C}_5\rangle$, where $\mathcal{S}_5$ is of class Vi.1, then $X^{\#}$ is a 6-flat which contains $X$ and which meets $\mathrm{Rk}_2$ in 43 points.
2. If $X = \langle \mathcal{C}_5\rangle$, $\mathcal{S}_5 \in$ class Ve.1, then $X$ is self-associate: $X^{\#} = X$.

3. If $X := \mathcal{E}(\mathcal{C}_6)$, $\mathcal{S}_6 \in$ class VIc.2, then:
   (i) $X = \mathcal{E}(\mathcal{C}_6)$ is of type $(6, 25)$ with stabilizer $\mathcal{G}_X \cong \mathrm{Sym}(3)$;
   (ii) $X^{\#}$ is of type $(4, 27)$ with stabilizer $\mathcal{G}_{X^{\#}} \cong (Z_3 \times Z_3).Z_2$;
   (iii) $X^{\#\#}$ is of type $(4, 27)$ with stabilizer $\mathcal{G}_{X^{\#\#}} \cong (Z_3 \times Z_3).Z_2$;
   (iv) $X^{\#\#\#} = X^{\#}$. (3.3)
4. If $X := \mathcal{E}(\mathcal{C}_6)$, $\mathcal{S}_6 \in$ class VIf.1, then:
   (i) $X := \mathcal{E}(\mathcal{C}_6)$, $X^{\#}$ and $X^{\#\#}$ are all of type $(2, 29)$;
   (ii) $\mathcal{G}_X = \mathcal{G}_{X^{\#}} = \mathcal{G}_{X^{\#\#}} \cong Z_2$, and $X^{\#\#\#} = X$. (3.4)

Finally we consider external 4-flats. All such 4-flats were determined in [6], where the following results were proved .
(i) The external 4-flats form precisely two $\mathrm{GL}(5, 2)$-orbits, orb(4+), orb(4−).
(ii) The stabilizer $\mathcal{G}_X$ of any external 4-flat $X$ is the normalizer $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5, 2)$.
(iii) Under the action $T$ of $\mathcal{N}$ upon $V_{10} = \wedge^2 V_5$, given by $T_C = \wedge^2 C$, $C \in \mathcal{N}$, the space $V_{10}$ has a unique decomposition as the direct sum $V_+ \oplus V_-$ of two 5-dimensional spaces upon which $\mathcal{N}$ acts irreducibly and inequivalently.
(iv) Both the 4-flats $X_+ := \mathbb{P}V_+$ and $X_- := \mathbb{P}V_-$ are external flats, and belong to different $\mathrm{GL}(5, 2)$-orbits.

It is now a short step, see [4], to the proof of the following theorem.

**Theorem 3.4** *If $X$ is any external 4-flat in* $\mathrm{PG}(9, 2)$ *then its associate $X^{\#}$ is also an external 4-flat, with $X$ and $X^{\#}$ belonging to different* $\mathrm{GL}(5, 2)$-*orbits. Moreover $(X^{\#})^{\#} = X$.*

Given an external 4-flat $X$ see [4, Theorem 4.10] for an interesting construction of its associate $X^{\#}$.

# 4 Further results, and some speculations

## 4.1 Direct use of the quintic $Q$

As shown in [3] the 155 internal points of the Grassmannian $\mathcal{G}_{1,4,2}$ are precisely those points $x \in \mathrm{PG}(9, 2) = \mathbb{P}(\wedge^2 V_5)$ which satisfy a certain quintic equation $Q(x) = 0$. Only those terms $Q_5$ in $Q$ whose reduced degree is 5 will contribute to the alternating quinquelinear form $Q^{\ddagger}$ obtained by completely polarizing $Q$. Since, for $x = \Sigma_{1 \le i < j \le 5}\, x_{ij} e_i \wedge e_j$, we have $Q_5(x) = \Sigma x_{ij} x_{jk} x_{kl} x_{lm} x_{mi}$ (12 terms), it follows that $Q^{\ddagger}$ has $12 \times 5! = 1440$ terms. Explicitly

$$Q^{\ddagger}(a, b, c, d, y) = \sum\nolimits_{1 \le i < j \le 5} f_{ij}(a, b, c, d) y_{ij} \tag{4.1}$$

where, for given $a, b, c, d$, the coefficient of $y_{ij}$ consists of the $4! \times 3! = 144$ terms

$$f_{ij}(a, b, c, d) = \sum u_{ir} v_{rs} w_{st} x_{tj}, \tag{4.2}$$

with $uvwx$ running through the 4! permutations of $abcd$ and $rst$ running through the 3! permutations of $\{\{1,2,3,4,5\}\setminus\{i,j\}\}$.

Up till now we have made use of lemma 1.5, and its off-shoots, in order to avoid dealing with the complicated coordinate forms (4.1), (4.2). Nevertheless there do exist cases where $X^{\#}$ can be simply determined direcly from the explicit expressions (4.1), (4.2). Several of the examples which we now give involve Latin solids and Latin and Greek planes; see [1, Ch. 24]. Recall that the 7 lines of a plane $\alpha \subset \mathrm{PG}(4,2)$ map onto the 7 points of a 'Greek' plane $P(\alpha) \subset \mathcal{G}_{1,4,2}$, and the 15 lines of $\mathrm{PG}(4,2)$ which pass through the point $z$, forming let us say the *star* $\mathrm{st}(z)$, map onto the 15 points of a 'Latin' solid, say $\mathrm{St}(z) \subset \mathcal{G}_{1,4,2}$. We also denote by $\mathrm{St}(z,\sigma), z \in \sigma$, the Latin plane which is the Grassmann image of the seven lines of the solid $\sigma \subset \mathrm{PG}(4,2)$ which pass through the point $z \in \sigma$. The following theorems are easily derived, see [4], from the explicit form (4.1), (4.2) of $Q^{\ddagger}$.

**Theorem 4.1** *If $D$ is any 3-flat $\langle \mathrm{St}(z,\sigma), h\rangle$ which contains the Latin plane* $\mathrm{St}(z,\sigma)$*, then* $D^{\#} = \mathrm{PG}(9,2)$*. In particular* $\mathrm{St}(z)^{\#} = \mathrm{PG}(9,2)$.

**Theorem 4.2** *If $X$ is any 4-flat $\langle \mathrm{St}(z), h\rangle$ which contains the Latin solid* $\mathrm{St}(z)$ *then* $X^{\#} = \mathrm{PG}(9,2)$.

**Theorem 4.3** *If $D$ is any 3-flat $\langle P(\alpha), h\rangle$ which contains the Greek plane $P(\alpha)$ then* $D^{\#} = \mathrm{PG}(9,2)$.

**Theorem 4.4** *Let $D = \langle a_1,a_2,a_3,a_4\rangle$ be a 3-flat in* $\mathrm{PG}(9,2)$ *such that, for some fixed solid* $\sigma \subset \mathrm{PG}(4,2)$*, the following holds for each* $i = 1,2,3,4$*: if* $a_i \in \mathrm{Rk}_2$ *then $a_i$ is the image of a line of $\sigma$, while if* $a_i \in \mathrm{Rk}_4$ *then* $\mathrm{im}\, a_i = \sigma$*. Then* $D^{\#} = \mathrm{PG}(9,2)$.

## 4.2 Flats arising from a regulus-free $\mathcal{S}_8$

In $\mathrm{PG}(4,2)$ the partial spreads of size 8 which are regulus-free constitute a single class VIIIa.1 of especial interest; in particular a partial spread $\mathcal{S}_8$ of class VIIIa.1 has a large stabilizer $\mathcal{G}_{\mathcal{S}_8} \cong 2^3 : (7:3)$ which is 2-transitive on the eight lines of $\mathcal{S}_8$; see [2, Section 6 and Table B.2]. (The stabilizer groups for the other eight classes of $\mathcal{S}_8$ are all of order $\leq 6$, and are not transitive.) The seven points of $\mathrm{PG}(4,2)$ not on any of the lines of $\mathcal{S}_8$ constitute the *invariant plane* $\alpha$ of $\mathcal{S}_8$, the only plane in $\mathrm{PG}(4,2)$ which is stabilized by $\mathcal{G}_{\mathcal{S}_8}$. The Grassmann images in $\mathcal{G}_{1,4,2}$ of the seven lines $\lambda \subset \alpha$ form the Greek plane $P(\alpha)$, and the Grassmann image in $\mathcal{G}_{2,4,2}$ of the plane $\alpha$ is an internal point $t$ of $\mathrm{PG}(9,2)^* = \mathbb{P}(\wedge^3 V_5)$.

Let $\mathrm{orb}(2\gamma)^*$ denote the orbit of external planes in $\mathrm{PG}(9,2)^*$ which is the analogue of the orbit $\mathrm{orb}(2\gamma)$, see [6, Section 4], of external planes in $\mathrm{PG}(9,2)$. See [4] for the construction of a pair $\{\mathcal{S}_8, P^*\}$ consisting of a partial spread $\mathcal{S}_8$ of class VIIIa.1 and a plane $P^*$ of $\mathrm{orb}(2\gamma)^*$ such that $\mathcal{G}_{P^*} = \mathcal{G}_{\mathcal{S}_8}$.

As well as the external plane $P^*$ we will also be interested in the solid $\langle t, P^*\rangle \subset \mathrm{PG}(9,2)^*$, which is of type $(1,14)$. The stabilizer $\mathcal{G}_{\langle t,P^*\rangle}$ of the solid $\langle t, P^*\rangle$ is larger than $\mathcal{G}_{P^*}$; it has the structure $2^6 : (7:3)$. Let $\mathcal{C}_8 = \{m_a\}_{a\in V_3} \subset \mathcal{G}_{1,4,2}$ be the 8-cap arising from the regulus-free partial spread $\mathcal{S}_8$. Because the $m_a$ satisfy the linear relation $m_\Sigma = 0$, see [7, Lemma 3.1], $\mathcal{E}(\mathcal{C}_8)$ is a 5-flat. As shown in [7, Theorem 3.2], $\mathcal{E}(\mathcal{C}_8)$ is of type $(7,56)$, meeting $\mathcal{G}_{1,4,2}$ in the plane $P(\alpha)$, and $\langle\mathcal{C}_8\rangle$ is a 6-flat of type $(15,112)$, meeting $\mathcal{G}_{1,4,2}$ in the 15-set $\mathcal{C}_8 \cup P(\alpha)$.

See [4] for the proof of the following theorem.

**Theorem 4.5** *(i) If* $X = \langle\mathcal{C}_8\rangle$ *then* $X^O = P^*$, $\mathcal{G}_{\langle\mathcal{C}_8\rangle} = \mathcal{G}_{P^*}$ *and* $X^\# = \emptyset$.
*(ii) If* $X = \mathcal{E}(\mathcal{C}_8)$ *then* $X^O = \langle t, P^*\rangle$, $\quad \mathcal{G}_{\mathcal{E}(\mathcal{C}_8)} = \mathcal{G}_{\langle t,P^*\rangle}$ *and* $X^\# = X$.

**Remark 4.6** *In theorem 3.1 it is noted that for a 4-flat* $X$ *the associate* $X^\#$ *is the intersection of five "hyperplanes"* $H_i$. *In an obvious generalization, and obvious notation, if* $X$ *is a 5-flat then its associate* $X^\#$ *is the intersection of* $\binom{6}{2} = 15$ *"hyperplanes"* $H_{ij}$. *Consequently one usually expects that for a 5-flat* $X$ *there are enough independent hyperplanes to entail that* $X^\# = \emptyset$. *The result* $X^\# = X$ *in part (ii) is therefore remarkable. That* $\mathcal{E}(\mathcal{C}_8)^\#$ *is at least as big as* $\mathcal{E}(\mathcal{C}_8)$ *can in fact be simply seen as follows. Since* $\mathcal{E}(\mathcal{C}_8) \cap \mathcal{G}_{1,4,2} = P(\alpha)$ *every 4-flat of* $\mathcal{E}(\mathcal{C}_8)$ *is odd, either meeting* $\mathcal{G}_{1,4,2}$ *in* $P(\alpha)$ *or else in a line of* $P(\alpha)$. *Hence* $\mathcal{E}(\mathcal{C}_8)$ *is contained in* $\mathcal{E}(\mathcal{C}_8)^\#$.

### 4.3 Some speculations concerning even 4-flats

Because we are exploring new territory there is a temptation to speculate prematurely! For example, provoked by the result in eq. (3.3), we initially wondered whether the following held in general: *if* $Y$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *such that* $G_{Y^\#} = G_Y$, *then* $Y^{\#\#} = Y$. But while this holds in the case $Y := X^\#$, with $X$ as in (3.3), it is false for $Y = X$ in (3.4). Nevertheless (albeit very tentatively!) we put forward the following conjectures/speculations concerning even 4-flats in $\mathrm{PG}(9,2)$. All we can say in support of these is that we have checked that they hold up for those even 4-flats treated in the present paper.

**Conjecture 4.7** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then its associate* $X^\#$ *is an even 4-flat.*

**Conjecture 4.8** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then its annihilator* $X^O$ *is an even 4-flat in* $\mathrm{PG}(9,2)^*$.

**Conjecture 4.9** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *of type* $(n_1,n_2)$ *then its annihilator* $X^O$ *is a 4-flat in* $\mathrm{PG}(9,2)^*$ *of type* $(n_1,n_2)$.

**Conjecture 4.10** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then* $(X^\#)^O = (X^O)^\#$.

For any flat $X$ we can consider the '#-sequence' $(X_0, X_1, X_2, \ldots)$, where $X_0 = X$ and $X_{r+1} = (X_r)^\#$. Let us assume now that conjecture 4.7 holds. Then if $X$ is an even 4-flat its sequence is such that each member $X_r$ is also an even 4-flat. (Moreover, corollary 3.2, $X_r$ is disjoint from $X_{r+1}$.) For any even 4-flat $X$ we can therefore define an ordered pair $(r, s)$ of integers $r, s$, with $0 \leq r < s$, such that the members of the finite sequence $(X_0, X_1, \ldots X_s)$ are distinct and such that $(X_s)^\# = X_r$. From theorem 3.4, and from eqs. (3.3), (3.4), we have examples of even 4-flats $X$ for which (i) $(r, s) = (0, 1)$, (ii) $(r, s) = (1, 2)$ and (iii) $(r, s) = (0, 2)$. Do other values of $(r, s)$ occur?

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