# The associate $X^{\#}$ of a flat $X$ in $\mathrm{PG}(n,2)$ with respect to a given hypersurface

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### Abstract

After briefly considering general hypersurfaces in $\mathrm{PG}(n,2)$ we concentrate upon the case of the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2) = \mathbb{P}V(5,2)$, whose 155 points are those points $x \in \mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which satisfy a certain quintic equation $Q(x) = 0$. (The quintic polynomial $Q$ is given explicitly in [12].) A projective flat $X \subset \mathrm{PG}(9,2)$ will be termed *odd* or *even* according as $X$ intersects $\mathcal{G}_{1,4,2}$ in an odd or even number of points. Let $Q^{\ddagger}(x_1, \ \ldots \ x_5)$ denote the alternating quinquelinear form obtained by completely polarizing $Q$. We define the *associate* $Y = X^{\#}$ of a $r$-flat $X \subset \mathrm{PG}(9,2)$ by

$$Y = \{y \in \mathrm{PG}(n,2) \,|\, Q^{\ddagger}(x_1, x_2, x_3, x_4, y) = 0 \text{ for all } x_1, x_2, x_3, x_4 \in X\}.$$

Because $Q^{\ddagger}$ is quinquelinear, the associate $X^{\#}$ of an $r$-flat $X$ is an $s$-flat for some $s$. The cases where $r = 4$ are of particular interest: if $X$ is an odd 4-flat then $X \subseteq X^{\#}$ while if $X$ is an even 4-flat then $X^{\#}$ is necessarily also a 4-flat which is moreover disjoint from $X$. We give an example of an odd 4-flat $X$ which is self-associate: $X^{\#} = X$. An example of an even 4-flat $X$ such that $(X^{\#})^{\#} = X$ is provided by any 4-flat $X$ which is external to $\mathcal{G}_{1,4,2}$; in fact there are just two $\mathrm{GL}(5,2)$-orbits of external 4-flats, and if $X$ is on one orbit $X^{\#}$ is on the other. However it appears that the two possibilities just illustrated, namely $X^{\#} = X$ for an odd 4-flat and $(X^{\#})^{\#} = X$ for an even 4-flat, are the exception rather than the rule. Indeed, we provide examples of odd 4-flats for which $X^{\#} = \mathrm{PG}(9,2)$ and of even 4-flats for which $X^{\#\#\#} = X$.

Many of our examples of associates $X^{\#}$ are for $r$-flats $X \subset \mathrm{PG}(9,2)$ constructed from partial spreads in $\mathrm{PG}(4,2)$, including some obtained from the (highly symmetric) regulus-free partial spreads $\mathcal{S}_8$ of size 8.

**Keywords** hypersurfaces in $\mathrm{PG}(n,2)$, associate of a projective flat, Grassmannian $\mathcal{G}_{1,4,2}$, partial spreads in $\mathrm{PG}(4,2)$

# Contents

# 1 Introduction

Throughout we work over GF(2), and so *we may identify the nonzero elements of a vector space* $V(n+1,2)=V_{n+1}$ *with the points* $S_0$ *of the associated projective space* $\mathbb{P}V_{n+1}=\mathrm{PG}(n,2)$. Consequently we identify $\mathrm{GL}(V_{n+1})=\mathrm{GL}(n+1,2)$ with the group $\mathrm{PGL}(n+1,2)$ of collineations of $\mathrm{PG}(n,2)$. Similarly nonzero elements of the dual vector space $V_{n+1}^*$ will be identified with the points of the dual projective space $\mathbb{P}V_{n+1}^*=\mathrm{PG}(n,2)^*$. We use $\prec u,v,\ \dots\ \succ$ for the vector subspace spanned by vectors $u,v,\ \dots$ , and $\langle u,v,\ \dots\ \rangle$ for the flat (projective subspace) generated by projective points $u,v,\ \dots$ . Recall that the annihilator $U^{\mathrm{O}}:=\{f\in V_{n+1}^*|\ f(u)=0,\text{ for all } u\in U\}$ of any subset $U\subset V_{n+1}$ is always a subspace of $V_{n+1}^*$; moreover if $\dim\prec U\succ=r+1$ then $\dim U^{\mathrm{O}}=n-r$. We use the same notation also for projective subspaces. Thus if $\alpha$ is a plane in $\mathrm{PG}(4,2)$ then $\alpha^{\mathrm{O}}$ is a line in $\mathrm{PG}(4,2)^*$.

The vector space $F(S_0)$ of all functions $S_0\to\mathrm{GF}(2)$ is of dimension $|S_0|=2^{n+1}-1$, one basis being $\{\chi_a\}_{a\in S_0}$, where $\chi_a$ is the characteristic function of the singleton subset $\{a\}\subset S_0$. Given a choice of coordinates $x_1,x_2,\ \dots\ ,x_{n+1}$ in $V_{n+1}$, there are $\binom{n+1}{r}$ monomials $\{x_{i_1}x_{i_2}\dots x_{i_r}\}_{1\le i_1<\ \dots\ <i_r\le n+1}$ of (reduced) degree $r$. So altogether we have $\Sigma_{r=1}^{n+1}\binom{n+1}{r}=|S_0|$ linearly independent monomials, and these form another basis for $F(S_0)$. Given a point-set $\psi\subset\mathrm{PG}(n,2)$ it follows that it has equation $Q(x)=0$ for some *uniquely determined* polynomial function $Q$ on $V_{n+1}$ (of reduced degree $\le n+1$ and satisfying $Q(0)=0$). Briefly stated, *every point-set of* $\mathrm{PG}(n,2)$ *is a hypersurface.*

**Remark 1.1** *The foregoing should be contrasted with the situation concerning subsets* $\psi$ *of the points* $S_0$ *of* $\mathrm{PG}(n,q)$ *for* $q>2$*. Here the vector space* $F(S_0)$ *of all functions* $S_0\to\mathrm{GF}(q)$ *is of dimension* $|S_0|=(q^{n+1}-1)/(q-1)$*, and it is still true, see for example [9, Lemma 3.5(a)], that any subset* $\psi\subset S_0$ *can be described to consist of those points* $X=\prec x\succ\in S_0$ *such that* $x\in V(n+1,q)$ *satisfies a single polynomial equation* $p(x)=0$*. However, for* $q>2$*, the polynomial* $p$ *is far from unique. Concerning uniqueness, let us restrict attention to the spaces* $F_1,\ F_2,\ \dots,\ F_{n+1}$*, where* $F_r$ *consists of those homogeneous polynomials* $p(x_1,\ \dots\ ,x_{n+1})$ *of degree* $r(q-1)$*. Such a polynomial* $p$ *satisfies* $p(\lambda x)=p(x)$ *for all* $\lambda\in\mathrm{GF}(q)$*, and so gives rise to a function* $f$ *on* $S_0$ *properly defined by* $f(X)=p(x)$*. Moreover, see [9, Lemma 3.5(a)], each* $f\in F(S_0)$ *arises from some polynomial* $p\in F_{n+1}$*. However for a given subset* $\psi\subset S_0$ *there are of course* $(q-1)^{|\psi^c|}$ *different functions* $f\in F(S_0)$ *such that* $\psi=\{X|f(X)=0\}$*, one such* $f$ *being the characteristic function* $f=\chi_{\psi^c}$ *of* $\psi^c$*, satisfying* $f(X)=0$ *for* $X\in\psi$ *and* $f(X)=1$ *for* $X\in\psi^c$*.*

We will deal with situations where some particular choice of a non-empty point-set $\psi\subset\mathrm{PG}(n,2)=\mathbb{P}V_{n+1}$ has been made. Points, lines, ... , which lie inside $\psi$ will be called *internal* points, lines, ... , and those which lie in $\psi^c$ are termed *external.* Let $\mathcal{G}_\psi<\mathrm{GL}(n+1,2)$ be the stabilizer of the set $\psi$, and denote by $\mathcal{G}_X$ that subgroup of $\mathcal{G}_\psi$ which stabilizes a subset $X\subset\mathrm{PG}(n,2)$. Let us suppose that the subset $\psi$ has equation $Q(x)=0$ where the polynomial $Q$ has (reduced) degree $d$.

In section 1.2 below we consider a few promising possibilities for the choice of $\psi\subset\mathrm{PG}(n,2)$. But in fact our chief concern, from section 2 onwards, will be with the case $\psi=\mathcal{G}_{1,4,2}\subset\mathrm{PG}(9,2)$ of the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2)$, for which $\mathcal{G}_\psi \cong \mathrm{GL}(5,2)$ and for which, see [12], $\deg Q = 5$. However, prior to this, let us describe some results valid for any choice of subset $\psi \subset \mathrm{PG}(n,2)$. It will prove convenient to term a subset $X \subset \mathrm{PG}(n,2)$ *odd* if it contains an odd number of internal points:

**Definition 1.2** *Given the choice of subset $\psi \subset \mathrm{PG}(n,2)$ a subset $X \subset \mathrm{PG}(n,2)$ is termed $\psi$-odd whenever $|X \cap \psi|$ is odd and $\psi$-even whenever $|X \cap \psi|$ is even. Once a particular $\psi$ has been agreed then we omit the "$\psi$-".*

In the case when $X$ is a flat note that $X$ is odd whenever $X$ contains an even number of external points.

**Theorem 1.3** *If the subset $\psi \subset \mathrm{PG}(n,2)$ has equation $Q(x) = 0$ where $\deg Q = d$, then for $r \geq d$ every $r$-flat $X$ in $\mathrm{PG}(n,2)$ is $\psi$-odd.*

**Proof.** See the theorem in [18]. ∎

## 1.1 The associate $X^{\#}$ of a flat $X \subset \mathrm{PG}(n,2)$

Let $Q^{(r)}$ be the $r$th polarization of $Q$. Thus

$$\begin{aligned} Q^{(1)}(x_1, x_2) &= Q(x_1 + x_2) + Q(x_1) + Q(x_2) = Q(x_2, x_1), \\ Q^{(2)}(x_1, x_2, x_3) &= Q^{(1)}(x_1, x_2 + x_3) + Q^{(1)}(x_1, x_2) + Q(x_1, x_3) \\ &= \textstyle\sum_{i=1}^{3} Q(x_i) + \sum_{1\leq i<j\leq 3} Q(x_i + x_j) + Q(x_1 + x_2 + x_3), \end{aligned}$$

and so on, with

$$\begin{aligned} Q^{(r)}(x_1, \ldots, x_r, x_{r+1}) &= Q^{(r-1)}(x_1, \ldots, x_r + x_{r+1}) + Q^{(r-1)}(x_1, \ldots, x_r) + Q^{(r-1)}(x_1, \ldots, x_{r+1}) \\ &= \textstyle\sum_{i=1}^{r+1} Q_i + \sum_{1\leq i<j\leq r+1} Q_{ij} + \sum_{1\leq i<j\leq r+1} Q_{ijk} + \ldots + Q_{12\ldots r(r+1)} \, . \end{aligned} \tag{1.1}$$

(Here we have abbreviated $Q(x_i), Q(x_i{+}x_j), Q(x_i{+}x_j{+}x_k), \ldots$ as $Q_i, Q_{ij}, Q_{ijk}, \ldots$ etc.) Consequently the following lemma holds.

**Lemma 1.4** *If the points $x_1, \ldots, x_{r+1}$ generate a projective $r$-flat $U$ then*

$$Q^{(r)}(x_1, \ldots, x_{r+1}) = \textstyle\sum_{x\in U} Q(x), \tag{1.2}$$

*while if $x_1, \ldots, x_{r+1}$ are dependent then $Q^{(r)}(x_1, \ldots, x_{r+1}) = 0$.*

Granted that the (reduced) degree of $Q$ is $d$, observe that $Q^{(d-1)}$, the completely polarized form of $Q$, is a multilinear function of its $d$ arguments. In the following we will adopt the abbreviation $Q^{\ddagger} := Q^{(d-1)}$. Note therefore that $Q^{\ddagger}(x_1, \ldots, x_d)$ *is an alternating multilinear form.*

**Definition 1.5** *If $X \subset \mathrm{PG}(n,2)$ is an $r$-flat $(r > 0)$, then (with respect to the subset $\psi$) its* associate *$X^{\#}$ is the following subset of $\mathrm{PG}(n,2)$:*

$$X^{\#} = \{y \in \mathrm{PG}(n,2) \,|\, Q^{\ddagger}(x_1, \ldots, x_{d-1}, y) = 0 \textit{ for all } x_1, \ldots, x_{d-1} \in X\}. \tag{1.3}$$

In the rest of this section we consider some immediate consequences of this definition.

**Lemma 1.6** *Let $X$ be an $r$-flat in* $\mathrm{PG}(n,2)$*. Then*

*(i) $X^{\#}$ is always a flat;*

*(ii) if $r$ is $< d-2$ then $X^{\#} = \mathrm{PG}(n,2)$;*

*(iii) if $r = d-2$ then $X \subset X^{\#}$;*

*(iv) if $W$ is a flat then* $\quad W \subset X \Longrightarrow X^{\#} \subseteq W^{\#}$;

*(v)* $\mathcal{G}_X \leq \mathcal{G}_{X^{\#}}$.

**Proof.** (i) Since $\deg Q = d$, $Q^{\ddagger}$ is multilinear in its $d$ arguments.

(ii) If $r < d-2$ then any $d-1$ points $x_1, ..., x_{d-1}$ of $X$ are dependent.

(iii) If $r = d-2$ and $y \in X$ then $x_1, ..., x_{d-1}, y$ in (1.3) are dependent.

(iv) If $Q^{\ddagger}(x_1, ..., x_{d-1}, y) = 0$ holds for all $x_1, ..., x_{d-1} \in X$ then it holds for all $x_1, ..., x_{d-1}$ in the subspace $W$ of $X$.

(v) $X^{\#}$ is uniquely determined by $X$. ■

**Remark 1.7** *In most cases of interest the polynomial $Q$, and even more its various polarizations $Q^{(r)}$, will be quite complicated. Consequently, leaving aside the use of the computer, the determination of $X^{\#}$ directly from definition 1.5 will usually be an extremely daunting task. Fortunately, as in the next lemma, the definition of $X^{\#}$ may be rephrased in terms of certain incidence properties.*

**Lemma 1.8** *Let $X \subset \mathrm{PG}(n,2)$ be an $r$-flat, $r \geq d-2$. Then*

*(i) a point $y \in X^c$ is in the associate $X^{\#}$ of $X$ if and only if for each $(d-2)$-flat $H$ of $X$ the join $\langle y, H\rangle$ is odd;*

*(ii) a point $y \in X$ is in the associate $X^{\#}$ of $X$ if and only if the following holds: if $K$ is any $(d-1)$-flat of $X$ which contains $y$ then $K$ is odd.*

**Proof.** The lemma follows from the following two observations. Firstly, if $x_1, \ldots, x_{d-1}, y$ are independent, generating a $(d-1)$-flat $U = \langle y, H\rangle$, where $H = \langle x_1, ..., x_{d-1}\rangle$, then, see (1.2), $Q^{(d-1)}(x_1, ..., x_{d-1}, y) = \sum_{u \in U} Q(u)$, which last $= 0$ if and only if $Q(u) = 0$ for an odd number of points $u \in U$. Secondly, if $x_1, \ldots, x_{d-1}, y$ are dependent, then, lemma 1.4, $Q^{(d-1)}(x_1, ..., x_{d-1}, y) = 0$. ■

**Corollary 1.9** *(i) If $X$ is a $(d-2)$-flat then a point $y \in X^c$ is in the associate $X^{\#}$ of $X$ if and only if $\langle y, X\rangle$ is odd.*

*(ii) If $X$ is an even $(d-1)$-flat then $X^{\#}$ is disjoint from $X$.*

*(iii) If $X$ is an odd $(d-1)$-flat then $X^{\#} \supseteq X$; moreover $X^{\#}$ is odd.*

**Proof.** Parts (i) and (ii) follow immediately from parts (i) and (ii) of the last lemma. Concerning part (iii), the relation $X^{\#} \supseteq X$ also follows from part (ii) of the last lemma; hence either $X^{\#} = X$, and so $X^{\#}$ is odd, or else $X^{\#}$ is a $r$-flat for some $r \geq d$, whence, by theorem 1.3, $X^{\#}$ is odd. ■

## 1.2 Which hypersurfaces should we investigate?

We can hardly expect to find many further properties of the associate $X^{\#}$ which hold for a general choice of point-set $\psi$. (Recall that even in the familiar case when $Q$ has degree $d = 2$ results such as

$$\dim U + \dim U^{\perp} = n+1, \tag{1.4}$$

where $U$ is a vector subspace of an $(n+1)$-dimensional vector space $V(n+1, \mathbb{F})$, are valid only if the bilinear form $Q^{\ddagger}$ is non-degenerate. In particular (1.4) never holds if $\mathbb{F} = \mathrm{GF}(2)$ and $n$ is even.) So it seems sensible to investigate properties of the associate $X^{\#}$ for some (hopefully interesting) particular choices of point-set $\psi$.

Out of the enormous number of choices available for $\psi$ we drew up a provisional list of ones which we intend to investigate further. Our list includes the following.

1. The Segre variety $\psi = \mathcal{S}_{1,2,2} \subset \mathrm{PG}(5,2)$, for which $\deg Q = 3$. The 21 points of $\psi$ are the 21 decomposable elements of a tensor product space $V_6 = V_2 \otimes V_3$.
2. The Segre variety $\mathcal{S}_{1,3,2} \subset \mathrm{PG}(7,2)$, for which $\deg Q = 4$.
3. The Segre variety $\mathcal{S}_{2,2,2} \subset \mathrm{PG}(8,2)$, for which $\deg Q = 6$.
4. A double-five $\psi \subset \mathrm{PG}(5,2)$, see [19], [20], [21], for which $d = 3$, see [19, Section 5], [20, Section 2.2].
5. An icosahedral twenty-seven $\phi \subset \mathrm{PG}(5,2)$, for which $d = 3$, see [19, Section 4.2, Theorem 5.4].
6. Various other cubic hypersurfaces in $\mathrm{PG}(5,2)$, see [21, Section 1.4]. One such is $\psi = \mathrm{PG}(5,2) \diagdown \{2 \text{ skew planes}\}$, see [21, line before Section 6.2].
7. The hypersurface $\psi \subset \mathrm{PG}(15,2) = \mathbb{P}(\operatorname{End} V_{4,2})$ with equation $\det A = 0$, $A \in \operatorname{End}(V_{4,2})$. So $d = 4$. For this choice of $\psi$ it may be especially interesting to investigate the associate in the case of external flats $X$, namely the linear sections of $\mathrm{GL}(4,2)$ which were classified in [5]. In particular the maximal 3-dimensional linear sections which lies inside an $\mathrm{Alt}(7)$ subgroup of $\mathrm{GL}(4,2)$, see [22], may be particularly worthy of consideration.
8. The Grassmannian $\mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$, for which $d = 5$, see [12].

For most of the $d = 3$ cases it would presumably be a reasonably simple task to determine the associate for all flats. Nevertheless we delayed looking at these and concentrated instead upon the choice 8. In part this was because we envisaged that $\deg Q = 5$ could be more interesting than $\deg Q = 3$, and give a better idea of the richness to be expected concerning the associate with respect to more general hypersurfaces. We were also attracted to the choice 8 because we, jointly with J.G. Maks, had recently succeeded in classifying all flats in $\mathrm{PG}(9,2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$, see [14], [15]; see Appendix E for a brief summary of the classification. As the rest of this paper will demonstrate, our choice 8 involved us in a prodigious amount of (exciting!) work. Despite the length of this paper, many further aspects remain to be investigated, see for example the speculations in section 9. Consequently investigation of any of the other choices 1 - 7 must remain a task for the future!

# 2 The quintic Grassmannian $\mathcal{G}_{1,4,2}$ in $\mathrm{PG}(9,2)$

So we now leave aside the case of a general subset $\psi \subset \mathrm{PG}(n,2)$ and concentrate upon the special case of the quintic Grassmannian $\psi = \mathcal{G}_{1,4,2}$ in the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$, $V_5 = V(5,2)$, for which $\mathcal{G}_\psi \cong \mathrm{GL}(5,2)$ and for which the reduced degree of the polynomial function $Q$ is 5; see [12], and also eq. (5.1) below. However it should become clear that some of the results which we enunciate for $\mathcal{G}_{1,4,2}$ are in fact special cases of valid general results: see remark 4.3.

Since $d = 5$ and $n = 9$, theorem 1.3 now reads as follows.

**Theorem 2.1** *For $r \geq 5$ every $r$-flat $X$ in* $\mathrm{PG}(9,2)$ *is odd.*

## 2.1 Preliminaries

### 2.1.1 Bivectors, trivectors and duality

We are starting out from a vector space $V_5 = V(5,2)$, and its projective space $\mathrm{PG}(4,2) = \mathbb{P}V_5$, along with the concomitant space $V_{10} := \wedge^2 V_5$ of bivectors. Whilst the dual $(V_{10})^*$ of $V_{10}$ may be viewed as the space $\wedge^2 V_5^*$ of dual bivectors, we will usually view it as the space $V_{10}^* := \wedge^3 V_5$ of trivectors, by means of the natural nondegenerate bilinear pairing $\lceil \cdot,\cdot \rfloor$ of $\wedge^3 V_5$ with $\wedge^2 V_5$ defined by

$$t \wedge b = \lceil t, b \rfloor\, e, \quad t \in \wedge^3 V_5,\ b \in \wedge^2 V_5. \tag{2.1}$$

Here $e$ is the (unique!) basis vector for the 1-dimensional space $\wedge^5 V_5$. For more material surrounding this choice of $\wedge^3 V_5$ as dual of $V_{10}$ see Appendix A, where details may also be found concerning:
(i) the fact that $\wedge^3 V_5$ and $\wedge^2 V_5^*$ are images of each other under the (unique!) Poincaré isomorphisms $\perp : \wedge^2 V_5^* \to \wedge^3 V_5$ and $\perp' = (\perp)^{-1} : \wedge^3 V_5 \to \wedge^2 V_5^*$;
(ii) the alternating bilinear form $t(\cdot,\cdot)$ determined by $t \in \wedge^3 V_5$;
(iii) the 'polar' $\alpha^t$ of a flat $\alpha \subset \mathrm{PG}(9,2)$.

The (projective) annihilator of a subset $U$ of $\mathrm{PG}(9,2)^* = \mathbb{P}V_{10}^*$ is thus that flat $U^{\mathrm{O}}$ of $\mathrm{PG}(9,2) = \mathbb{P}V_{10}$ defined by

$$U^{\mathrm{O}} = \{b \in \mathrm{PG}(9,2) |\ u \wedge b = 0, \text{ for all } u \in U\}. \tag{2.2}$$

If $U$ is a single point $\langle t \rangle$ then $U^{\mathrm{O}}$ is a hyperplane, which we denote

$$H(t) := \langle t \rangle^{\mathrm{O}}. \tag{2.3}$$

### 2.1.2 The action of $\mathrm{GL}(5,2)$ upon $\mathrm{PG}(9,2)$ and $\mathrm{PG}(9,2)^*$

Each $A \in \mathrm{GL}(5,2)$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\mathrm{GL}(V_{10})$ whose effect on the decomposable bivectors $u \wedge v \in V_{10}$ is $T_A(u \wedge v) = Au \wedge Av$, $A \in \mathrm{GL}(5,2)$. Similarly we put $\hat{T}_A = \wedge^3 A \in \mathrm{GL}(V_{10}^*)$. Since $(\wedge^5 A)e = e$ for all $A \in \mathrm{GL}(5,2)$, note the invariance property

$$\hat{T}_A t \wedge T_A b = t \wedge b, \quad t \in \wedge^3 V_5,\ b \in \wedge^2 V_5, \text{ for all } A \in \mathrm{GL}(5,2). \tag{2.4}$$

Thus $\hat{T}_A$ is the contragredient of $T_A$ : $\lceil \hat{T}_A t, T_A b \rfloor = \lceil t, b \rfloor$.

If $X$ is an object of some $\mathrm{GL}(5,2)$-space then $\mathcal{G}_X < \mathrm{GL}(5,2)$ denotes its stabilizer group. In particular if $X$ is an object in $\wedge^2 V_5$ its stabilizer is $\mathcal{G}_X =$

$\{A \in \mathrm{GL}(5,2) | T_A(X) = X\}$. Under the action $T$ of $\mathrm{GL}(5,2)$ the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$ is the union $\mathrm{Rk}_2 \cup \mathrm{Rk}_4$ of two $\mathrm{GL}(5,2)$-orbits, consisting of those bivectors having rank 2 and rank 4, respectively. The Grassmann map $\prec u, v \succ \mapsto \prec u \wedge v \succ$ sends the 2-spaces of $V_5$ to those 1-spaces of $\wedge^2 V_5$ which are spanned by decomposable bivectors. Projectively, the lines of $\mathrm{PG}(4,2)$ are mapped onto the points of the orbit $\mathrm{Rk}_2$, the latter, being the Grassmannian $\mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$ of lines of $\mathrm{PG}(4,2)$, having length 155. Consequently $|\,\mathrm{Rk}_4\,| = 1023 - 155 = 868$. Throughout this paper the images in $G_{1,4,2} \subset \mathrm{PG}(9,2)$ of lines $\lambda, \mu$ in $\mathrm{PG}(4,2)$ will be denoted $l, m$. Similarly under the action $\hat{T}$ the projective space $\mathrm{PG}(9,2)^* = V_{10}^* \diagdown \{0\}$ splits into two $\mathrm{GL}(5,2)$-orbits, say $\mathrm{Rk}_2^*$ and $\mathrm{Rk}_4^*$, of lengths 155 and 868. Here the orbit $\mathrm{Rk}_2^*$ of length 155 is the Grassmannian $\mathcal{G}_{2,4,2}$ consisting of the 155 decomposable trivectors, the Grassmann images of the 155 planes of $\mathrm{PG}(4,2)$, with $a \wedge b \wedge c$ being the image of the plane $\langle a, b, c\rangle$. Of course the decomposition $\mathrm{PG}(9,2)^* = \mathrm{Rk}_2^* \cup \mathrm{Rk}_4^*$ is the image under the Poincaré isomorphism $\perp$ of the decomposition, say $\mathbb{P}(\wedge^2 V_5^*) = \mathrm{Rk}_2(V_5^*) \cup \mathrm{Rk}_4(V_5^*)$, of $\mathbb{P}(\wedge^2 V_5^*)$ into dual bivectors of ranks 2 and 4.

A bivector $b \in \mathrm{Rk}_4$ can be expressed $b = u \wedge v + x \wedge y$, for linearly independent $x, y, u, v \in V_5$ and defines a solid $\mathrm{im}\, b := \langle u, v, x, y\rangle \subset \mathrm{PG}(4,2)$, and also a point $k^* := \ker b \in \mathrm{PG}(4,2)^*$, where $\mathrm{im}\, b$ has equation $k^*(x) = 0$. Similarly a dual bivector $b^* \in \mathrm{Rk}_4(V_5^*)$ can be expressed $b^* = u^* \wedge v^* + x^* \wedge y^*$, for linearly independent $x^*, y^*, u^*, v^* \in V_5^*$ and defines a solid $\mathrm{im}\, b^* := \langle u^*, v^*, x^*, y^*\rangle \subset \mathrm{PG}(4,2)^*$, and also a point $k := \ker b^* \in \mathrm{PG}(4,2)$, namely $k = (\mathrm{im}\, b^*)^{\mathrm{O}}$. If $b^* = e^1 \wedge e^2 + e^3 \wedge e^4 \in \mathrm{Rk}_4(V_5^*)$, then the corresponding trivector $t = \perp b^* \in \mathrm{Rk}_4^*$ is, see (A.7),

$$t = \perp (e^1 \wedge e^2 + e^3 \wedge e^4) = (e_3 \wedge e_4 + e_1 \wedge e_2) \wedge e_5. \tag{2.5}$$

Observe that $t$ is of the form $b \wedge k$ where $k$ is the kernel $e_5$ of $b^*$. For a trivector $t = \perp b^* \in \mathrm{Rk}_4^*$ we define the point $\ker t \in \mathrm{PG}(4,2)$ to be $\ker b^*$; Alternatively, for $t \in \mathrm{Rk}_4^*$, we may define $\ker t$ to be that unique point $k \in \mathrm{PG}(4,2)$ which satisfies $t \wedge k = 0$.

## 2.2 The associate $X^{\#}$ of a flat $X \subset \mathrm{PG}(9,2)$

The completely polarized form $Q^{\ddagger}$ of $Q$ is an alternating quinquelinear function of its five arguments and so definition 1.5 of the associate $X^{\#}$ of an $r$-flat $X$ now reads:

**Definition 2.2** *If $X \subset \mathrm{PG}(9,2)$ is an $r$-flat ($r > 0$), then its* associate $X^{\#}$ *is the following subset of* $\mathrm{PG}(9,2)$*:*

$$X^{\#} = \{y \in \mathrm{PG}(9,2) \,|\, Q^{\ddagger}(x_1, ..., x_4, y) = 0 \textit{ for all } x_1, ..., x_4 \in X\}. \tag{2.6}$$

Since $d = 5$ and $n = 9$, lemmas 1.6 and 1.8, and corollary 1.9, now read as follows.

**Lemma 2.3** *Let $X$ be an $r$-flat in* $\mathrm{PG}(9,2)$*. Then*
*(i) $X^{\#}$ is always a flat;*
*(ii) if $r$ is $< 3$ then $X^{\#} = \mathrm{PG}(9,2)$;*
*(iii) if $r = 3$ then $X \subseteq X^{\#}$;*
*(iv) if $W$ is a flat then $W \subset X \Longrightarrow X^{\#} \subseteq W^{\#}$.*
*(v) $\mathcal{G}_X \leq \mathcal{G}_{X^{\#}}$.*

**Remark 2.4** *The explicit coordinate form of* $Q^\ddagger$ *is quite complicated, see eqs. (5.2), (5.3), and so when seeking to determine the associate* $X^\#$ *of an* $r$*-flat* $X \subset \mathrm{PG}(9,2)$ *it is usually best to avoid direct appeal to the definition 2.2. Instead, in most cases, it would appear sensible to attempt to determine* $X^\#$ *by appeal to the incidence properties of lemma 2.5, see below, and its off-shoots. Nevertheless there do exist a few cases where one can make direct use of the explicit form of* $Q^\ddagger$ *— see section 5.*

**Lemma 2.5** *Let* $X$ *be an* $r$*-flat in* $\mathrm{PG}(9,2)$, $r \geq 3$. *Then*

*(i) a point* $y \in X^c$ *is in the associate* $X^\#$ *of* $X$ *if and only if for each* 3*-flat* $D$ *of* $X$ *the join* $\langle y, D\rangle$ *(a 4-flat) is odd;*

*(ii) a point* $y \in X$ *is in the associate* $X^\#$ *of* $X$ *if and only if the following holds: if* $K$ *is any* 4*-flat of* $X$ *which contains* $y$ *then* $K$ *is odd.*

**Corollary 2.6** *(i) If* $X$ *is a* 3*-flat then a point* $y \in X^c$ *is in the associate* $X^\#$ *of* $X$ *if and only if* $\langle y, X\rangle$ *is odd.*

*(ii) If* $X$ *is an even* 4*-flat then* $X^\#$ *is disjoint from* $X$.

*(iii) If* $X$ *is an odd* 4*-flat then* $X^\# \supseteq X$; *moreover* $X^\#$ *is odd.*

Notwithstanding the foregoing rephrasing of the definition of $X^\#$, it has to be said that the determination of the associate of a given $r$-flat in $\mathrm{PG}(9,2)$ is in general a far from trivial task, and moreover it appears that we are embarking upon a journey through previously unexplored terrain. For a start, there are, under the action of $\mathrm{GL}(5,2)$, a large number of different kinds of flats in $\mathrm{PG}(9,2)$, but at present a classification exists only in the case of those flats which are external to the Grassmannian $\mathcal{G}_{1,4,2}$. (See [14], [15], and also Appendix E, for a description of the ten $\mathrm{GL}(5,2)$-orbits of external flats.) Thus the territory to be explored is quite vast. In the present paper, at least we succeed in proving a few general results and in determining the associate $X^\#$ for a considerable number of flats $X$ in $\mathrm{PG}(9,2)$. Many of the flats $X$ which we consider will be ones constructed out of partial spreads in $\mathrm{PG}(4,2)$; see section 2.3. Hopefully future explorations will reveal a fuller picture.

## 2.3 Flats in $\mathrm{PG}(9,2)$ from partial spreads in $\mathrm{PG}(4,2)$

Which flats in $\mathrm{PG}(9,2)$ should we investigate? Certainly one good source of flats in $\mathrm{PG}(9,2)$ arises from the partial spreads in $\mathrm{PG}(4,2)$. A *partial spread* $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ of size $r(>0)$ is a set $\{\mu_1, \ldots, \mu_r\}$ of $r$ pairwise disjoint lines. Such partial spreads have recently been completely classified: see [7], where, under the action of $\mathrm{GL}(5,2)$, it is shown that they fall into 64 distinct classes ($\mathrm{GL}(5,2)$-orbits). See Appendix D for tables extracted from [7]. Given a partial spread $\mathcal{S}_r = \{\mu_1, \ldots, \mu_r\}$ in $\mathrm{PG}(4,2)$, let the corresponding $r$-set of points of $\mathrm{Rk}_2 \subset \mathrm{PG}(9,2)$ be $\mathcal{C}_r = \{m_1, \ldots., m_r\}$. For $1 \leq i < j < \ \ldots \ \leq r$ we define $m_{ij} = m_i + m_j$, $m_{ijk} = m_i + m_j + m_j$, ...., $m_\Sigma = m_{12\ldots r} = \sum_{i=1}^r m_i$. One should be aware that some of these vectors may be zero, and that coincidences may therefore occur amongst the points (= nonzero vectors). For example, if $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ is a spread for a solid $\sigma \subset \mathrm{PG}(4,2)$ then $m_\Sigma = 0$, whence $m_{123} = m_{45}$, $m_{1234} = m_5$, etc.

**Theorem 2.7** *([13]) (i)* $\mathcal{C}_r$ *is a* $r$*-cap: that is, no three points of* $\mathcal{C}_r$ *are collinear.*

*(ii) The $\binom{r}{2}$ points $m_{ij}$ are distinct and are external.*
*(iii) The $\binom{r}{3}$ points $m_{ijk}$ are distinct and are external.*

*Caution.* As in the example given prior to the theorem, the points $m_{ij}$ in (ii) are not always distinct from the points $m_{ijk}$ in (iii).

If $m_1, \ldots., m_r$ are independent then in addition to the $(r-1)$-flat $\langle \mathcal{C}_r \rangle = \langle m_1, \ldots., m_r \rangle$ we may also consider the $(r-2)$-flat $\mathcal{E}(\mathcal{C}_r)$ which is generated by the $\binom{r}{2}$ points $m_{ij}$. The flats in $\mathrm{PG}(9,2)$ of the kind $\mathcal{E}(\mathcal{C}_r)$ were used in [14], [15] to construct representatives for seven out of the ten orbits of external flats which exist in $\mathrm{PG}(9,2)$.

If for a flat $X \subset \mathrm{PG}(9,2)$ we have $|X \cap \mathrm{Rk}_2| = n_1$ and $|X \cap \mathrm{Rk}_4| = n_2$ then we will say that $X$ is a flat of type $(n_1, n_2)$. For flats of the form $X = \langle \mathcal{C}_5 \rangle$ or $X = \mathcal{E}(\mathcal{C}_5)$, where $\mathcal{C}_5$ is the Grassmann image of a partial spread $\mathcal{S}_5$, the value of $(n_1, n_2)$ can be read off from Table 1 below. See Table 2 for flats of the form $X = \langle \mathcal{C}_6 \rangle$ or $X = \mathcal{E}(\mathcal{C}_6)$. The labelling, Vc.1, VId.2, etc., of the different classes of partial spreads is as in [7, Table B.2].

Table 1: values of $n_1(X) = |X \cap \mathrm{Rk}_2|$ for $X$ arising from an $\mathcal{S}_5$

| Class Vx | x = | a.1 | b.1 | c.1 | d.1 | e.1 | f.1 | g.1 | h.1 | i.1 | j.1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_5 \rangle$ | $n_1(X) =$ | — | 7 | 9 | 6 | 5 | 6 | 5 | 7 | 7 | — |
| $X = \mathcal{E}(\mathcal{C}_5)$ | $n_1(X) =$ | 5 | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 1 | 5 |

Table 2: values of $n_1(X) = |X \cap \mathrm{Rk}_2|$ for $X$ arising from an $\mathcal{S}_6$

| Class VIx | x = | a.1 | b.1 | c.1 | c.2 | d.1 | d.2 | e.1 | e.2 |
|---|---|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_6 \rangle$ | $n_1(X) =$ | 9 | 11 | 13 | — | 11 | 15 | 9 | 7 |
| $X = \mathcal{E}(\mathcal{C}_6)$ | $n_1(X) =$ | 3 | 3 | 4 | 6 | 4 | 6 | 2 | 1 |

Table 2 (continued)

| Class VIx | x = | f.1 | g.1 | h.1 | h.2 | i.1 | j.1 |
|---|---|---|---|---|---|---|---|
| $X = \langle \mathcal{C}_6 \rangle$ | $n_1(X) =$ | 9 | 7 | 9 | 11 | 13 | 11 |
| $X = \mathcal{E}(\mathcal{C}_6)$ | $n_1(X) =$ | 2 | 1 | 2 | 3 | 4 | 3 |

*Notes.* (i) In Table 1, $\mathcal{E}(\mathcal{C}_5)$ is a 3-flat, and so it is of type $(n_1, 15-n_1)$, where $n_1$ is as given in the last row. With the exception of an $\mathcal{S}_5$ of class Va.1 or of class Vj.1, $\langle \mathcal{C}_5 \rangle$ is a 4-flat, and so it is of type $(n_1, 31-n_1)$, where $n_1$ is as given in the penultimate row. But for an $\mathcal{S}_5$ of class Va.1 or of class Vj.1 we have $m_\Sigma = \Sigma_{i=1}^{5} m_i = 0$, whence $\langle \mathcal{C}_5 \rangle$ coincides with $\mathcal{E}(\mathcal{C}_5)$, and, for both of these classes, $\langle \mathcal{C}_5 \rangle$ is of type $(5, 10)$.
(ii) In Table 2, $\mathcal{E}(\mathcal{C}_6)$ is a 4-flat, and so it is of type $(n_1, 31-n_1)$, where $n_1$ is as given in the last row. With the exception of an $\mathcal{S}_6$ of class VIc.2, $\langle \mathcal{C}_6 \rangle$ is a 5-flat, and so it is of type $(n_1, 63-n_1)$, where $n_1$ is as given in the penultimate row. But for an $\mathcal{S}_6$ of class VIc.2, $\langle \mathcal{C}_6 \rangle$ coincides with $\mathcal{E}(\mathcal{C}_6)$, and is of type $(6, 25)$.

**Remark 2.8** *In a sequence of the kind $\mathcal{C}_r, \langle \mathcal{C}_r \rangle, \langle \mathcal{C}_r \rangle^{\#}, \langle \mathcal{C}_r \rangle^{\#\#}, \langle \mathcal{C}_r \rangle^{\#\#\#}, \ldots$ , it should be noted that* symmetry can never decrease: *if $X$ precedes $Y$ in the sequence then $Y$ is uniquely determined by $X$, whence $\mathcal{G}_X \leq \mathcal{G}_Y$. A corresponding* *remark applies to the members of a sequence* $\mathcal{C}_r, \mathcal{E}(\mathcal{C}_r), \mathcal{E}(\mathcal{C}_r)^{\#}, \mathcal{E}(\mathcal{C}_r)^{\#\#}$, .... . *An example where* $\mathcal{G}_{\mathcal{C}_r}$ *is strictly included in* $\mathcal{G}_{\langle\mathcal{C}_r\rangle}$ *is provided by a partial spread* $\mathcal{S}_4$ *of class IVa.1; see section 3.2. However we always have* $\mathcal{G}_{\mathcal{C}_r} = \mathcal{G}_{\langle\mathcal{C}_r\rangle}$ *in cases where* $\langle\mathcal{C}_r\rangle \cap \mathcal{G}_{1,4,2} = \mathcal{C}_r$. *We also always have* $\mathcal{G}_{X^O} = \mathcal{G}_X$, *this following from the fact that* $(X^O)^O = X$.

# 3 The associate $D^{\#}$ of a 3-flat $D \subset \mathrm{PG}(9,2)$

Bearing in mind lemma 2.3(ii), (iii), let us first concentrate upon finding $D^{\#}$ for certain 3-flats $D$.

## 3.1 The linear form $f_D$ and trivector $t_D$ of a 3-flat $D$

Let $D = \langle a_1, a_2, a_3, a_4 \rangle$ be a 3-flat of $\mathrm{PG}(9,2) = \mathbb{P}V_{10}$. Then we define its *associated linear form* $f_D \in (V_{10})^*$ by

$$f_D(x) = Q^{\ddagger}(a_1, a_2, a_3, a_4, x), \quad x \in \wedge^2 V_5. \tag{3.1}$$

This is a good definition: because $Q^{\ddagger}$ is alternating and quinquelinear, any choice of independent points $a_1, a_2, a_3, a_4 \in D$ yield the same linear form. As in eq. (A.3), the linear form $f_D$ determines, via $f_D(x) = \lceil t_D, x \rfloor$, a trivector $t_D \in V_{10}^* := \wedge^3 V_5$, the *associated trivector* $t_D$ of the 3-flat $D$. Thus

$$t_D \wedge x = f_D(x)e, \quad x \in \wedge^2 V_5. \tag{3.2}$$

In the foregoing notation the next lemma is now immediate. Note that part (i) justifies the strict inclusion $D \subset D^{\#}$ which was asserted in lemma 2.3(iii).

**Lemma 3.1** *Let $D$ be a 3-flat in* $\mathrm{PG}(9,2)$.

*(i) If $f_D$ is nonzero, then $D^{\#}$ is the hyperplane $H(t_D) := \langle t_D \rangle^O$; if $f_D$ is the zero form then $D^{\#} = \mathrm{PG}(9,2)$.*

*(ii) The trivector $t_D$ is invariant under the action $\hat{T}$ of $\mathcal{G}_D$.*

*(iii) If $t_D \in \mathrm{Rk}_2^*$, and so is the Grassmann image of a plane $\alpha_D \subset \mathrm{PG}(4,2)$, then $\alpha_D$ is stabilized by $\mathcal{G}_D$.*

*(iv) If $t_D \in \mathrm{Rk}_4^*$ let $k_D = \ker t_D$ (see after eq. (2.5)); then the point $k_D$ of* $\mathrm{PG}(4,2)$ *is a fixed point of $\mathcal{G}_D$.*

**Remark 3.2** *Cases where $t_D = 0$, that is where $f_D$ is the zero form, do arise, see for example theorems 3.9, 5.1 and 5.3 below. In such cases it is sometimes convenient to interpret* $\mathrm{PG}(9,2)$ *as the "hyperplane" $H(0)$.*

**Remark 3.3** *In the foregoing notation lemma 2.5(i) asserts that a point $y \in X^c$ is in the associate $X^{\#}$ of the $r$-flat $X$ if and only if $f_D(y) = 0$ for each 3-flat $D$ of $X$.*

### 3.1.1 The two kinds of hyperplanes in $\mathrm{PG}(9,2)$

On account of lemma 3.1(i) we need to know about the hyperplanes in $\mathrm{PG}(9,2)$. In fact, since $\mathrm{PG}(9,2)^*$ consists of just two $\mathrm{GL}(5,2)$-orbits, $\mathrm{Rk}_2^*$ and $\mathrm{Rk}_4^*$, it follows, as in the next theorem, that there are just two $\mathrm{GL}(5,2)$-orbits of hyperplanes in $\mathrm{PG}(9,2)$.

**Theorem 3.4** *Under the natural action of* $\mathrm{GL}(5,2)$ *there are two orbits of hyperplanes in* $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$, *say $\mathcal{H}_{155}$ and $\mathcal{H}_{868}$, of lengths 155 and 868. A hyperplane $H \in \mathcal{H}_{155}$ intersects $\mathcal{G}_{1,4,2}$ in 91 points while a hyperplane $H \in \mathcal{H}_{868}$ intersects $\mathcal{G}_{1,4,2}$ in 75 points. The hyperplanes in $\mathcal{H}_{155}$ are in bijective correspondence $\pi \leftrightarrow H(\pi)$ with the planes $\pi \subset \mathrm{PG}(4,2)$, the elements of the 91-set $H(\pi) \cap \mathrm{Rk}_2$ being the Grassmann images of the 91 lines of* $\mathrm{PG}(4,2)$ *which meet* *the plane $\pi$. The hyperplanes in $\mathcal{H}_{868}$ are in bijective correspondence $t \leftrightarrow H(t)$ with the trivectors $t \in \mathrm{Rk}_4^*$, the elements of the 75-set $H(t) \cap \mathrm{Rk}_2$ being the Grassmann images of the 75 lines of* $\mathrm{PG}(4,2)$ *which are isotropic, see after eq. (A.10), for the alternating bilinear form $t(\cdot,\cdot)$.*

**Proof.** Each hyperplane in $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$ is of the form $\langle t\rangle^{\mathrm{O}}$ for some point $t \in \mathrm{PG}(9,2)^* = \mathbb{P}(\wedge^3 V_5) = \mathrm{Rk}_2^* \cup \mathrm{Rk}_4^*$. The orbit $\mathcal{H}_{155}$ consists of the 155 hyperplanes $\langle t\rangle^{\mathrm{O}}$ with $t \in \mathrm{Rk}_2^*$ and the orbit $\mathcal{H}_{868}$ consists of the 868 hyperplanes $\langle t\rangle^{\mathrm{O}}$ with $t \in \mathrm{Rk}_4^*$. If $t_\pi \in \mathrm{Rk}_2^*$ is the Grassmann image of the plane $\pi$ in $\mathrm{PG}(4,2)$, then by (2.2) and (A.11) a point $l$ of $\mathcal{G}_{1,4,2}$ lies in $\langle t_\pi\rangle^{\mathrm{O}}$ if and only if $l$ is the image of a line $\lambda$ which meets $\pi$. Such lines $\lambda$ number $7 + 7 \times 12 = 91$, since $\pi$ contains 7 lines, and through each of the 7 points of $\pi$ there pass 12 lines not in $\pi$.

Consider the hyperplane $H = \langle t\rangle^{\mathrm{O}} \in \mathcal{H}_{868}$ where $t \in \mathrm{Rk}_4^*$. By (2.2) a point $l = x \wedge y$ of $\mathcal{G}_{1,4,2}$ lies in $H$ if and only if $t \wedge x \wedge y = 0$, that is, see (A.9), if and only if $t(x,y) = 0$. But this last is the condition for the line $\lambda = \langle x, y\rangle$ to be isotropic, and by lemma A.2 there are 75 of these lines. ■

## 3.2 The associate $D^{\#}$ of $D = \langle \mathcal{C}_4\rangle$

First we look at certain 3-flats in $\mathrm{PG}(9,2)$ arising from partial spreads of size 4 in $\mathrm{PG}(4,2)$. There exist, see [7, Table B.2], four distinct classes of partial spreads $\mathcal{S}_4$ in $\mathrm{PG}(4,2)$. For an $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ in each of these classes we consider the solid $D = \langle \mathcal{C}_4\rangle$ in $\mathrm{PG}(9,2)$, where $\mathcal{C}_4 = \{m_1, m_2, m_3, m_4\} \subset \mathrm{Rk}_2$ denotes the Grassmann image of $\mathcal{S}_4$. We aim to determine the associate $D^{\#} = \langle t_D\rangle^{\mathrm{O}}$ of $D$. To this end we make use of lemma 3.1 in conjunction with the following result.

**Lemma 3.5** *Consider the associate $D^{\#} = \langle t_D\rangle^{O}$ of a solid in* $\mathrm{PG}(9,2)$ *which is of the form $D = \langle \mathcal{C}_4\rangle$ just described. Then each line $\mu_i \in \mathcal{S}_4$ is isotropic for the alternating bilinear form $t_D(\cdot,\cdot)$, see eq. (A.9). In particular, if $t_D \in \mathrm{Rk}_2^*$, and so is the Grassmann image of a plane $\alpha_D \subset$* $\mathrm{PG}(4,2)$, *then $\alpha_D$ meets each line $\mu_i \in \mathcal{S}_4$.*

**Proof.** Since, lemma 2.3(iii), $D \subseteq D^{\#}$ these results follow immediately from theorem 3.4. ■

Let us give a brief description of the four classes of $\mathcal{S}_4$. If $\mathcal{S}_4$ is regulus-free then, see [7, Table B.2], it is either of class IVa.1 or of class IVb.1. If of class IVa.1, with stabilizer $\mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(4)$, then it possesses a unique extension to a spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\mu_5\}$ on a parabolic quadric $\mathcal{P}_4$. Since $m_5 = \Sigma_{i=1}^4 m_i$, note that $D = \langle \mathcal{C}_4\rangle$ can be expressed $D = \langle \mathcal{C}_5\rangle = \mathcal{E}(\mathcal{C}_5)$. So $D$ is of type $(5,10)$, and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_5} \cong \mathrm{Sym}(5)$. The stabilizer $\mathcal{G}_D$ possesses a unique fixed point $p_0 \in \mathrm{PG}(4,2)$, namely the nucleus of $\mathcal{P}_4$.

If $\mathcal{S}_4$ is of class IVb.1, then $D = \langle \mathcal{C}_4\rangle$ is of type $(4,11)$, and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(3)$. The partial spread $\mathcal{S}_4$ has a unique decomposition $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3\} \cup \{\mu_4\}$ such that $\mathcal{G}_{\mathcal{S}_4}$ effects all permutations of $\mu_1, \mu_2, \mu_3$ but leaves $\mu_4$ invariant. If the transversal of $\{\mu_1, \mu_2, \mu_3\}$ meets $\mu_i$ in $c_i$, $i = 1,2,3$ then we have $\mu_i = \langle a_i, c_i\rangle$ where the points $a_i \in \mu_i$, $i = 1,2,3$, are uniquely determined by the requirement that the invariant line $\mu_4$ is of the form $\{a_2 + a_3, a_3 + a_1, a_1 + a_2\}$. Then $\mathcal{G}_D = \langle A, J\rangle$ where $A$ effects the permutation $(a_1a_2a_3)(c_1c_2c_3)$ and $J$

effects $(a_1a_2)(c_1c_2)(a_3)(c_3)$. Observe that $\mathcal{G}_D$ possesses a unique fixed point $p_0 \in \mathrm{PG}(4,2)$, namely $p_0 = a_1 + a_2 + a_3$.

There are just two other classes, IVc.1 and IVd.1, of partial spread $\mathcal{S}_4$. A partial spread $\mathcal{S}_4$ of class IVc.1 consists of a regulus $\rho = \{\mu_1, \mu_2, \mu_3\}$ together with a line $\mu_4$ not lying in the ambient solid $\sigma$ of $\rho$. In this case $D = \langle \mathcal{C}_4 \rangle$ is of type $(4, 11)$, and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}(3) \times Z_2$. The stabilizer $\mathcal{G}_D$ has a unique fixed point $p_0 \in \mathrm{PG}(4,2)$, namely the point where $\mu_4$ meets $\sigma$.

Finally a partial spread $\mathcal{S}_4$ of class IVd.1 is of the form $\mathcal{S}_5 \diagdown \{\mu_5\}$ where $\mathcal{S}_5$ is a spread for some solid $\sigma \subset \mathrm{PG}(4,2)$. Since $m_5 = \Sigma_{i=1}^4 m_i$, note that $D := \langle \mathcal{C}_4 \rangle$ can be expressed $D = \langle \mathcal{C}_5 \rangle = \mathcal{E}(\mathcal{C}_5)$. So $D$ is of type $(5, 10)$, and $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_5} \cong 2^4 : \Gamma\,\mathrm{L}(2,4)$.

**Theorem 3.6** *Let* $\mathcal{S}_4$ *be of class IVa.1, and let* $\mathcal{P}_4$ *be that parabolic quadric in* $\mathrm{PG}(4,2)$ *such that* $\mathcal{S}_4$ *extends to a spread* $\mathcal{S}_5$ *on* $\mathcal{P}_4$*. Then* $\langle \mathcal{C}_4 \rangle^{\#}$ *is the hyperplane* $H(t)$*, where* $t(\cdot,\cdot)$ *is that alternating bilinear form (whose kernel is the nucleus of* $\mathcal{P}_4$*) which is obtained by polarizing the quadratic form of* $\mathcal{P}_4$*.*

**Proof.** The possibility $t_D = 0$, that is $D^{\#} = \mathrm{PG}(9,2)$, is quickly ruled out. For example if $\mu$ is any bisecant of $\mathcal{P}_4$ then one checks that the 4-flat $\langle m, D \rangle$ is of even type $(10, 21)$, whence $m \notin D^{\#}$. But in its action upon $\wedge^3 V_5$ the stabilizer $\mathcal{G}_D$ has a unique nonzero fixed point, namely $t$. Hence $t_D = t$. ■

**Theorem 3.7** *Let* $\mathcal{S}_4$*, of class IVb.1, be* $\{\mu_1, \mu_2, \mu_3\} \cup \{\mu_4\}$ *as in the preamble. Then* $D = \langle \mathcal{C}_4 \rangle$ *has for its associate* $D^{\#}$ *the hyperplane* $H(t_D)$, $t_D = m_{123} \wedge p_0$*, where* $p_0$ *is the fixed point of* $\mathcal{G}_{\mathcal{S}_4}$ *and* $m_{123} = m_1 + m_2 + m_3$*.*

**Proof.** The kernel of the alternating bilinear form $t(\cdot,\cdot)$ is $\mathcal{G}_D$-invariant, and so must contain the unique fixed point $p_0$ of $\mathcal{G}_D \cong \mathrm{Sym}(3)$. Moreover, lemma 3.5, each line $\mu_i$ of $\mathcal{S}_4$ must be isotropic. Upon imposing these conditions we see that $t_D$ is forced to be of the form

$$t_D = \kappa(m_{123} \wedge p_0) + \kappa' a_1 \wedge a_2 \wedge a_3, \qquad \kappa, \kappa' \in \mathrm{GF}(2). \tag{3.3}$$

Now $t_D \neq 0$, since for the Grassmann image $l$ of the line $\lambda = \langle a_1 + a_2, a_1 + c_1 \rangle$ the 4-flat $\langle l, D \rangle$ is seen to be of even type $(8, 23)$. On the other hand the Grassmann image $m = c_1 \wedge c_2$ of the transversal $\mu = \{c_1, c_2, c_3\}$ of $\{\mu_1, \mu_2, \mu_3\}$ is such that the 4-flat $\langle m, D \rangle$ is of odd type $(9, 22)$; hence $m \in H(t_D)$, and so $t_D \wedge m = 0$. But from (3.3) we see that $t_D \wedge m = \kappa' e$, whence $\kappa' = 0$, and the theorem follows. ■

**Theorem 3.8** *Let* $\mathcal{S}_4 = \rho \cup \{\mu_4\}$ *be of class IVc.1. Let* $\Sigma_5 = \rho \cup \{\xi, \eta\}$ *be the extension of the regulus* $\rho$ *to a spread* $\Sigma_5$ *for the ambient solid* $\sigma$ *of* $\rho$*, and suppose that* $\mu_4$ *meets* $\sigma$ *in the point* $p_0 \in \xi$*. Then* $D = \langle \mathcal{C}_4 \rangle$ *has for its associate* $D^{\#}$ *the hyperplane* $H(t)$ *where* $t$ *is the Grassmann image of the plane* $\langle p_0, \eta \rangle$*.*

**Proof.** Arguing as in the proof of theorem 3.7 we find that $t_D$ is of the form $t_D = \kappa t + \kappa' t' + \kappa'' t''$. Upon imposing the condition that $t_D$ is $\mathcal{G}$-invariant we find that $\kappa' = \kappa'' = 0$, and so either $t_D = t$ as in the theorem, or else $t_D = 0$. But this last possibility, that $D^{\#} = \mathrm{PG}(9,2)$, does not hold since not all 4-flats through $D$ are odd. ■

**Theorem 3.9** *If* $\mathcal{S}_4$ *is of class IVd.1 then* $D = \langle \mathcal{C}_4 \rangle$ *has for its associate* $D^{\#}$ *the whole of* $\mathrm{PG}(9,2)$*.*

**Proof.** Possibility (iii) in lemma 3.1 does not hold since no plane in PG$(4,2)$ is stabilized by $\mathcal{G}_D$. Moreover no point of PG$(4,2)$ is fixed by $\mathcal{G}_D$. (Note in particular that $\mathcal{G}_D$ contains a subgroup $\cong (Z_2)^4$ of transvections which acts transitively on the 16 points of $\sigma^c$.) Consequently possibility (iv) in lemma 3.1 also does not hold. Hence $D^\# = \mathrm{PG}(9,2)$. ■

The theorem may also be proved using incidence considerations. Thus if $m$ is the image of any line $\mu$ which meets $\sigma$ in a point, then one easily sees that $\langle m, D\rangle$ is of odd type $(7,24)$, and so $D^\#$ contains all such $m$, whence $D^\# = \mathrm{PG}(9,2)$. Actually the theorem is just a special case of a more general result: see theorem 5.4 below.

## 3.3 The associate $D^\#$ of a 3-flat of the form $D = \mathcal{E}(\mathcal{C}_5)$

We may also construct 3-flats in PG$(9,2)$ from partial spreads of size 5 in PG$(4,2)$. Consider a solid $D$ of the form $D = \mathcal{E}(\mathcal{C}_5)$, where $\mathcal{C}_5 = \{m_i\}_{1\le i\le 5} \subset \mathrm{Rk}_2$ is the Grassmann image of a partial spread $\mathcal{S}_5 = \{\mu_i\}_{1\le i\le 5}$. Since $D = \langle m_{15}, m_{25}, m_{35}, m_{45}\rangle$ its associated linear form is $f_D$, where

$$f_D(x) = Q^\ddagger(m_{15}, m_{25}, m_{35}, m_{45}, x). \tag{3.4}$$

Setting $\mathcal{C}(i) := \mathcal{C}_5 \diagdown \{m_i\}$ and $D_i = \langle \mathcal{C}(i)\rangle$, let $t_i := t_{D_i}$ be the associated trivector of the solid $D_i$.

**Theorem 3.10** *(i) The trivector of the solid $D = \mathcal{E}(\mathcal{C}_5)$ is $t_D = \Sigma_{i=1}^5 t_i$.*

*(ii) With respect to the bilinear form $t_D(\cdot,\cdot)$, see eq. (A.9), there are just two possibilities for the five lines $\mu_i$ of $\mathcal{S}_5$. Either*
*(a) each line $\mu_i$ is isotropic: $t_D \wedge m_i = 0$, $i = 1, \dots, 5$, or*
*(b) each line $\mu_i$ is non-isotropic: $t_D \wedge m_i = e$, $i = 1, \dots, 5$.*

**Proof.** (i) Upon expanding $f_D(x)$ in (3.4) we obtain $f_D = \Sigma_{i=1}^5 f_{D_i}$.
(ii) By lemma 2.3(iii), $m_{ij} \in D^\# = \langle t_D\rangle^{\mathrm{O}}$, and so $t_D \wedge (m_i + m_j) = 0$. ■

### 3.3.1 Example 1: $D = \mathcal{E}(\mathcal{C}_5)$, $\mathcal{S}_5$ of class Ve.1

Consider the solid $D = \mathcal{E}(\mathcal{C}_5)$ with $\mathcal{C}_5 = \{m_1, m_2, m_3, m_4, m_5\}$ the image of the partial spread $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are

$$\{1, 24, 124\},\ \{2, 35, 235\},\ \{3, 41, 341\},\ \{4, 52, 452\},\ \{5, 13, 513\}. \tag{3.5}$$

(Here we adopt an abbreviated notation, using $i$ for the basis vector $e_i$ and writing $24 = e_2 + e_4$, $124 = e_1 + e_2 + e_4$, etc.; we also write $u = 12345$.) Upon observing that the profile, see [7, Section 1.1], of $\mathcal{S}_5$ is $(2,3,4)^5$, it follows, see [7, Table B.2], that $\mathcal{S}_5$ is of class Ve.1. Consequently, see [14, Theorem 3.3], $D$ is an external solid $\in$ orb$(3\alpha)$. It has stabilizer $\mathcal{G}_D = \mathcal{G}_{\mathcal{S}_5} = \langle B\rangle \cong Z_5$, where $B \in \mathrm{GL}(5,2)$ effects the permutation (12345) of the basis vectors and has $u = 12345$ for its unique fixed point. Note that $B$ effects the cyclic permutation $(\mu_1\mu_2\mu_3\mu_4\mu_5)$ of the lines of $\mathcal{S}_5$.

In the notation of theorem 3.10 we have $t_D = t_1 + t_2 + t_3 + t_4 + t_5$. Using theorem 3.7 we find $t_5$ to be $m_{234} \wedge n$ where (in abbreviated notation) $n = 12$ and $m_{234} = 2 \wedge 35 + 3 \wedge 41 + 4 \wedge 52$. By use of the $Z_5$ symmetry we find that

$$t_D = \Sigma_{i=1}^5 t_i = \Sigma_{1\le i<j<k\le 5} e_i \wedge e_j \wedge e_k. \tag{3.6}$$

**Theorem 3.11** *Let* $\mathcal{C}_5 = \{m_1, m_2, m_3, m_4, m_5\}$ *be of class Ve.1, and so* $D = \mathcal{E}(\mathcal{C}_5)$ *belongs to* $\mathrm{orb}(3\alpha)$*. Let* $u \in \mathrm{PG}(4,2)$ *be the unique fixed point of the stabilizer* $\mathcal{G}_D \cong Z_5$*. Then the associate of the solid* $D$ *is the hyperplane*

$$D^{\#} = H(t_D) \quad \textit{where } t_D = u \wedge m_\Sigma. \tag{3.7}$$

**Proof.** For $\mathcal{S}_5$ in (3.5) we have $m_\Sigma := \Sigma_{i=1}^5 m_i = \Sigma_{1\le i<j\le 5} e_i \wedge e_j$ and so $u \wedge m_\Sigma$ equals $t_D$ in (3.6). ■

**Remark 3.12** *See Appendix B.3 for information concerning the orbits of* $T_B$ *upon* $\mathrm{PG}(9,2)$*. There are precisely three fixed points, one being* $\Sigma_{1\le i<j\le 5} e_i \wedge e_j$*.*

### 3.3.2 Example 2: $D = \mathcal{E}(\mathcal{C}_5)$, $\mathcal{S}_5$ of class Vd.1

Let $\langle C\rangle \cong Z_3$ be the stabilizer $\mathcal{G}_{\mathcal{S}_5}$ of a partial spread $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ of class Vd.1, and let $p_0$ be the unique fixed point of $C$. Let the lines of $\mathcal{S}_5$ be labelled so that $C$ stabilizes $\mu_4$ and $\mu_5$ and effects the permutation $(\mu_1\mu_2\mu_3)$; see theorem C.2. Let $\mathcal{S}_8 = \{\mu_1, \dots, \mu_8\}$ be the unique extension of $\mathcal{S}_5$ to a regulus-free partial spread $\mathcal{S}_8$; see theorem C.1(vi). Then $C$ cyclically permutes the lines $\mu_6, \mu_7, \mu_8$. The Grassmann images $t_{123}, t_{678} \in \mathcal{G}_{1,4,2}$ of the transversals $\tau_{123}$, $\tau_{678}$ are then fixed points of $T_C$. Also the image $t_\pi$ of the invariant plane $\pi := (\mu_1 \cup \dots \cup \mu_8)^c$ of $\mathcal{S}_8$ is a fixed point of $\hat{T}_C$.

**Theorem 3.13** *If* $D = \mathcal{E}(\mathcal{C}_5)$*, where* $\mathcal{S}_5$ *is of class Vd.1, then*

$$D^{\#} = H(t_D) \quad \textit{where } t_D = t_\pi + p_0 \wedge (t_{123} + t_{678}). \tag{3.8}$$

**Proof.** As in theorem 3.10, $t_D = t_1 + t_2 + t_3 + t_4 + t_5$. Choosing a particular $\mathcal{S}_5$ of class Vd.1 it is a straightforward, if somewhat lengthy, task to use theorem 3.7 to compute $t_1, t_2, t_3$, and also one of $t_4, t_5$, and to use theorem 3.6 to compute the other one of $t_4, t_5$, and thereby to confirm the result (3.8). ■

# 4 The associate $X^{\#}$ of a 4-flat $X \subset \mathrm{PG}(9,2)$

## 4.1 Some general results for 4-flats

Given a 4-flat $X \subset \mathrm{PG}(9,2)$ let $\mathcal{B} = \{a_1, a_2, a_3, a_4, a_5\}$ be *any* choice of points which generate $X$, and consider the five independent solids $D_i = \langle \mathcal{B}_i \rangle \subset X$, where $\mathcal{B}_i := \mathcal{B} \diagdown \{a_i\}$, $i = 1, 2, 3, 4, 5$. Let $t_i := t_{D_i}$ be the associated trivector of $D_i$ and consider the five "hyperplanes" $H_i := H(t_i) = \langle t_i \rangle^{\mathrm{O}}$. (Recall, see remark 3.2, that $H_i = \mathrm{PG}(9,2)$ if $t_i = 0$.)

**Theorem 4.1** *Given a 4-flat $X = \langle a_1, a_2, a_3, a_4, a_5 \rangle$ in* $\mathrm{PG}(9,2)$, *let the notation be as in the preamble. Then*

$$X^{\#} = T^{O}, \qquad \textit{where } T := \langle t_1, t_2, t_3, t_4, t_5 \rangle \subset \mathrm{PG}(9,2)^{*}. \tag{4.1}$$

**Proof.** Since $Q^{\ddagger}$ is alternating and quinquelinear the 4-flat $X^{\#}$ is the intersection $\cap_{i=1}^{5} \langle t_i \rangle^{\mathrm{O}} = \langle t_1, t_2, t_3, t_4, t_5 \rangle^{\mathrm{O}}$ of the five "hyperplanes" $H_i$. ■

**Corollary 4.2** *If $X$ is an even 4-flat then $X^{\#}$ is a disjoint 4-flat.*

**Proof.** By the theorem, $X^{\#}$ is the intersection of at most five independent hyperplanes; hence $X^{\#}$ is an $s$-flat for some $s \geq 4$. But if $X$ is even then, see corollary 2.6(ii), $X \cap X^{\#} = \emptyset$, and so $s \leq 4$. Hence $s = 4$. ■

**Remark 4.3** *Some results which we enunciate for the case of the quintic Grassmannian $\mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$ do have obvious generalizations, valid for the case of a general subset $\psi \subset \mathrm{PG}(n,2)$ with equation $Q(x) = 0$ with $\deg Q = d$. For example, lemma 3.1(i) generalizes to:* if $D \subset \mathrm{PG}(n,2)$ is a $(d-2)$ flat then its $\psi$-associate $D^{\#}$ is a hyperplane if its associated linear form $f_D$ is nonzero, or the whole of $\mathrm{PG}(n,2)$ if $f_D$ is zero.

*As another example, theorem 4.1 generalizes to:* if $X \subset \mathrm{PG}(n,2)$ is a $(d-1)$ flat then its $\psi$-associate $X^{\#}$ is the intersection of at most $d$ independent hyperplanes — *leading to the following generalization of corollary 4.2:* if $X$ is an even $(d-1)$-flat then $X^{\#}$ is a disjoint $(n-d)$-flat.

Next we consider 4-flats $X$ of the form $X = \mathcal{E}(\mathcal{C}_6)$ where $\mathcal{C}_6 = \{m_i\}_{1 \leq i \leq 6} \subset \mathrm{Rk}_2$ is the Grassmann image of a partial spread $\mathcal{S}_6 = \{\mu_i\}_{1 \leq i \leq 6}$. Setting $\mathcal{C}_5(i) = \mathcal{C}_6 \diagdown \{m_i\}$ and $\mathcal{C}_4(ij) = \mathcal{C}_6 \diagdown \{m_i, m_j\}$ then we may define, for $1 \leq i \neq j \leq 6$, solids $D_i := \mathcal{E}(\mathcal{C}_5(i))$ and $D_{ij} := \langle \mathcal{C}_4(ij) \rangle$. Let the associated trivectors of these solids be $t_i := t_{D_i}$ and $t_{ij} := t_{D_{ij}}$. In the next theorem we choose $m_6$ as preferred element of $\mathcal{C}_6$.

**Theorem 4.4** *The associate of the 4-flat $X = \mathcal{E}(\mathcal{C}_6)$ in the preamble is*

$$X^{\#} = T^{O}, \qquad \textit{where } T := \langle t_1, t_2, t_3, t_4, t_5 \rangle \subset \mathrm{PG}(9,2)^{*}. \tag{4.2}$$

*Moreover* $t_1 = \Sigma_{j \neq 1} t_{1j}$, ... , $t_5 = \Sigma_{j \neq 5} t_{5j}$.

**Proof.** Since $X = \langle m_{16}, m_{26}, m_{36}, m_{46}, m_{56} \rangle$ the result (4.2) follows immediately from that in theorem 4.1. Further, from theorem 3.10, we obtain the stated expressions for $t_1, \ldots, t_5$. ■

## 4.2 Some results for particular 4-flats

Let us illustrate the use of theorems 3.6 - 3.10 and theorems 4.1, 4.4.

### 4.2.1 The associate of $X = \langle \mathcal{C}_5 \rangle$, $\mathcal{S}_5$ of class Vi.1

Suppose $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \lambda\}$ where $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ is a spread for a solid $\sigma \subset \mathrm{PG}(4,2)$ and where $\lambda$ meets $\sigma$ in the point $a \in \mu_5$. We wish to determine $X^{\#}$ for $X = \langle \mathcal{C}_5 \rangle$, where $\mathcal{C}_5 = \{m_1, m_2, m_3, m_4, l\}$ is the Grassmann image of $\mathcal{S}_5$. Set $\mathcal{C}(i) = \mathcal{C}_5 \diagdown \{\mu_i\}$ for $i = 1, 2, 3, 4$, and $\mathcal{C}(5) = \mathcal{C}_5 \diagdown \{\lambda\}$. Then $X^{\#}$ is the intersection of the five flats $H_i := \langle \mathcal{C}(i) \rangle^{\#}$. Now, by theorem 3.9, $H_5 = \mathrm{PG}(9,2)$, and, by theorem 3.8, $H_i$ for $i = 1, 2, 3, 4$ is the hyperplane $H(t_i)$ where $t_i$ is the Grassmann image of the plane $\langle a, \mu_i \rangle$.

**Theorem 4.5** *If $X = \langle \mathcal{C}_5 \rangle$, where $\mathcal{S}_5$ is of class Vi.1, then $X^{\#}$ is a 6-flat which contains $X$ and which meets* $\mathrm{Rk}_2$ *in 43 points.*

**Proof.** Since $X$ is odd then $X \subset X^{\#}$. By the lead-in, $X^{\#} = H_1 \cap H_2 \cap H_3 \cap H_4$. Now, for $i = 1, 2, 3, 4$, $H_i = \langle t_i \rangle^{\mathrm{O}}$ where $t_i := a \wedge m_i$ is the trivector corresponding to the plane $\alpha_i = \langle a, \mu_i \rangle$. Observe that these four trivectors satisfy the one linear relation $\Sigma_{i=1}^{4} t_i = 0$. [*Proof:* since $m_1 + m_2 + m_3 + m_4 = m_5$ it follows that $\Sigma_{i=1}^{4} t_i = a \wedge m_5$, which last is 0 since $a \in \mu_5$.] So $X^{\#}$ is the intersection of three independent hyperplanes and hence is a 6-flat. The lines in $\mathrm{PG}(4,2)$ which meet all four of the planes $\alpha_i$ are precisely the following: (i) the 35 lines of $\sigma$, and (ii) the 8 lines which meet $\sigma$ in the point $a$. It follows that $|X^{\#} \cap \mathrm{Rk}_2| = 35 + 8 = 43$. ■

### 4.2.2 The associate of $X = \langle \mathcal{C}_5 \rangle$, $\mathcal{S}_5$ of class Ve.1

**Theorem 4.6** *If $X = \langle \mathcal{C}_5 \rangle$, $\mathcal{S}_5 \in$ class Ve.1, then $X$ is self-associate: $X^{\#} = X$.*

**Proof.** Let $\mathcal{S}_5$ be as in section 3.3, Example 1. Then the associate of the 4-flat $X = \langle \mathcal{C}_5 \rangle$ is $X^{\#} = \langle t_1, t_2, t_3, t_4, t_5 \rangle^{\mathrm{O}}$, see Theorem 4.1. By the $Z_5$ symmetry, the only possibility for the $t_i$ to be dependent is that they satisfy $\Sigma_i t_i = 0$. This is not so, see eq. (3.6). It follows that $X^{\#}$ is a 4-flat. But we have $X \subset X^{\#}$ since $X$ is odd. Hence $X^{\#} = X$. ■

### 4.2.3 The associate $X^{\#}$ of a 4-flat of the form $X = \mathcal{E}(\mathcal{C}_6)$

By making use of the general results of theorems 3.10 and 4.4, in conjunction with the particular results in section 3.2, the determination of the associate $X^{\#}$ of 4-flats of the form $X = \mathcal{E}(\mathcal{C}_6)$ would appear to be a reasonably straightforward task. However it is certainly not a quick task. Moreover in the cases when $X = \mathcal{E}(\mathcal{C}_6)$ is an even 4-flat we would also like to determine $X^{\#\#} := (X^{\#})^{\#}$, $X^{\#\#\#} := (X^{\#\#})^{\#}$, ... . But it may prove difficult to pin down the 4-flat $X^{\#}$; furthermore $X^{\#}$ may not be of the form $\mathcal{E}(\mathcal{C}_6)$, and so theorems 3.10 and 4.4 may not be of help in determining $X^{\#\#}$. Consequently arriving at a reasonable understanding of such a sequence $X$, $X^{\#}$, $X^{\#\#}$, $X^{\#\#\#}$, ... is in general quite a tough undertaking.

In the case where $\mathcal{S}_6$ is a partial spread of class VIc.2, and so, Table 2 of section 2.3, $X = \mathcal{E}(\mathcal{C}_6)$ is an even 4-flat, of type $(6, 25)$, we did succeed — after a considerable struggle! — in unravelling the intricacies of the sequence

$X$, $X^{\#}$, $X^{\#\#}$, ... . See section 6 below for a description of the results of our investigation, noting in particular the result $X^{\#\#\#} = X^{\#}$.

We were also successful in the case where $\mathcal{S}_6$ is a partial spread of class VIf.1, and so, Table 2 of section 2.3, $X = \mathcal{E}(\mathcal{C}_6)$ is an even 4-flat, of type $(2, 29)$. See section 7 below for a description of our results, noting in particular the result $X^{\#\#\#} = X$.

## 4.3 The associate of an external 4-flat

The 4-flats $X$ considered in this section are *not* of the form $X = \langle \mathcal{C}_5 \rangle$ or $X = \mathcal{E}(\mathcal{C}_6)$, and we will use quite different methods from those in the preceding section to determine their associates.

### 4.3.1 Using the $N(Z_{31})$-construction of external 4-flats

Given any subgroup $\mathcal{Z} \cong Z_{31}$ of $\mathrm{GL}(5, 2)$, let $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ be its normalizer. Consider the representation $T$ of $\mathcal{N}$ upon $V_{10} = \wedge^2 V_5$ given by $T_C = \wedge^2 C$, $C \in \mathcal{N}$. Since $2 \nmid |\mathcal{N}|$, Maschke's theorem applies, and so the representation $T$ is completely reducible. Now no one-dimensional subspace of $V_{10}$ is $\mathcal{N}$-invariant, and only for $n \geq 5$ does $\mathrm{GL}(n, 2)$ possess a subgroup $\cong Z_{31}$. Hence the 10-dimensional representation $T$ of $\mathcal{N}$ decomposes as the direct sum $T_+ \oplus T_-$ of two 5-dimensional irreducible representations $T_+$ and $T_-$, arising from some $\mathcal{N}$-invariant direct sum decomposition $V_+ \oplus V_-$ of $V_{10}$.

**Theorem 4.7** *Given such a direct sum decomposition $V_+ \oplus V_-$ of $V_{10}$, consider the corresponding 4-flats $X_+ = \mathbb{P}V_+$ and $X_- = \mathbb{P}V_-$.*

*(i) The 4-flats $X_+$, $X_-$ are both external.*

*(ii) $\mathcal{G}_{X_+} = \mathcal{G}_{X_-} = \mathcal{N}$.*

*(iii) $X_+$ and $X_-$ belong to different $\mathrm{GL}(5, 2)$-orbits, say $\mathrm{orb}(4+)$ and $\mathrm{orb}(4-)$.*

*(iv) There are no other orbits of external 4-flats.*

*(v) The stabilizer $\mathcal{G}_X$ of any external 4-flat $X$ is the normalizer $N(\mathcal{Z}) = \mathcal{N} \cong Z_{31} \rtimes Z_5$ of some Singer cyclic subgroup $\mathcal{Z} \cong Z_{31}$ of $\mathrm{GL}(5, 2)$.*

*(vi) The 5-dimensional irreducible representations $T_+$, $T_-$ of $\mathcal{N}$, see the preamble to the theorem, are inequivalent, and the decomposition $V_{10} = V_+ \oplus V_-$ of $V_{10}$ into irreducible $\mathcal{N}$-spaces is unique.*

**Proof.** (i) This is so because $\mathrm{Rk}_2$ is a single $T_{\mathcal{N}}$-orbit.

(ii) The stabilizers $\mathcal{G}_{X_+}, \mathcal{G}_{X_-}$ are proper subgroups of $\mathrm{GL}(5, 2)$ which contain $\mathcal{N}$; but $\mathcal{N}$ is a maximal subgroup of $\mathrm{GL}(5, 2)$.

(iii) Suppose that there exists $P \in \mathrm{GL}(5, 2)$ such that $T_P$ maps $X_+$ onto $X_-$. It follows that $P\mathcal{G}_{X_+}P^{-1} = \mathcal{G}_{X_-}$, that is $P\mathcal{N}P^{-1} = \mathcal{N}$. But $\mathcal{N}$, being maximal, is its own normalizer, whence $P \in \mathcal{N}$; so $T_P$ stabilizes $X_+$, contradicting the supposition that $T_P$ maps $X_+$ onto $X_-$.

(iv) See [14, Theorem 4.6].

(v) This follows immediately from (ii), (iv).

(vi) This is so, for otherwise the decomposition $V_+ \oplus V_-$ of $V_{10}$ would not be unique, resulting in the existence of more than 2 orbits of external 4-flats. ■

**Theorem 4.8** *If $X$ is any external 4-flat in $\mathrm{PG}(9, 2)$ then its associate $X^{\#}$ is also an external 4-flat, with $X$ and $X^{\#}$ belonging to different $\mathrm{GL}(5, 2)$-orbits. Moreover $(X^{\#})^{\#} = X$.*

**Proof.** Since $X$ is an even 4-flat, and since $X^{\#}$ is stabilized by $\mathcal{G}_X \cong Z_{31} \rtimes Z_5$, it follows that $X^{\#}$ is an external 4-flat which is disjoint from $X$. The theorem now follows from the uniqueness of the decomposition $V_{10} = V_+ \oplus V_-$ in theorem 4.7. ■

### 4.3.2 Another construction of $X^{\#}$

With the aid of the next theorem we are able to give, see theorem 4.10, another construction of $X^{\#}$.

**Theorem 4.9** *(See [14, Section 4.1]) Let $X$ be any external 4-flat and let $F$ be any 5-flat which contains $X$. Put $\mathcal{C}_F = F \cap \mathrm{Rk}_2$, with $\mathcal{S}_F$ the corresponding line-set in* PG(4, 2). *Then*

*(i) $\mathcal{C}_F$ is a cap, and $\mathcal{S}_F$ is a regulus-free partial spread;*
*(ii) if $F \neq F'$ then $\mathcal{C}_F \cap \mathcal{C}_{F'} = \emptyset$;*
*(iii) $\{\mathcal{S}_F \,|\, F$ a 5-flat $\supset X\}$ is a partition of the 155 lines of* PG(4, 2)*;*
*(iv) for each 5-flat $F \supset X$, $|\mathcal{C}_F| = 5$;*
*(v) $\mathcal{S}_F$ is of class Ve.1 and $\mathcal{G}_{\mathcal{C}_F} \cong Z_5$.*

Setting $y_F := \Sigma_{m \in \mathcal{C}_F} m$, we claim that $y_F \in X^{\#}$. To justify our claim we need to show that each 4-flat $\langle y_F, D\rangle$, where $D$ is a solid of $X$, meets $\mathcal{C}_F$ in an odd number of points. This is of course true if $D = \mathcal{E}(\mathcal{C}_F)$, when $\langle y_F, D\rangle = \langle \mathcal{C}_F\rangle$ contains all five points of $\mathcal{C}_F$. If $D$ is any other solid of $X$ then $\langle y_F, D\rangle$ will meet $\mathcal{E}(\mathcal{C}_F)$ in a plane $P$. By examining the possibilities for $P$ we find that $\langle y_F, P\rangle$ meets $\mathcal{C}_F$ in one or in three points. So all 31 points $y_F$ lie in $X^{\#}$. Since $X^{\#}$ is a 4-flat, we thus have the following result.

**Theorem 4.10** *Let $X$ be any external 4-flat, and let the notation be as in the preceding theorem. Then the associate $X^{\#}$ of $X$ is*

$$X^{\#} = \{y_F \,|\, F \text{ a 5-flat containing } X\}, \qquad \text{where } y_F := \textstyle\sum_{m \in \mathcal{C}_F} m. \tag{4.3}$$

**Remark 4.11** *Observe that the foregoing proof of the result (4.3) did not require a knowledge of the stabilizer group $\mathcal{G}_X \cong Z_{31} \rtimes Z_5$ of the external 4-flat $X$.*

## 5 Direct use of the quintic $Q$

The 155 internal points of the Grassmannian $\mathcal{G}_{1,4,2}$ are precisely those points $x \in \mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$ which satisfy a certain quintic equation $Q(x) = 0$. As was shown in [12], the polynomial function $Q(x)$ is, for $x = \Sigma_{1\le i<j\le 5}\, x_{ij} e_i \wedge e_j$, explicitly as follows:

$$\begin{aligned} Q(x) &= Q_2(x) + Q_3(x) + Q_4(x) + Q_5(x), \text{ where} \\ Q_2(x) &= \Sigma x_{ij}x_{kl} \qquad (15 \text{ terms}), \\ Q_3(x) &= \Sigma x_{ij}x_{jk}x_{lm} \qquad (30 \text{ terms}), \\ Q_4(x) &= \Sigma x_{ij}x_{jk}x_{ki}x_{lm} \qquad (10 \text{ terms}), \\ Q_5(x) &= \Sigma x_{ij}x_{jk}x_{kl}x_{lm}x_{mi} \qquad (12 \text{ terms}). \end{aligned} \tag{5.1}$$

Only the terms $Q_5$ in $Q$ contribute to the alternating quinquelinear form $Q^\ddagger$ obtained by completely polarizing $Q$. So $Q^\ddagger$ has $12 \times 5! = 1440$ terms. Explicitly

$$Q^\ddagger(a,b,c,d,y) = \sum\nolimits_{1\le i<j\le 5} f_{ij}(a,b,c,d)y_{ij} \tag{5.2}$$

where, for given $a,b,c,d$, the coefficient of $y_{ij}$ consists of the $4! \times 3! = 144$ terms

$$f_{ij}(a,b,c,d) = \sum u_{ir}v_{rs}w_{st}x_{tj}, \tag{5.3}$$

with $uvwx$ running through the 4! permutations of $abcd$ and $rst$ running through the 3! permutations of $\{\{1,2,3,4,5\}\diagdown\{i,j\}\}$. Consequently, as we pointed out in section 2.2, the determination of $X^\#$ directly from definition 2.2 will usually be an extremely daunting task, and in most cases it would appear sensible to attempt to determine $X^\#$ by appeal instead to the incidence properties of lemma 2.5 and its off-shoots.

Nevertheless there do exist cases where the associate can be simply determined by direct appeal to the explicit form (5.2), (5.3) of $Q^\ddagger$. Several of the examples which we now give involve Latin solids and Latin and Greek planes; see [10, Ch. 24]. Recall that the 7 lines of a plane $\alpha \subset \mathrm{PG}(4,2)$ map onto the 7 points of a 'Greek' plane $P(\alpha) \subset \mathcal{G}_{1,4,2}$, and the 15 lines of $\mathrm{PG}(4,2)$ which pass through the point $z$, forming let us say the *star* $\mathrm{st}(z)$, map onto the 15 points of a 'Latin' solid, say $\mathrm{St}(z) \subset \mathcal{G}_{1,4,2}$. We also denote by $\mathrm{St}(z,\sigma)$, $z \in \sigma$, the Latin plane which is the Grassmann image of the seven lines of the solid $\sigma \subset \mathrm{PG}(4,2)$ which pass through the point $z \in \sigma$.

**Theorem 5.1** *If $D$ is any 3-flat $\langle \mathrm{St}(z,\sigma), h\rangle$ which contains the Latin plane* $\mathrm{St}(z,\sigma)$*, then $D^\# = \mathrm{PG}(9,2)$. In particular* $\mathrm{St}(z)^\# = \mathrm{PG}(9,2)$.

**Proof.** For $z = e_1$ and $\sigma = \langle e_1,e_2,e_3,e_4\rangle$ we have $D := \langle \mathrm{St}(z,\sigma), h\rangle = \langle a,b,c,h\rangle$, where $a = e_1 \wedge e_2,\ b = e_1 \wedge e_3,\ c = e_1 \wedge e_4$. Consider the associated linear form $f_D(y) = Q^\ddagger(a,b,c,h,y)$ of $D$ as given in equations (5.2), (5.3). Because the only nonzero coordinates of $a,b,c$ are $a_{12}, b_{13}, c_{14}$, respectively, and since the product $a_{12}b_{13}c_{14}$ does not occur in any of the coefficients (5.3), it follows that each coefficient $f_{ij}(a,b,c,h)$ is zero. So $f_D$ is the zero form, whence, see lemma 3.1, $D^\# = \mathrm{PG}(9,2)$. ■

**Theorem 5.2** *If $X$ is any 4-flat $\langle \mathrm{St}(z), h\rangle$ which contains the Latin solid* $\mathrm{St}(z)$ *then $X^\# = \mathrm{PG}(9,2)$.*

**Proof.** For $z = e_1$ we have $X := \langle \mathrm{St}(z), h\rangle = \langle a, b, c, d, h\rangle$, where $a = e_1 \wedge e_2,\ b = e_1 \wedge e_3,\ c = e_1 \wedge e_4$ and $d = e_1 \wedge e_5$. By theorem 4.1 $X^{\#}$ is the intersection of five "hyperplanes". But by theorem 5.1 each of these "hyperplanes", such as $\langle a, b, c, h\rangle^{\#}$, is in fact the whole of $\mathrm{PG}(9, 2)$. $\blacksquare$

**Theorem 5.3** *If $D$ is any 3-flat $\langle P(\alpha), h\rangle$ which contains the Greek plane $P(\alpha)$ then* $D^{\#} = \mathrm{PG}(9, 2)$.

**Proof.** For $\alpha = \langle e_1, e_2, e_3\rangle$ we have $D := \langle P(\alpha), h\rangle = \langle a, b, c, h\rangle$ where $a = e_1 \wedge e_2,\ b = e_1 \wedge e_3,\ c = e_2 \wedge e_3$. Since the product $a_{12}b_{13}c_{23}$ does not occur in any of the coefficients $f_{ij}(a, b, c, h)$, it follows that $f_D$ is the zero form. $\blacksquare$

**Theorem 5.4** *Let $D = \langle a_1, a_2, a_3, a_4\rangle$ be a 3-flat in* $\mathrm{PG}(9, 2)$ *such that, for some fixed solid $\sigma \subset$* $\mathrm{PG}(4, 2)$, *the following holds for each $i = 1, 2, 3, 4$: if $a_i \in \mathrm{Rk}_2$ then $a_i$ is the image of a line of $\sigma$, while if $a_i \in \mathrm{Rk}_4$ then $\mathrm{im}\, a_i = \sigma$. Then* $D^{\#} = \mathrm{PG}(9, 2)$.

**Proof.** If $\sigma = \langle e_1, e_2, e_3, e_4\rangle$ then $D = \langle a, b, c, d\rangle$ where $a_{i5} = b_{i5} = c_{i5} = d_{i5} = 0,\ i = 1, 2, 3, 4$. Consequently each coefficient $f_{ij}(a, b, c, d)$ in eq. (5.3) is 0. So $f_D$ is the zero form. $\blacksquare$

Note that theorem 3.9 holds as a particular case of the present theorem.

# 6 Some even 4-flats exemplifying $X^{\#\#\#} = X^{\#}$

We are particularly interested in 4-flats $X$ which are even since, corollary 4.2, the associate $X^{\#}$ is necessarily also 4-flat, which is moreover disjoint from $X$. So, upon recalling Table 1 in section 2.3, our attention is drawn to flats of the form $X = \langle \mathcal{C}_5 \rangle$ where $\mathcal{S}_5$ is of class Vd.1 or class Vf.1, since they are of even type $(6, 25)$. However it turns out, see lemma 6.2 below, that consideration of such even 4-flats may be subsumed by considering flats $X = \langle \mathcal{C}_6 \rangle = \mathcal{E}(\mathcal{C}_6)$ where $\mathcal{S}_6$ is of class VIc.2. This is helpful, since thereby we may take advantage of the higher symmetry of the latter flats, namely Sym(3) as compared to $Z_3$ or $Z_2$.

## 6.1 Orientation and summary of results

Partial spreads $\mathcal{S}_6$ of class IVc.2 are unusual in at least one respect: the elements of the corresponding 6-cap $\mathcal{C}_6 \subset \mathcal{G}_{1,4,2}$ are not independent. This comes about because such a partial spread is of the form $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\mu_6\}$ with $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ of class Va.1. Now, alone amongst the non-maximal partial spreads $\mathcal{S}_5$ of size 5, those of class Va.1 (which are spreads on some parabolic quadric $\mathcal{P}_4$ in PG$(4, 2)$) have 5-caps $\mathcal{C}_5 = \{m_1, \ldots, m_5\}$ whose elements $m_i$ are dependent, satisfying $m_{12345} = 0$. Moreover every extension $\mathcal{S}_6$ of a partial spread $\mathcal{S}_5$ of class Va.1 is of class VIc.2. Consequently if $X = \mathcal{E}(\mathcal{C}_6)$, where $\mathcal{S}_6$ is of class VIc.2, then $X$ coincides with $\langle \mathcal{C}_6 \rangle$ and is even, of type $(6, 25)$; further $\mathcal{G}_X = \mathcal{G}_{\mathcal{S}_6} \cong$ Sym(3).

In the case of $\mathcal{S}_6 \in$ class VIc.2 our chief results may be summarized:

(i) $X := \langle \mathcal{C}_6 \rangle$ is of type $(6, 25)$ with stabilizer $\mathcal{G}_X \cong$ Sym(3);

(ii) $X^{\#}$ is of type $(4, 27)$ with stabilizer $\mathcal{G}_{X^{\#}} \cong (Z_3 \times Z_3).Z_2$;

(iii) $X^{\#\#}$ is of type $(4, 27)$ with stabilizer $\mathcal{G}_{X^{\#\#}} \cong (Z_3 \times Z_3).Z_2$;

(iv) $X^{\#\#\#} = X^{\#}$. (6.1)

*Notes. (i)* Notice the increase in symmetry, recall remark 2.8, on passing from $X$ to $X^{\#}$. The appearance of a group isomorphic to $(Z_3 \times Z_3).Z_2$ was unforeseen, but it goes along with the fact that there exists a $Z_3$-triplet $\{\mathcal{S}_6, \mathcal{S}'_6, \mathcal{S}''_6\}$ of partial spreads of class VIc.2 such that $\langle \mathcal{C}_6 \rangle^{\#} = \langle \mathcal{C}_6 \rangle^{\#} = \langle \mathcal{C}_6 \rangle^{\#}$; see theorem 6.7.
*(ii)* The tie-in with partial spreads $\mathcal{S}_5$ of class Vd.1 or class Vf.1, is as follows. If, as above, $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\mu_6\}$ with $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ of class Va.1, then $\mathcal{S}_5(i) := \mathcal{S}_6 \diagdown \{\mu_i\}$ is of class Vd.1, with stabilizer $\cong Z_3$, for two choices of $i \in \{1, 2, 3, 4, 5\}$; also $\mathcal{S}_5(i)$ is of class Vf.1, with stabilizer $\cong Z_2$, for three choices of $i$. Moreover $\langle \mathcal{C}_5(i) \rangle = \langle \mathcal{C}_6 \rangle$ for each $i = 1, 2, 3, 4, 5$. Consequently $\langle \mathcal{C}_5 \rangle^{\#\#\#} = \langle \mathcal{C}_5 \rangle^{\#}$ for the even entries in the middle row of Table 1, section 2.3.

## 6.2 Partial spreads $\mathcal{S}_6$ in PG$(4, 2)$ of class VIc.2

If a partial spread $\mathcal{S}_6$ in PG$(4, 2)$ is of type $\mathsf{I}$ (that is $\mathcal{S}_6$ contains precisely one regulus) then it has a unique decomposition $\mathcal{S}_6 = \rho \cup \mathcal{S}_3$, where $\rho = \{\rho_1, \rho_2, \rho_3\}$ is the regulus of $\mathcal{S}_6$ and $\mathcal{S}_3 = \{\lambda_1, \lambda_2, \lambda_3\}$ is a non-regulus. Let $\xi$ be the transversal of $\mathcal{S}_3$, and let $\sigma$ be the ambient PG$(3, 2)$ of $\rho$. As explained in [7, Section 7.3.1], under the action of GL$(5, 2)$ the partial spreads $\mathcal{S}_6$ in PG$(4, 2)$ which are of type I fall into seven distinct classes (orbits), and four of these, namely classes VIc.1, VIc.2, VId.1, VId.2, are such that $\xi$ lies inside $\sigma$. So if $\mathcal{H}_3$ denotes the hyperbolic quadric $\subset \sigma$ which has $\rho$ as one set of generators then, for these four classes, we have

$$\sigma = \xi \cup \eta \cup \mathcal{H}_3 \tag{6.2}$$

where $\eta$ denotes the fifth line of a spread $\Sigma_5 = \{\xi, \eta, \rho_1, \rho_2, \rho_3\}$ for $\sigma$. In each case the non-regulus $\mathcal{S}_3$ determines, see [23, Section 2.2.1], a symplectic polarity in the solid $\sigma$, and the foregoing four classes are distinguished, see [7, Section 7.3.1], as follows: $\mathcal{S}_6$ is allocated to class VIc.1, VIc.2, VId.1 or VId.2 according as the number of lines of $\rho$ which are selfpolar is $0, 2, 1$ or $3$, respectively.

We now wish to give an example of a partial spread $\mathcal{S}_6$ of class VIc.2. To this end, let us in (6.2) set $\xi = \{x_1, x_2, x_3\}$, with $x_i \in \lambda_i$, and $\eta = \{y_1, y_2, y_3\}$. Then

$$\mathcal{H}_3 = \begin{pmatrix} z_{23} & z_{31} & z_{12} \\ z_{32} & z_{13} & z_{21} \\ z_{11} & z_{22} & z_{33} \end{pmatrix}, \qquad \text{where } z_{ij} := x_i + y_j. \tag{6.3}$$

Let $\rho$ denote the regulus $\{\rho_1, \rho_2, \rho_3\}$ formed by the three rows of this array and let $\rho^{\mathrm{o}} = \rho^{\mathrm{opp}}$ denote the opposite regulus $\{\rho_1^{\mathrm{o}}, \rho_2^{\mathrm{o}}, \rho_3^{\mathrm{o}}\}$ formed by the three columns. Given some fixed choice of a point $p_0$ lying outside $\sigma$, let $\mathcal{S}_3$ be the non-regulus, with transversal $\xi$, defined by

$$\mathcal{S}_3 = \{\lambda_1, \lambda_2, \lambda_3\}, \qquad \text{where } \lambda_i := \langle x_i, p_0 + y_i \rangle. \tag{6.4}$$

With respect to the symplectic polarity induced in $\sigma$ by the hyperbasis $(\lambda_1 \cup \lambda_2 \cup \lambda_3) \diagdown \xi$ we see that precisely two, namely $\rho_1$ and $\rho_2$, of the lines of $\rho$ are self-polar. Hence the partial spread

$$\mathcal{S}_6 := \rho \cup \mathcal{S}_3 \tag{6.5}$$

is of class VIc.2. Let us define elements $C, C', J$ of $\mathrm{GL}(5,2)$ by requiring them to permute the points of $\xi$ and $\eta$ in the manner

$$C : (x_1 x_2 x_3)(y_1 y_2 y_3), \quad C' : (x_1 x_2 x_3)(y_1 y_3 y_2), \quad J : (x_2 x_3)(y_2 y_3), \tag{6.6}$$

and by requiring each of them to fix the point $p_0$. Then $C$ stabilizes each line of $\rho$ and effects the permutation $(\rho_1^{\mathrm{o}} \rho_2^{\mathrm{o}} \rho_3^{\mathrm{o}})$ of the lines of $\rho^{\mathrm{o}}$, while $C'$ stabilizes each line of $\rho^{\mathrm{o}}$ and effects the permutation $(\rho_1 \rho_2 \rho_3)$ of the lines of $\rho$; also the involution $J$ effects the permutations $(\rho_1 \rho_2)$ and $(\rho_2^{\mathrm{o}} \rho_3^{\mathrm{o}})$, and stabilizes both $\rho_3$ and $\rho_1^{\mathrm{o}}$. Moreover both $C$ and $J$ stabilize $\mathcal{S}_3$, with $C$ effecting the permutation $(\lambda_1 \lambda_2 \lambda_3)$ and $J$ effecting the permutation $(\lambda_2 \lambda_3)$. Since it is known, [7, Lemma 7.4], that the stabilizer $\mathcal{G}_{\mathcal{S}_6}$ of $\mathcal{S}_6$ is isomorphic to $\mathrm{Sym}(3)$ it follows that

$$\mathcal{G}_{\mathcal{S}_6} = \langle C, J \rangle \cong \mathrm{Sym}(3). \tag{6.7}$$

Observe that the $Z_3$ subgroup $\langle C \rangle$ of $\mathcal{G}_{\mathcal{S}_6}$ stabilizes every line of the spread $\Sigma_5 = \{\xi, \eta, \rho_1, \rho_2, \rho_3\}$ for $\sigma$, and has $p_0$ for its unique fixed point.

### 6.2.1 Triplets of $\mathcal{S}_6$ of class VIc.2

The three elements $C, C', J$ of $\mathrm{GL}(5,2)$ generate a subgroup

$$\mathcal{G}_0(p_0, \xi, \eta) = \langle C, C', J \rangle \cong (Z_3 \times Z_3).Z_2 \tag{6.8}$$

which consists precisely of all elements of $\mathrm{GL}(5,2)$ which (i) fix $p_0$ (ii) stabilize both $\xi$ and $\eta$, and (iii) stabilize both $\rho$ and $\rho^{\mathrm{o}}$. Note that $\mathcal{G}_0 := \mathcal{G}_0(p_0,\xi,\eta)$ is of index 2 in that group $\mathcal{G}(p_0,\xi,\eta) \cong \mathrm{GL}(2,2) \times \mathrm{GL}(2,2) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ which consists of all elements of $\mathrm{GL}(5,2)$ which fix $p_0$ and stabilize both $\xi$ and $\eta$. Element of $\mathcal{G}\diagdown\mathcal{G}_0$ effect the interchange $\rho \leftrightarrow \rho^{\mathrm{o}}$, one such element being that involution $J^*$ which fixes $p_0$, effects the interchange $y_2 \leftrightarrow y_3$ of two points of $\eta$, and keeps $\xi$ pointwise fixed. Of course $\mathcal{G} := \mathcal{G}(p_0,\xi,\eta)$ itself is of index 2 in a group $\bar{\mathcal{G}}$ isomorphic to the orthogonal group $\mathrm{O}(\mathcal{H}_3)$ of the hyperbolic quadric $\mathcal{H}_3$, with elements of $\bar{\mathcal{G}}$ fixing $p_0$ and stabilizing the pair $\{\xi,\eta\}$ of external lines of $\mathcal{H}_3$. Elements of $\bar{\mathcal{G}}\diagdown\mathcal{G}$ effect the interchange $\xi \leftrightarrow \eta$, one such element being the involution $K$ defined by $Kp_0 = p_0$, $Kx_i = y_i$ and $Ky_i = x_i$. Note that $K$ stabilizes $\rho$ and $\rho^{\mathrm{o}}$ separately, while the element $A = J^*K \in \bar{\mathcal{G}}$ effects both of the interchanges $\xi \leftrightarrow \eta$ and $\rho \leftrightarrow \rho^{\mathrm{o}}$. Summarizing, the elements $J^*, K, A(= J^*K)$ fix $p_0$ and effect the following permutations of the points of $\xi$ and $\eta$:

$$J^* : (y_2y_3), \quad K : (x_1y_1)(x_2y_2)(x_3y_3), \quad A : (x_1y_1)(x_2y_3x_3y_2). \tag{6.9}$$

Note that $A$ is of order 4 and satisfies $A^2 = J$.

Under the action of the $Z_3$ subgroup $\langle C'\rangle$ the non-regulus $\mathcal{S}_3$ is one of a triplet $\mathcal{S}_3, \mathcal{S}'_3, \mathcal{S}''_3$ of non-reguli:

$$\begin{aligned}
\mathcal{S}_3 &= \{\lambda_1,\lambda_2,\lambda_3\}, &&\text{where } \lambda_i := \langle x_i, p_0 + y_i\rangle,\\
\mathcal{S}'_3 &= \{\lambda'_1,\lambda'_2,\lambda'_3\}, &&\text{where } \lambda'_i := C'(\lambda_i) = \langle x_{i+1}, p_0 + y_{i-1}\rangle,\\
\mathcal{S}''_3 &= \{\lambda''_1,\lambda''_2,\lambda''_3\}, &&\text{where } \lambda''_i := (C')^2(\lambda_i) = \langle x_{i-1}, p_0 + y_{i+1}\rangle,
\end{aligned} \tag{6.10}$$

each of which has $\xi$ as its transversal. Since $C'$ preserves the regulus $\rho$, we see that $\mathcal{S}_6$ is one of a $\langle C'\rangle$-triplet

$$\mathcal{S}_6 = \mathcal{S}_3 \cup \rho, \quad \mathcal{S}'_6 = \mathcal{S}'_3 \cup \rho, \quad \mathcal{S}''_6 = \mathcal{S}''_3 \cup \rho \tag{6.11}$$

of partial spreads of class VIc.2, with respective stabilizers $\langle C, J\rangle$, $\langle C, J'\rangle$ and $\langle C, J''\rangle$, where $J' := C'J(C')^{-1}$ and $J'' := (C^{\prime)})^{-1}JC'$.

If the $\langle C'\rangle$-triplet $\mathcal{S}_3, \mathcal{S}'_3, \mathcal{S}''_3$ of non-reguli (6.10) are displayed as the rows of the following $3 \times 3$ array of lines

$$\begin{array}{cccc}
 & \mathcal{S}^*_3 & \mathcal{S}^{*\prime}_3 & \mathcal{S}^{*\prime\prime}_3 \\
\mathcal{S}_3 & \lambda_1 := \langle x_1, p_0 + y_1\rangle & \lambda_2 := \langle x_2, p_0 + y_2\rangle & \lambda_3 := \langle x_3, p_0 + y_3\rangle \\
\mathcal{S}'_3 & \lambda'_1 := \langle x_2, p_0 + y_3\rangle & \lambda'_2 := \langle x_3, p_0 + y_1\rangle & \lambda'_3 := \langle x_1, p_0 + y_2\rangle \\
\mathcal{S}''_3 & \lambda''_1 := \langle x_3, p_0 + y_2\rangle & \lambda''_2 := \langle x_1, p_0 + y_3\rangle & \lambda''_3 := \langle x_2, p_0 + y_1\rangle
\end{array} \tag{6.12}$$

then observe that the columns comprise a $\langle C\rangle$-triplet $\mathcal{S}^*_3, \mathcal{S}^{*\prime}_3, \mathcal{S}^{*\prime\prime}_3$ of non-reguli, all six non-reguli having $\xi$ as their common transversal. This $\langle C\rangle$-triplet $\mathcal{S}^*_3, \mathcal{S}^{*\prime}_3, \mathcal{S}^{*\prime\prime}_3$ of non-reguli will be needed in theorem 6.11 below. We shall also make use of the following $3 \times 3$ array of lines

$$\begin{array}{cccc}
 & \overline{\mathcal{S}}_3 & \overline{\mathcal{S}}'_3 & \overline{\mathcal{S}}''_3 \\
\widetilde{\mathcal{S}}_3 & \widetilde{\lambda}_1 := \langle y_1, p_0 + x_1\rangle & \widetilde{\lambda}_2 := \langle y_2, p_0 + x_2\rangle & \widetilde{\lambda}_3 := \langle y_3, p_0 + x_3\rangle \\
\widetilde{\mathcal{S}}'_3 & \widetilde{\lambda}'_1 := \langle y_3, p_0 + x_2\rangle & \widetilde{\lambda}'_2 := \langle y_1, p_0 + x_3\rangle & \widetilde{\lambda}'_3 := \langle y_2, p_0 + x_1\rangle \\
\widetilde{\mathcal{S}}''_3 & \widetilde{\lambda}''_1 := \langle y_2, p_0 + x_3\rangle & \widetilde{\lambda}''_2 := \langle y_3, p_0 + x_1\rangle & \widetilde{\lambda}''_3 := \langle y_1, p_0 + x_2\rangle
\end{array}\ . \tag{6.13}$$

Again the rows, denoted $\widetilde{\mathcal{S}}_3, \widetilde{\mathcal{S}}'_3, \widetilde{\mathcal{S}}''_3$, comprise a $\langle C'\rangle$-triplet of non-reguli, and the columns, denoted $\overline{\mathcal{S}}_3, \overline{\mathcal{S}}'_3, \overline{\mathcal{S}}''_3$, comprise a $\langle C\rangle$-triplet of non-reguli; however the six non-reguli given by this second array have $\eta$ as their common transversal.

The second array (6.13) can be obtained from the first array (6.12) by defining

$$\widetilde{\mathcal{S}}_3 = K(\mathcal{S}_3), \qquad \widetilde{\mathcal{S}}'_3 = C'(\widetilde{\mathcal{S}}_3), \qquad \widetilde{\mathcal{S}}''_3 = (C')^2(\widetilde{\mathcal{S}}_3). \tag{6.14}$$

This $\langle C'\rangle$-triplet $\widetilde{\mathcal{S}}_3, \widetilde{\mathcal{S}}'_3, \widetilde{\mathcal{S}}''_3$ of non-reguli will be needed in theorem 6.11 below. Note that for either array the involution $J^* \in \mathrm{GL}(5,2)$ interchanges the $i$ th row with the $i$ th column, $i = 1,2,3$. Upon using $A$ in (6.9) observe also that the three non-reguli $\overline{\mathcal{S}}_3, \overline{\mathcal{S}}''_3, \overline{\mathcal{S}}'_3$ in (6.13) are (**in that order!**) the $A$-images of the three non-reguli $\mathcal{S}_3, \mathcal{S}'_3, \mathcal{S}''_3$ in (6.12):

$$\overline{\mathcal{S}}_3 = A(\mathcal{S}_3), \quad \overline{\mathcal{S}}''_3 = A(\mathcal{S}'_3), \quad \overline{\mathcal{S}}'_3 = A(\mathcal{S}''_3). \tag{6.15}$$

Alternatively, since $A = J^*K$, the first of these relations follows from $\overline{\mathcal{S}}_3 = J^*(\widetilde{\mathcal{S}}_3)$ and $\widetilde{\mathcal{S}}_3 = K(\mathcal{S}_3)$; the third relation $\overline{\mathcal{S}}'_3 = A(\mathcal{S}''_3)$ then follows:

$$\overline{\mathcal{S}}'_3 = C(\overline{\mathcal{S}}_3) = CA(\mathcal{S}_3) = A(C')^2(\mathcal{S}_3) = A(\mathcal{S}''_3),$$

after making use of $A^{-1}CA = (C')^2$.

In theorem 6.7 below we will have need of the partial spreads

$$\bar{\mathcal{S}}_6 = \bar{\mathcal{S}}_3 \cup \rho^{\mathrm{o}}, \quad \bar{\mathcal{S}}'_6 = \bar{\mathcal{S}}'_3 \cup \rho^{\mathrm{o}}, \quad \bar{\mathcal{S}}''_6 = \bar{\mathcal{S}}''_3 \cup \rho^{\mathrm{o}}; \tag{6.16}$$

these form a $\langle C\rangle$-triplet, with $C$ effecting the permutation $(\bar{\mathcal{S}}_6\bar{\mathcal{S}}'_6\bar{\mathcal{S}}''_6)$. Since $A$ effects the interchange $\rho \leftrightarrow \rho^{\mathrm{o}}$, the partial spreads $\bar{\mathcal{S}}_6$, $\bar{\mathcal{S}}''_6$, $\bar{\mathcal{S}}'_6$ are (**in that order!**) the $A$-images of the partial spreads $\mathcal{S}_6, \mathcal{S}'_6, \mathcal{S}''_6$:

$$\overline{\mathcal{S}}_6 = A(\mathcal{S}_6), \quad \overline{\mathcal{S}}''_6 = A(\mathcal{S}'_6), \quad \overline{\mathcal{S}}'_6 = A(\mathcal{S}''_6). \tag{6.17}$$

Consequently $\bar{\mathcal{S}}_6$, $\bar{\mathcal{S}}''_6$, $\bar{\mathcal{S}}'_6$ are all of class VIc.2, and, since $ACA^{-1} = C'$, have respective stabilizers $\langle C', \bar{J}\rangle$, $\langle C', \bar{J}'\rangle$ and $\langle C', \bar{J}''\rangle$, where $\bar{J} = AJA^{-1} = J$, $\bar{J}' := AJ'A^{-1}$ and $\bar{J}'' := AJ''A^{-1}$.

## 6.3 The even 4-flat $X = \langle \mathcal{C}_6 \rangle = \mathcal{E}(\mathcal{C}_6)$

For $\mathcal{S}_6$ as in (6.4), (6.5), consider its Grassmann image

$$\mathcal{C}_6 = \{r_1, r_2, r_3\} \cup \{l_1, l_2, l_3\} \subset \mathrm{Rk}_2 \tag{6.18}$$

in $\mathrm{PG}(9,2)$. Because $\mathcal{S}_6$ is of class VIc.2 the flat $X := \langle \mathcal{C}_6 \rangle$ generated by the 6-cap $\mathcal{C}_6$ *is a 4-flat, and not a 5-flat as would have been the case for $S_6$ belonging to any one of the other thirteen classes of partial spread of size 6.* This exceptional feature of partial spreads of class VIc.2 comes about because:
(i) such $\mathcal{S}_6$ are of the form $\mathcal{S}_5 \cup \{\mu\}$ where $\mathcal{S}_5$ is of class Va.1;
(ii) every extension to an $\mathcal{S}_6$ of an $\mathcal{S}_5$ of class Va.1 is, see [7, Lemma A.10(ii)], of class VIc.2;
(iii) if $\mathcal{C}_5 = \{m_1, \ldots, m_5\}$ is the Grassmann image of an $\mathcal{S}_5$ then the $m_i$ are linearly independent — *except, see [16, Remark 1.2], for those $S_5$ of class Va.1 or class Vj.1, when they satisfy the single relation* $\Sigma_{i=1}^{5} m_i = 0$.
(iv) an $\mathcal{S}_5$ of class Vj.1, being a spread in some solid, has no extension to an $\mathcal{S}_6$.
For $\mathcal{S}_6$ as in (6.4), (6.5), the only line of $\mathcal{S}_6$ which is stabilized by $\mathcal{G}_{\mathcal{S}_6}$ is $\rho_3$. Consequently, for $\mathcal{C}_6$ as in (6.18) the linear relation has to be

$$r_1 + r_2 + l_1 + l_2 + l_3 = 0, \tag{6.19}$$

which of course can be verified directly. Thus $\mathcal{S}_6$ has a distinguished decomposition

$$\mathcal{S}_6 = \mathcal{S}_5^{(a)} \cup \{\rho_3\}, \qquad \text{with } \mathcal{S}_5^{(a)} = \{\rho_1, \rho_2, \lambda_1, \lambda_2, \lambda_3\} \text{ of class Va.1.} \tag{6.20}$$

**Remark 6.1** *(i) The partial spreads* $\mathcal{S}_5^{(d_1)} = \mathcal{S}_6 \diagdown \{\rho_1\}$ *and* $\mathcal{S}_5^{(d_2)} = \mathcal{S}_6 \diagdown \{\rho_2\}$ *are of class Vd.1, and have* $\langle C \rangle \cong Z_3$ *as common stabilizer.*

*(ii) The partial spreads* $\mathcal{S}_5^{(f_i)} = \mathcal{S}_6 \diagdown \{\lambda_i\}$, $i = 1, 2, 3$, *are of class Vf.1, the stabilizer of* $\mathcal{S}_5^{(f_i)}$ *being* $\langle J_i \rangle \cong Z_2$, *where* $J_1 = J$, $J_2 = CJC^{-1}$, $J_3 = C^2JC^{-2}$.

**Lemma 6.2** *(i) If* $X = \langle \mathcal{C}_6 \rangle$, *where* $\mathcal{C}_6$, *the Grassmann image of a partial spread* $\mathcal{S}_6$ *of class VIc.2, is as in (6.18), then* $X$ *coincides with* $\mathcal{E}(\mathcal{C}_6)$ *and is an even 4-flat of type* $(6, 25)$.

*(ii) If* $\mathcal{C}_5^{(d_i)}, i = 1, 2,$ *and* $\mathcal{C}_5^{(f_i)}, i = 1, 2, 3,$ *are the Grassmann images of the partial spreads in remark 6.1, then* $X = \langle \mathcal{C}_5^{(d_i)} \rangle = \langle \mathcal{C}_5^{(f_i)} \rangle$.

**Proof.** (i) The six elements of $\mathcal{C}_6$ satisfy just one linear relation, as in (6.19), whence $\langle \mathcal{C}_6 \rangle = \mathcal{E}(\mathcal{C}_6)$ is a 4-flat. Let us, temporarily changing the notation, write $\mathcal{C}_6 = \{m_1, \ldots, m_6\}$ with $m_6 = \rho_3$. Then the 25 elements of $\langle \mathcal{C}_6 \rangle \diagdown \mathcal{C}_6$ comprise 15 elements $m_{ij}, 1 \leq i < j \leq 6$, and 10 elements $m_{ij6}, 1 \leq i < j \leq 5$, all 25 thus being of rank 4. So $X$ is of type $(6, 25)$.

(ii) On account of (6.19), $\langle \mathcal{C}_5^{(d_i)} \rangle = \langle \mathcal{C}_5^{(f_i)} \rangle = \langle \mathcal{C}_6 \rangle$. ∎

## 6.4 The associate of $X = \langle \mathcal{C}_6 \rangle$, ($\mathcal{S}_6 \in$ classVIc.2)

We now wish to determine the associate $X^{\#}$ of the even 4-flat $X = \langle \mathcal{C}_6 \rangle$ in lemma 6.2. We will find that $X^{\#}$ and $X^{\#\#}$ are both even 4-flats, but that $X^{\#\#} \neq X$; however it will turn out that $X^{\#\#\#} = X^{\#}$. It will be of help if we first sketch out some preliminary (and presumably familiar) material.

### 6.4.1 Preliminaries: some structures in a $\mathrm{PG}(5, 2)$

Let us first fix upon some particular solid $\sigma = \mathrm{PG}(3, 2) = \mathbb{P}V_4$ and consider the projective space $\mathrm{PG}(5, 2) = \mathbb{P}(\wedge^2 V_4)$. In the $\mathrm{Rk}_2 \cup \mathrm{Rk}_4$ decomposition $\mathrm{PG}(5, 2) = \mathcal{H} \cup \mathcal{W}$ the 35 points of the Grassmannian $\mathcal{H} = \mathcal{H}(\sigma) = \mathcal{G}_{1,3,2}$ are those points $b = \Sigma_{i<j} b_{ij} e_i \wedge e_j$ which satisfy the single quadratic Grassmann condition $Q(b) := b_{12}b_{34} + b_{13}b_{24} + b_{14}b_{23} = 0$ and they constitute a nondegenerate hyperbolic quadric $\mathcal{H}_5$ (the Klein quadric) in $\mathrm{PG}(5, 2)$. The quadratic form $Q$ polarizes to yield a nondegenerate symplectic form $< \cdot, \cdot >$ on $V_6 = \wedge^2 V_4$ which satisfies

$$< x_1 \wedge x_2, y_1 \wedge y_2 >= \begin{cases} 0 & \text{if } \langle x_1, x_2 \rangle \text{ and } \langle y_1, y_2 \rangle \text{ meet,} \\ 1 & \text{if } \langle x_1, x_2 \rangle \text{ is skew to } \langle y_1, y_2 \rangle. \end{cases} \tag{6.21}$$

A point $b \in \mathcal{W} = \mathcal{W}(\sigma)$ gives rise to a nondegenerate alternating bilinear form $b(\cdot, \cdot)$ on $V_4$, defined by $b(x_1, x_2) =< b, x_1 \wedge x_2 >$, and so in this manner the 28 points of $\mathcal{W}$ are in correspondence with the 28 $\mathrm{Sp}(4, 2)$ subgroups of $\mathrm{GL}(4, 2)$, that is with the 28 distinct symplectic polarities on $\mathrm{PG}(3, 2)$.

When dealing with the decomposition $\mathrm{PG}(9, 2) = \mathrm{Rk}_2 \cup \mathrm{Rk}_4$ of $\mathrm{PG}(9, 2) = \mathbb{P}(\wedge^2 V_5)$, each of the 31 solids $\sigma = \mathbb{P}(V_4)$ in $\mathrm{PG}(4, 2) = \mathbb{P}V_5$ defines a $\mathrm{PG}(5, 2)$ in $\mathrm{PG}(9, 2)$, namely the 5-flat $\Pi(\sigma) = \mathbb{P}(\wedge^2 V_4)$. Upon setting

$$\mathcal{H}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_2 \qquad \text{and} \qquad \mathcal{W}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_4, \tag{6.22}$$

the 63 points of the 5-flat $\Pi(\sigma)$ consist of the 35 points of the Grassmannian $\mathcal{G}_{1,3,2} = \mathcal{H}(\sigma) = \mathrm{Rk}_2(V_4)$ together with the 28 points $\mathcal{W}(\sigma) = \mathrm{Rk}_4(V_4)$ which are external to $\mathcal{H}(\sigma)$. It should be noted, see [14, Section 2.1.1], that the 31 subsets $\mathcal{W}(\sigma)$, $\sigma$ a solid in $\mathrm{PG}(4,2)$, are disjoint, and so *yield a partition of the 868 points of* $\mathrm{Rk}_4$ *into 31 subsets of size 28:* $868 = 31 \times 28$.

Let us now briefly describe features of the $\mathrm{PG}(5,2) = \Pi(\sigma)$ which are determined by the decomposition $\sigma = \xi \cup \eta \cup \mathcal{H}_3$, see (6.2), of $\sigma = \mathrm{PG}(3,2)$. Denote by $x = x_1 \wedge x_2$ and $y = y_1 \wedge y_2$ the points of $\mathrm{PG}(5,2)$ which are the Plücker images of the lines $\xi$ and $\eta$ of $\sigma$, and let $\mathcal{C}_\rho = \{r_1, r_2, r_3\}$ and $\mathcal{C}_{\rho^{\mathrm{o}}} = \{r_1^{\mathrm{o}}, r_2^{\mathrm{o}}, r_3^{\mathrm{o}}\}$ be the 3-caps on $\mathcal{H}_5 := \mathcal{H}(\sigma)$ corresponding to the reguli $\rho$ and $\rho^{\mathrm{o}}$. Then note that $\xi$ and $\eta$ determine a line $L_{\{\xi,\eta\}} = \langle x, y\rangle$ of $\mathrm{PG}(5,2)$ which is a bisecant of the quadric $\mathcal{H}_5$. The third point $b = x + y$ of this bisecant $\{x, y, b\}$ defines a symplectic polarity on $\sigma$ with respect to which $\eta$ is the polar of $\xi$.

By using the scalar product (6.21), next note that the line $L_{\{\xi,\eta\}}$ gives rise to a solid $D := (L_{\{\xi,\eta\}})^\perp$ in $\mathrm{PG}(5,2)$ whose 15 point are as follows. Nine of the points are the Grassmann images of the nine tangent lines $\langle x_i, y_j\rangle$ to $\mathcal{H}_3$, these constituting the hyperbolic quadric $\mathfrak{H}_3 \subset D$ given by the array

$$\mathfrak{H}_3 = \begin{pmatrix} x_1 \wedge y_1 & x_1 \wedge y_2 & x_1 \wedge y_3 \\ x_2 \wedge y_1 & x_2 \wedge y_2 & x_2 \wedge y_3 \\ x_3 \wedge y_1 & x_3 \wedge y_2 & x_3 \wedge y_3 \end{pmatrix}. \tag{6.23}$$

Observe that the lines of one set of generators of $\mathfrak{H}_3$ are the images of the three pencils $(x_i, \eta)$, and the lines of the other set are the images of the three pencils $(y_i, \xi)$. The other six points of $D$, comprising the two lines of $D$ external to $\mathfrak{H}_3$, are seen to be

$$L_\rho := \mathcal{E}(\mathcal{C}_\rho) = \{r_{23}, r_{31}, r_{12}\}, \qquad L_{\rho^{\mathrm{o}}} := \mathcal{E}(\mathcal{C}_{\rho^{\mathrm{o}}}) = \{r_{23}^{\mathrm{o}}, r_{31}^{\mathrm{o}}, r_{12}^{\mathrm{o}}\}, \tag{6.24}$$

where $r_{ij} := r_i + r_j$ and $r_{ij}^{\mathrm{o}} := r_i^{\mathrm{o}} + r_j^{\mathrm{o}}$. For, from (6.3), we see that

$$\begin{aligned} L_\rho &= \{x_3 \wedge y_2 + x_1 \wedge y_1,\ x_1 \wedge y_1 + x_2 \wedge y_3,\ x_2 \wedge y_3 + x_3 \wedge y_2\}, \\ L_{\rho^{\mathrm{o}}} &= \{x_2 \wedge y_2 + x_3 \wedge y_3,\ x_3 \wedge y_3 + x_1 \wedge y_1,\ x_1 \wedge y_1 + x_2 \wedge y_2\}. \end{aligned} \tag{6.25}$$

Note that $L_\rho, L_{\rho^{\mathrm{o}}}$ both lie in $\mathcal{W}(\sigma)$, (and so, from a $\mathrm{PG}(9,2)$ point of view, belong to $\mathrm{orb}(1\beta)$).

Summarizing, we can express $\Pi(\sigma) = \mathrm{PG}(5,2)$ in the form

$$\mathrm{PG}(5,2) = \langle L_{\{\xi,\eta\}}, L_\rho, L_{\rho^{\mathrm{o}}}\rangle = \langle L_{\{\xi,\eta\}}, D\rangle, \tag{6.26}$$

where the line $L_{\{\xi,\eta\}}$ is a bisecant of $\mathcal{H}_5$, where the lines $L_\rho, L_{\rho^{\mathrm{o}}}$ are external to $\mathcal{H}_5$ and where $D$ is the solid

$$D := (L_{\{\xi,\eta\}})^\perp = \langle L_\rho, L_{\rho^{\mathrm{o}}}\rangle = L_\rho \cup L_{\rho^{\mathrm{o}}} \cup \mathfrak{H}_3. \tag{6.27}$$

We will shortly be interested in the structure $\mathfrak{S} = \{x, y, L_\rho, L_{\rho^{\mathrm{o}}}\}$ in $\mathrm{PG}(5,2)$, and in particular in that subgroup $\mathcal{G}(\mathfrak{S})$ of $\mathrm{GL}(4,2)$ which — under the natural action $T : A \mapsto T_A = \wedge^2 A$ of $\mathrm{GL}(V_4)$ upon $\mathrm{GL}(\wedge^2 V_4)$ — separately preserves each of the four elements of $\mathfrak{S}$. Clearly $\mathcal{G}(\mathfrak{S})$ contains $C_4, C_4', J_4$, that is the restrictions to $V_4$ of $C, C', J$ in (6.6). In fact it is not hard to see that $\mathcal{G}(\mathfrak{S}) = \langle C_4, C_4', J_4\rangle$, and that is of index 2 in that subgroup $\mathcal{G}(\xi,\eta) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ of $\mathrm{GL}(4,2)$ consisting of all elements of $\mathrm{GL}(4,2)$ which stabilize both $\xi$ and $\eta$.

The next lemma summarizes some of the foregoing in the language of vector spaces and group representations.

**Lemma 6.3** *(i) The group $\mathcal{G} := \mathcal{G}(\xi, \eta)$ acts irreducibly upon $V_4$. Under the action of the subgroup $\mathcal{G}_0 := \mathcal{G}(\mathfrak{S}) = \langle C_4, C_4', J_4 \rangle$ of $\mathcal{G}$ the space $V_4$ decomposes as the direct sum $V_4 =\prec \xi \succ \oplus \prec \eta \succ$ of two irreducible invariant subspaces upon which $\mathcal{G}_0$ acts inequivalently.*

*(ii) The representation of $\mathcal{G}$ naturally induced in the space $\wedge^2 V_4$ is completely reducible, decomposing into a direct sum of irreducible representations, of dimensions 2 and 4, arising from the invariant decomposition*

$$\wedge^2 V_4 =\prec L_{\{\xi,\eta\}} \succ \oplus \prec D \succ . \tag{6.28}$$

*On passing to the subgroup $\mathcal{G}_0$, we have the further $\mathcal{G}_0$-invariant decompositions $\prec L_{\{\xi,\eta\}} \succ =\prec x \succ \oplus \prec y \succ$ and $\prec D \succ =\prec L_\rho \succ \oplus \prec L_{\rho^o} \succ$, leading to a complete reduction of the representation of $\mathcal{G}_0$ on $\wedge^2 V_4$ given by*

$$\wedge^2 V_4 =\prec x \succ \oplus \prec y \succ \oplus \prec L_\rho \succ \oplus \prec L_{\rho^o} \succ . \tag{6.29}$$

*Since $\mathcal{G}_0$ acts inequivalently upon $\prec L_\rho \succ$ and $\prec L_{\rho^o} \succ$, the decomposition (6.29) is unique —* except that in the decomposition $\prec L_{\{\xi,\eta\}} \succ =\prec x \succ \oplus \prec y \succ$, either of the one-dimensional spaces $\prec x \succ$, $\prec y \succ$ can be replaced by $\prec b \succ$ .

**Remark 6.4** *If $s \in \wedge^4 V_5$ is the quadrivector corresponding to the solid $\sigma = \mathbb{P}(V_4)$ then the nondegenerate alternating bilinear form $b(\cdot,\cdot)$ on $V_4$ determined, see after (6.21), by a point $b \in \mathcal{W}(\sigma)$ may alternatively be defined via*

$$b(x_1, x_2)s = b \wedge x_1 \wedge x_2, \qquad x_1, x_2 \in V_4. \tag{6.30}$$

*Given any point $k \in \sigma^c$ in* $\mathrm{PG}(4,2)$*, set $e = k \wedge s \in \wedge^5 V_5$ and $t = k \wedge b \in \wedge^3 V_5$. Then the form $b(\cdot,\cdot)$ on $V_4$ can be extended to give an alternating bilinear form $b^*(\cdot,\cdot)$ on $V_5$, of rank 4 and one-dimensional kernel $\prec k \succ$, via*

$$b^*(x_1, x_2)e = t \wedge x_1 \wedge x_2, \qquad x_1, x_2 \in V_5. \tag{6.31}$$

*Here the dual bivector $b^* \in \wedge^2 V_5^*$ is related to the trivector $t$ via the Poincaré isomorphism $\perp$:*

$$t = \perp b^*. \tag{6.32}$$

*For a given kernel $\prec k \succ$ it should be noted that $t$ can be expressed in the form $t = k \wedge b$ for 16 choices of $b$, since the original $b$ may be replaced by $b^* := b + a \wedge k$ for any $a \in \sigma$. These 16 choices corresponding to the 16 solids $\sigma^* := \mathrm{im}(b^*)$ which are disjoint from $k$, whence the restriction of $b^*$ to any such $\sigma^*$ is nondegenerate.*

### 6.4.2 Chief results concerning the sequence $X, X^{\#}, X^{\#\#}, \ldots$

**Theorem 6.5** *Let $Y = X^{\#}$ be the associate of the even 4-flat $X = \langle \mathcal{C}_6 \rangle$ in lemma (6.2), and let $Y^O \subset \mathrm{PG}(9,2)^* = \mathbb{P}(\wedge^3 V_5)$ be its annihilator. Then*
*(i) $Y$ is the even 4-flat*

$$Y = \langle x, p_0 \wedge \eta, L_{\rho^o} \rangle \tag{6.33}$$

*of type $(4, 27)$, with $Y \cap \mathrm{Rk}_2 = \{x\} \cup (p_0 \wedge \eta)$. Here $L_{\rho^o} = \mathcal{E}(\mathcal{C}_{\rho^o})$ as in (6.24).*
*(ii) $Y^O$ is the even 4-flat*

$$Y^O = \langle p_0 \wedge x, \xi \wedge y, p_0 \wedge L_\rho \rangle \tag{6.34}$$

*of type $(4, 27)$,with $Y^O \cap \mathrm{Rk}_2^* = \{p_0 \wedge x\} \cup (\xi \wedge y)$. Here $\xi \wedge y$ denotes the line in* $\mathrm{PG}(9,2)^*$ *whose points are $x_i \wedge y$, $i = 1, 2, 3$, and $p_0 \wedge L_\rho = p_0 \wedge \mathcal{E}(\mathcal{C}_\rho)$ denotes the line whose points are $p_0 \wedge r_{ij}$, $1 \leq i < j \leq 3$.*

**Proof.** On account of the relation (6.19) we have $X = \langle \mathcal{C}_6 \rangle = \langle \mathcal{C}_5^{d_2} \rangle = \langle l_1, l_2, l_3, r_1, r_3 \rangle$, where $\mathcal{C}_5^{d_2}$ is, see remark 6.1, of class Vd.1. Set, for $i = 1,2,3$, $\mathcal{C}_4^{(i)} = \{l_1, l_2, l_3, r_1, r_3\} \diagdown \{l_i\}$ and set $\mathcal{C}_4^{(4)} = \{l_1, l_2, l_3, r_3\}$ and $\mathcal{C}_4^{(5)} = \{l_1, l_2, l_3, r_1\}$. Consider the five solids $D_i = \langle \mathcal{C}_4^{(i)} \rangle$, $1 \leq i < 5$, and note that $\mathcal{C}_4^{(i)}$ is of class IVb.1 for $i = 1,2,3,4$, and of class IVa.1 for $i = 5$. By use of theorem 3.7 the associated trivectors $t_i$ of the solids $D_i$ are seen to be:

$$\begin{aligned} t_i &= p_0 \wedge x + x_i \wedge y + p_0 \wedge r_{13}, \quad i = 1,2,3, \\ t_4 &= p_0 \wedge r_{12}, \\ t_5 &= p_0 \wedge x + p_0 \wedge r_{12}. \end{aligned} \tag{6.35}$$

It follows that $\langle t_1, t_2, t_3, t_4, t_5 \rangle = \langle p_0 \wedge x, \xi \wedge y, p_0 \wedge L_\rho \rangle$, and so, see theorem 4.1, we have derived the result (6.34). The result (6.33) now follows, since an easy check confirms that each of $p_0 \wedge x, x_i \wedge y, p_0 \wedge r_{ij}$ annihilates each of $x, p_0 \wedge y_i, r_{ij}^{o}$.
(Points to note include the following:
(i) since $y_i \wedge y = 0$ (because $y_i \in \eta$), it follows that $(p_0 \wedge y_i) \wedge (x_i \wedge y) = 0$;
(ii) since $\rho_i^{\mathrm{o}}$ meets $\rho_j$ it follows that $r_i^{\mathrm{O}} \wedge r_j = 0$, for each $i, j$;
(iii) since $\rho_j^{\mathrm{o}}, x_i$ and $\eta$ all lie in the solid $\sigma = \langle \xi, \eta \rangle$ it follows that $r_j^{\mathrm{O}} \wedge x_i \wedge y = 0$;
(iv) since $\xi$ is skew to each line $\rho_i$ of $\rho$, and since $p_0$ lies outside the solid $\sigma = \langle \xi, \rho_i \rangle = \langle \xi, \rho_j \rangle$, it follows that $x \wedge p_0 \wedge (r_i + r_j) = e + e = 0$.) ■

**Theorem 6.6** *(i) If the 4-flats $Y$ and $Y^O$ are as in (6.33) and (6.34) then*

$$\mathcal{G}_Y = \mathcal{G}_{Y^O} = \mathcal{G}_0(p_0, \xi, \eta) = \langle C, C', J \rangle \cong (Z_3 \times Z_3).Z_2. \tag{6.36}$$

*(ii) Under the action of $\mathcal{G}(p_0, \xi, \eta)$ the space $\wedge^2 V_5$ has the decomposition*

$$\wedge^2 V_5 = \prec L_{\{\xi,\eta\}} \succ \oplus \prec p_0 \wedge \sigma \succ \oplus \prec D \succ \tag{6.37}$$

*into irreducible subspaces of dimensions $2,4,4$. Under the action of $\mathcal{G}_0(p_0, \xi, \eta)$ the space $\wedge^2 V_5$ decomposes further in the manner*

$$\wedge^2 V_5 = \prec x \succ \oplus \prec y \succ \oplus \prec p_0 \wedge \xi \succ \oplus \prec p_0 \wedge \eta \succ \oplus \prec L_\rho \succ \oplus \prec L_{\rho^o} \succ \tag{6.38}$$

*into irreducible subspaces of dimensions $1,1,2,2,2,2$.*

**Proof.** (i) Since $(Y^{\mathrm{O}})^{\mathrm{O}} = Y$, we have $\mathcal{G}_Y = \mathcal{G}_{Y^{\mathrm{O}}}$ for any flat $Y$. For $Y$ as in (6.33) clearly $\mathcal{G}_Y$ contains $\langle C, C', J \rangle$, and we can check that $\mathcal{G}_Y$ is no bigger than this.

(ii) Use lemma 6.3. ■

**Theorem 6.7** *(i) Each of the 4-flats $\langle \mathcal{C}_6 \rangle, \langle \mathcal{C}'_6 \rangle, \langle \mathcal{C}''_6 \rangle$ arising from the $\langle C' \rangle$-triplet (6.11) of partial spreads of class VIc.2 has for its associate the 4-flat $Y = \langle x, p_0 \wedge \eta, L_{\rho^o} \rangle$, of type $(4,27)$, the annihilator of $Y$ being the 4-flat $Y^O = \langle p_0 \wedge x, \xi \wedge y, p_0 \wedge L_\rho \rangle$, also of type $(4,27)$.*

*(ii) Each of the 4-flats $\langle \bar{\mathcal{C}}_6 \rangle, \langle \bar{\mathcal{C}}'_6 \rangle, \langle \bar{\mathcal{C}}''_6 \rangle$ arising from the $\langle C \rangle$-triplet (6.16) of partial spreads of class VIc.2 is of type $(6,25)$. Moreover each has for its associate the 4-flat*

$$\bar{Y} := \langle y, p_0 \wedge \xi, L_\rho \rangle, \tag{6.39}$$

*of type $(4,27)$, the annihilator of $\bar{Y}$ being the 4-flat*

$$\bar{Y}^O = \langle p_0 \wedge y, \eta \wedge x, p_0 \wedge L_{\rho^o} \rangle \tag{6.40}$$

*of type* $(4, 27)$. *Furthermore*

$$\mathcal{G}_{\bar{Y}} = \mathcal{G}_{\bar{Y}^o} = \mathcal{G}_0(p_0, \xi, \eta) = \langle C, C', J\rangle = \mathcal{G}_Y = \mathcal{G}_{Y^o}. \tag{6.41}$$

*(iii) (a)* $Y^{\#} = \bar{Y}$ *and (b)* $\bar{Y}^{\#} = Y$. *Thus*

$$Y^{\#\#} = Y, \qquad X^{\#\#\#} = X^{\#}. \tag{6.42}$$

**Proof.** Since $T_{C'}(Y) = Y$, part (i) follows from theorem 6.5.
Part (ii) is the image under $A \in \mathrm{GL}(5, 2)$ of part (i), and of (6.36).
(iii) Since $A$ effects the interchange $Y \leftrightarrow \bar{Y}$ we need to prove just one of (a) and (b). Consider (a). Now $\mathcal{G}_{Y^{\#}}$ contains $\mathcal{G}_Y = \mathcal{G}_0(p_0, \xi, \eta) = \langle C, C', J\rangle$, and moreover $Y^{\#}$ is disjoint from $Y = \langle x, p_0 \wedge \eta, L_{\rho^o}\rangle$. It follows from (6.38) that either $Y^{\#} = \langle y, p_0 \wedge \xi, L_\rho\rangle = \bar{Y}$ or else $Y^{\#} = \langle x + y, p_0 \wedge \xi, L_\rho\rangle$. But this last possibility can be seen not to hold. ■

**Remark 6.8** *Since for* $X = \langle \mathcal{C}_6\rangle$ *the 4-flats* $X, X^{\#}(= Y)$ *and* $X^{\#\#}(= Y^{\#} = \bar{Y})$ *are even, then the intersections* $X \cap X^{\#}$ *and* $X^{\#} \cap X^{\#\#}$ *are of course empty. However the intersection* $X \cap X^{\#\#}$ *is seen from (6.39) to be non-empty:*

$$X \cap X^{\#\#} = L_\rho.$$

### 6.4.3 Further aspects, involving regulus-free partial spreads $\mathcal{S}_8$

From remark 6.1 and lemma 6.2 we know that the even 4-flat $X$, of type $(6, 25)$, can be obtained, via $X = \langle \mathcal{C}_5^{(d_1)}\rangle$ and $X = \langle \mathcal{C}_5^{(d_2)}\rangle$, from two partial spreads $\mathcal{S}_5^{(d_1)} = \mathcal{S}_6 \setminus \{\rho_1\}$ and $\mathcal{S}_5^{(d_2)} = \mathcal{S}_6 \setminus \{\rho_2\}$ which are of class Vd.1, and which have $\langle C\rangle \cong Z_3$ as common stabilizer. In fact, upon applying $C' \in \mathrm{GL}(5, 2)$, we see that the 4-flat $X$ can be obtained from four further partial spreads $C'(\mathcal{S}_5^{(d_1)}) = \mathcal{S}'_6 \setminus \{\rho_2\}$, $C'(\mathcal{S}_5^{(d_2)}) = \mathcal{S}'_6 \setminus \{\rho_2\}$, $(C')^2(\mathcal{S}_5^{(d_1)}) = \mathcal{S}''_6 \setminus \{\rho_3\}$, $(C')^2(\mathcal{S}_5^{(d_2)}) = \mathcal{S}'_6 \setminus \{\rho_1\}$, , all of which are thus of class Vd.1 and have $\langle C\rangle \cong Z_3$ as common stabilizer. We now wish to make use of the fact, see theorem C.1, that *a partial spread* $S_5$ *of class Vd.1 has a unique extension to a regulus-free partial spread* $S_8$, that is to a partial spread of class VIIIa.1.

**Lemma 6.9** *Consider the partial spreads* $\mathcal{S}_5^{(d_1)} = \mathcal{S}_6 \setminus \{\rho_1\}$ *and* $\mathcal{S}_5^{(d_2)} = \mathcal{S}_6 \setminus \{\rho_2\}$ *of class Vd.1 in remark 6.1, which are obtained from the partial spread* $\mathcal{S}_6$ *of class VIc.2 in eq. (6.5). Then* $\mathcal{S}_5^{(d_1)} \cup \widetilde{\mathcal{S}}'_3$ *and* $\mathcal{S}_5^{(d_2)} \cup \widetilde{\mathcal{S}}''_3$ *are partial spreads of class VIIIa.1.*

**Proof.** The 7-set of PG$(4, 2)$ which is left uncovered by the eight lines of $\mathcal{S}_5^{(d_1)} \cup \widetilde{\mathcal{S}}'_3$ is a plane, namely $\langle p_0, \rho_1\rangle$. It follows from [7, Table B.1], that $\mathcal{S}_5^{(d_1)} \cup \widetilde{\mathcal{S}}'_3$ is of class VIIIa.1. Since the eight lines of $\mathcal{S}_5^{(d_2)} \cup \widetilde{\mathcal{S}}''_3$ also leave uncovered a plane, namely $\langle p_0, \rho_2\rangle$, it similarly follows that $\mathcal{S}_5^{(d_2)} \cup \widetilde{\mathcal{S}}''_3$ is of class VIIIa.1. ■

**Theorem 6.10** *Let* $\mathcal{S}_8 = \{\mu_1, \dots, \mu_8\}$ *be a partial spread of class VIIIa.1. Let* $\mathcal{S}_8 = \mathcal{S}_5 \cup \mathcal{S}_3$ *be any decomposition of* $\mathcal{S}_8$ *into partial spreads* $\mathcal{S}_5 = \{\mu_a, \mu_b, \mu_c, \mu_d, \mu_e\}$ *and* $\mathcal{S}_3 = \{\mu_f, \mu_g, \mu_h\}$. *Let* $\mathcal{C}_8 = \{m_1, \dots, m_8\}$, $\mathcal{C}_5$ *and* $\mathcal{C}_3$ *be the Grassmann images of these regulus-free partial spreads* $\mathcal{S}_8$, $\mathcal{S}_5$ *and* $\mathcal{S}_3$. *Then* $\mathcal{S}_5$ *is of class Vd.1, Moreover the external line* $\mathcal{E}(\mathcal{C}_3)$ *lies inside the associate* $\langle \mathcal{C}_5\rangle^{\#}$ *of* $\langle \mathcal{C}_5\rangle$.

**Proof.** By theorem C.1 $\mathcal{S}_5$ is of class Vd.1. So $\langle \mathcal{C}_5 \rangle$ is even, of type $(6, 25)$. Hence $Q^{\ddagger}(m_a, m_b, m_c, m_d, m_e) = 1$. Similarly, for all $\{a', b', c', d'\} \subset \{a, b, c, d, e\}$, we have $Q^{\ddagger}(m_{a'}, m_{b'}, m_{c'}, m_{d'}, m) = 1$ for each $m \in \mathcal{C}_3$. Hence, for all $\{a', b', c', d'\} \subset \{a, b, c, d, e\}$ and for each $b \in \mathcal{E}(\mathcal{C}_3)$, we have

$$Q^{\ddagger}(m_{a'}, m_{b'}, m_{c'}, m_{d'}, b) = 1 + 1 = 0.$$

Hence $\mathcal{E}(\mathcal{C}_3) \subset \langle \mathcal{C}_5 \rangle^{\#}$. ■

**Theorem 6.11** *(i) Let* $D$ *denote the solid* $\langle p_0 \wedge \eta, \mathcal{E}(\mathcal{C}_{\rho^o}) \rangle$ *which lies inside the 4-flat* $Y = X^{\#}$ *in (6.33). Then* $D = (p_0 \wedge \eta) \cup \mathcal{E}(\mathcal{C}_{\rho^o}) \cup \mathsf{H}_3$*, where one set of generators for the hyperbolic quadric* $\mathsf{H}_3$ *consists of the three lines* $\mathcal{E}(\widetilde{\mathcal{C}}_3)$, $\mathcal{E}(\widetilde{\mathcal{C}}'_3)$, $\mathcal{E}(\widetilde{\mathcal{C}}''_3)$*, belonging to* orb$(1\alpha)$*, which arise from the non-reguli* $\widetilde{\mathcal{S}}_3, \widetilde{\mathcal{S}}'_3, \widetilde{\mathcal{S}}''_3$ *in (6.14).*

*(ii) Let* $\bar{D}$ *denote the solid* $\langle p_0 \wedge \xi, \mathcal{E}(\mathcal{C}_{\rho}) \rangle$ *which lies inside the 4-flat* $\bar{Y}$ *in (6.39). Then* $\bar{D} = (p_0 \wedge \xi) \cup \mathcal{E}(\mathcal{C}_{\rho}) \cup \mathsf{H}^*_3$*, where one set of generators for the hyperbolic quadric* $\mathsf{H}^*_3$ *consists of the three lines* $\mathcal{E}(\mathcal{C}^*_3)$, $\mathcal{E}(\mathcal{C}^{*\prime}_3)$, $\mathcal{E}(\mathcal{C}^{*\prime\prime}_3)$*, belonging to* orb$(1\alpha)$*, which arise from the non-reguli* $\mathcal{S}^*_3, \mathcal{S}^{*\prime}_3, \mathcal{S}^{*\prime\prime}_3$ *in (6.12).*

**Proof.** (i) As in lemma 6.2 we have $X = \langle \mathcal{C}_5^{(d_1)} \rangle = \langle \mathcal{C}_5^{(d_2)} \rangle$. So it follows from lemma 6.9 and theorem 6.10 that the external lines $\mathcal{E}(\widetilde{\mathcal{C}}'_3)$ and $\mathcal{E}(\widetilde{\mathcal{C}}''_3)$ both lie inside $X^{\#}$. Now $T_{C'}(X) = X$ and, see (6.14), $T_{C'}(\mathcal{E}(\widetilde{\mathcal{C}}''_3)) = \mathcal{E}(\widetilde{\mathcal{C}}_3)$. Hence the line $\mathcal{E}(\widetilde{\mathcal{C}}_3)$ also lies inside $X^{\#}$. (Alternatively the line $\mathcal{E}(\widetilde{\mathcal{C}}_3)$, along indeed with the lines $p_0 \wedge \eta$ and $\mathcal{E}(\mathcal{C}_{\rho^o})$, lie inside $X^{\#}$, since these three lines lie in the solid $\langle \mathcal{E}(\widetilde{\mathcal{C}}'_3),\ \mathcal{E}(\widetilde{\mathcal{C}}''_3) \rangle$.)

(ii) This is the image under $A \in \mathrm{GL}(5, 2)$ of part (i). ■

**Remark 6.12** *An alternative proof of the result (6.33) for* $X^{\#}$ *could thus be constructed as follows. From theorem 6.11 we know that the 4-flat* $X^{\#}$ *contains the solid* $D = \langle \mathsf{H}_3 \rangle$*. So to complete the proof of (6.33) all we have to do is to check that* $X^{\#}$ *contains the point* $x$*.*

# 7 Some even 4-flats exemplifying $X^{\#\#\#} = X$

## 7.1 Introduction

### 7.1.1 Orientation and summary of results

As we explained at the beginning of section 6, we are particularly interested in 4-flats $X$ which are even. Now in Table 2 of section 2.3 the 4-flat $\mathcal{E}(\mathcal{C}_6)$ is of even type $(2, 29)$ for three classes of partial spread $\mathcal{S}_6$, namely VIe.1, VIf.1 and VIh.1. We will consider these even 4-flats in the present section.

In the case of $\mathcal{S}_6 \in$ class VIf.1 our chief results may be summarized:

(i) If $X := \mathcal{E}(\mathcal{C}_6)$, then $X,\ X^{\#}$ and $X^{\#\#}$ are all even 4-flats of type $(2, 29)$;
(ii) $\mathcal{G}_X = \mathcal{G}_{X^{\#}} = \mathcal{G}_{X^{\#\#}} \cong Z_2$;
(iii) $X^{\#\#\#} = X$. (7.1)

*Notes.* We will show that there exists an element $C \in \mathrm{GL}(5, 2)$ of order 3 such that $X^{\#} = T_C(X)$ and $X^{\#\#} = T_{C^2}(X)$. Thus $X^{\#} = \mathcal{E}(\mathcal{C}'_6)$ and $X^{\#\#} = \mathcal{E}(\mathcal{C}''_6)$ where $\mathcal{S}'_6 := C(\mathcal{S}_6)$ and $\mathcal{S}''_6 := C^2(\mathcal{S}_6)$ are also partial spreads of class Vf.1. We will also show that for a given $\mathcal{S}_6 \in$ class VIf.1 there exist two partial spreads $\mathcal{S}_6^{(1)}, \mathcal{S}_6^{(2)}$ of class VIe.1, and also two partial spreads $\mathcal{S}_6^{(3)}, \mathcal{S}_6^{(4)}$ of class VIh.1, such that $\mathcal{E}(\mathcal{C}_6^{(i)}) = \mathcal{E}(\mathcal{C}_6)$, $i = 1, 2, 3, 4$. The stabilizer of each $\mathcal{S}_6^{(i)}$ is trivial, and so the symmetry increases, from $\{I\}$ to $Z_2$, on passing from $\mathcal{S}_6^{(i)}$ to $\mathcal{E}(\mathcal{C}_6^{(i)})$.

### 7.1.2 Use of the computer

Recall that in section 6 we managed to unravel the intricacies of sequence $X,\ X^{\#},\ X^{\#\#},\ \ldots$ , only after a considerable struggle, despite the fact that we were able to take advantage of certain symmetry considerations. In the present section we are dealing with cases where there is little or no symmetry, and so it would appear to be even harder to pursue things to a successful conclusion. In the event we made use of the computer algebra system Magma [1] to help us find our bearings. Starting out from a 4-flat $X = \mathcal{E}(\mathcal{C}_6)$ arising from a particular choice of partial spread $\mathcal{S}_6$ of class VIf.1, we used Magma to compute $X,\ X^{\#},\ X^{\#\#},\ X^{\#\#\#}$, and also their respective annihilators $Z,\ Z',\ Z'',\ Z'''$, and discovered that $X^{\#\#\#} = X$. Guided by Magma we were able to spot a certain kernel pattern, see eq. (7.5), and from this pattern we were led to introduce a certain generating set $\{a, b, c, u, v, w\}$ for $\mathrm{PG}(4, 2)$, with $u + v + w = 0$, and hence to consider that element $C \in \mathrm{GL}(5, 2)$ which effects the permutation $(abc)(uvw)$. We were then led to give examples of partial spreads of classes VIf.1, VIe.1 and VIh.1 all expressed in terms of a common choice of $\{a, b, c, u, v, w\}$, as exhibited in eqs. (7.2), (7.7) and (7.10) below. This provided us with a proper framework in which to phrase various theorems, these then being checked by Magma using a particular choice of $\{a, b, c, u, v, w\}$.

## 7.2 Partial spreads $\mathcal{S}_6$ of class VIf.1

Let $\{a, b, c, u, v\}$ be a basis for $V_5$. Setting $w = u+v$, consider the partial spread $\mathcal{S}_6 = \{\mu_1, \ldots, \mu_6\}$, where

$$\begin{aligned} &\mu_1 = \{cu, av, acw\}, \quad \mu_2 = \{bu, bcw, cv\}, \quad \mu_3 = \{bc, abcu, au\}, \\ &\mu_4 = \{abcv, acu, bw\}, \quad \mu_5 = \{bv, abu, aw\}, \quad \mu_6 = \{b, a, ab\}. \end{aligned} \tag{7.2}$$

Here $c+u,\ a+c+w,\ \ldots$ are abbreviated to $cu,\ acw,\ \ldots$ . Then $\mathcal{S}_6$ is of type I, the sole regulus being $\rho = \{\mu_1, \mu_2, \mu_3\}$, whose extension to a spread in the ambient solid $\sigma$ of $\rho$ is $\rho \cup \{\lambda, \nu\}$ where

$$\lambda = \{ac, abcv, bv\}, \quad \nu = \{ab, w, abw\}. \tag{7.3}$$

Note that the lines $\mu_4, \mu_5$ meet $\sigma$ in the points $abcv, bv \in \lambda$ and the line $\mu_6$ meets $\sigma$ in the point $ab \in \nu$. The latter point is also where the transversal $\tau_{456} = \{bw, aw, ab\}$ of the non-regulus $\{\mu_4, \mu_5, \mu_6\}$ meets $\sigma$. Consequently, see [7, Section 7.3.1], $\mathcal{S}_6$ is of class VIf.1. Let $J \in \mathrm{GL}(5,2)$ be defined as follows:

$$\begin{aligned} &J \text{ fixes each of } a, b, c; \\ &Ju = bcu,\ Jv = acv \quad (\text{and so } Jw = abw). \end{aligned} \tag{7.4}$$

Then $J$ has order 2 and stabilizes $\mathcal{S}_6$ by effecting the permutation $(\mu_1\mu_2)(\mu_3)(\mu_4\mu_5)(\mu_6)$ of its members. So, see [7, Table B.2], $\mathcal{G}_{\mathcal{S}_6} = \langle J \rangle \cong Z_2$.

**Theorem 7.1** *Let $\mathcal{C}_6$ be the Grassmann image of the partial spread $\mathcal{S}_6$ in (7.2) of class VIf.1, and consider the 4-flat $X = \mathcal{E}(\mathcal{C}_6)$. Denote by $Z, Z', Z''$ the annihilators of $X, X^{\#}, X^{\#\#}$, respectively. Then*

*(i) each of $X, X^{\#}, X^{\#\#}$ is a 4-flat in $\mathrm{PG}(4,2)$ of even type $(2,29)$; moreover $X^{\#\#\#} = X$, and $\mathcal{G}_X = \mathcal{G}_{X^{\#}} = \mathcal{G}_{X^{\#\#}} = \langle J \rangle \cong Z_2$;*

*(ii) each of $Z, Z', Z''$ is a 4-flat in $\mathrm{PG}(4,2)^*$ of even type $(2,29)$; moreover $Z^{\#} = Z', Z^{\#\#} = Z''$ and $Z^{\#\#\#} = Z$.*

**Proof.** Checked using Magma for a particular choice of $a, b, c, u, v, w$. Concerning the stabilizers $\mathcal{G}_X,\ \mathcal{G}_{X^{\#}},\ \mathcal{G}_{X^{\#\#}}$, their equality follows from the result $X^{\#\#\#} = X$. See remark 7.3 below for the fact that $\mathcal{G}_X = \langle J \rangle$. ■

Each of the 29 external points $t$ of $Z$ determines a point $\ker t$ of $\mathrm{PG}(4,2)$, see after (2.5). Denote by $\mathcal{K}(Z)$ the multiset of these 29 kernels and denote by $\mathcal{K}_r(Z)$ that subset of $\mathrm{PG}(4,2)$ consisting of those kernels which occur with multiplicity $r$.

**Theorem 7.2** *For each $r$ the subsets $\mathcal{K}_r(Z), \mathcal{K}_r(Z'), \mathcal{K}_r(Z'')$ of $\mathrm{PG}(4,2)$ are of the same size, namely $10, 16, 2, 3$ or $0$ according as $r = 0, 1, 2, 3$ or $r > 3$. Moreover (for $\mathcal{S}_6$ of class VIf.1, as in (7.2)) the kernels which occur with multiplicity $r > 1$ are as given, in abbreviated notation, in the following table.*

$$\begin{array}{cccc} & \mathcal{K}_r(Z) & \mathcal{K}_r(Z') & \mathcal{K}_r(Z'') \\ r=2 & \{a, ca\} & \{b, ab\} & \{c, bc\} \\ r=3 & \{c, u, bcu\} & \{a, v, cav\} & \{b, w, abw\} \end{array} . \tag{7.5}$$

**Proof.** Checked using Magma for a particular choice of $a, b, c, u, v, w$. ■

Given such a kernel pattern (7.5) arising from a partial spread $\mathcal{S}_6$ of class VIf.1, then the ordered triple $(a, b, c)$ is uniquely determined. So is the pair of ordered triples $\{(u,v,w),(bcu,cav,abw)\}$ of collinear points arising from the $r=3$ row of the table. The pair of ordered bases $\{\{a,b,c,u,v\},\{a,b,c,bcu,cav\}\}$ will be referred to as the *preferred bases* for $\mathcal{S}_6$.

**Remark 7.3** *Note that the involution $J$ maps one basis to the other. Observe that a partial spread $S_6$ of class VIf.1 can be uniquely reconstituted, see (7.2), knowing either of its preferred ordered bases. Since therefore any element of the stabilizer $\mathcal{G}_X$ of $X=\mathcal{E}(\mathcal{C}_6)$ must preserve the preferred pair of ordered bases, it follows that $\mathcal{G}_X=\langle J\rangle\cong Z_2$.*

**Remark 7.4** *The kernels of the 29 external points of $X, X^{\#}, X^{\#\#}$ were found by Magma to exhibit a similar pattern to that in the table (7.5), but of course these kernels are points in the dual space* $\mathrm{PG}(4,2)^*$.

Guided by the kernel pattern (7.5), let $C$ be that element of order 3 in $\mathrm{GL}(5,2)$ which effects the permutation

$$C:\quad (abc)(uvw). \tag{7.6}$$

Then from $\mathcal{S}_6=\{\mu_i\}_{1\le i\le 6}$ we may obtain two further partial spreads $\mathcal{S}'_6=C(\mathcal{S}_6)=\{\mu'_i\}_{1\le i\le 6}$ and $\mathcal{S}''_6=C^2(\mathcal{S}_6)=\{\mu''_i\}_{1\le i\le 6}$, such that the three partial spreads $\mathcal{S}_6,\mathcal{S}'_6,\mathcal{S}''_6$ form a $\langle C\rangle$-triplet of partial spreads of class VIf.1. Theorem 7.2 strongly suggests that the following theorem should hold.

**Theorem 7.5** *Let $X=\mathcal{E}(\mathcal{C}_6), X'=\mathcal{E}(\mathcal{C}'_6), X''=\mathcal{E}(\mathcal{C}''_6)$ be the even 4-flats, of type* $(2,29)$, *which arise from the foregoing partial spreads $\mathcal{S}_6,\mathcal{S}'_6,\mathcal{S}''_6$. Then $X^{\#}=X'$ and $X^{\#\#}=X''$. Moreover $\mathcal{G}_X=\mathcal{G}_{X'}=\mathcal{G}_{X''}=\langle J\rangle$.*

**Proof.** The stabilizers are equal since $C$ commutes with $J$. The rest was checked by Magma, using a particular choice of $a,b,c,u,v,w$. ■

## 7.3 Partial spreads $\mathcal{S}_6$ of class VIe.1

Again let $\{a,b,c,u,v\}$ be a basis for $V_5$. Setting $w=u+v$, consider the partial spread ${}^{e}\mathcal{S}_6=\{\mu_1,\ldots,\mu_6\}$, where, in our abbreviated notation,

$$\begin{aligned}&\mu_1=\{abv,bc,acv\},\quad \mu_2=\{cu,cv,w\},\quad \mu_3=\{abcw,bv,acu\},\\ &\mu_4=\{bu,ab,au\},\quad \mu_5=\{abu,bw,av\},\quad \mu_6=\{abc,v,abcv\}.\end{aligned} \tag{7.7}$$

Then ${}^{e}\mathcal{S}_6$ is of type $\mathsf{I}$, the sole regulus being $\rho=\{\mu_1,\mu_2,\mu_3\}$, whose extension to a spread in the ambient solid $\sigma$ of $\rho$ is $\rho\cup\{\lambda,\nu\}$ where

$$\lambda=\{a,bu,abu\},\quad \nu=\{bcw,aw,abc\}. \tag{7.8}$$

Note that the lines $\mu_4,\mu_5$ meet $\sigma$ in the points $bu,abu\in\lambda$ and the line $\mu_6$ meets $\sigma$ in the point $abc\in\nu$. Since the transversal $\tau_{456}=\{bu,bw,v\}$ of the non-regulus $\{\mu_4,\mu_5,\mu_6\}$ meets $\sigma$ in the point $bu$, it follows, see [7, Section 7.3.1], that ${}^{e}\mathcal{S}_6$ is of class VIe.1 or VIe.2. Since we find that the 4-flat ${}^{e}X:=\mathcal{E}({}^{e}\mathcal{C}_6)$ is of even type $(2,29)$ it follows from Table 2 in section 2.3 that ${}^{e}\mathcal{S}_6$ is of class VIe.1. Recall from [7, Table B.2] that ${}^{e}\mathcal{S}_6$ has no symmetry: $\mathcal{G}_{{}^{e}\mathcal{S}_6}=\langle I\rangle$.

Upon applying the involution $J$ in (7.4) to ${}^e\mathcal{S}_6$ we obtain a further partial spread ${}^{ee}\mathcal{S}_6$ of class VIe.1:

$$\begin{aligned}&\mu_1 = \{bcv, bc, v\}, \quad \mu_2 = \{bu, av, abw\}, \quad \mu_3 = \{cw, abcv, abu\},\\ &\mu_4 = \{cu, ab, abcu\}, \quad \mu_5 = \{acu, aw, cv\}, \quad \mu_6 = \{abc, acv, bv\}.\end{aligned} \tag{7.9}$$

We now consider the 4-flats ${}^eX = \mathcal{E}({}^e\mathcal{C}_6), {}^{ee}X = \mathcal{E}({}^{ee}\mathcal{C}_6)$, where ${}^e\mathcal{C}_6, {}^{ee}\mathcal{C}_6$ are the Grassmann images of the partial spreads ${}^e\mathcal{S}_6, {}^{ee}\mathcal{S}_6$ , each of class VIe.1, in (7.7), (7.9).

**Theorem 7.6** *The 4-flats* ${}^eX = \mathcal{E}({}^e\mathcal{C}_6), {}^{ee}X = \mathcal{E}({}^{ee}\mathcal{C}_6)$ *arising from the partial spreads* ${}^e\mathcal{S}_6, {}^{ee}\mathcal{S}_6$ *of class VIe.1 coincide, each coinciding with the 4-flat* $X = \mathcal{E}(\mathcal{C}_6)$ *arising from the partial spread* $\mathcal{S}_6$ *of class VIf.1 given in (7.2).*

**Proof.** Checked using Magma, for a particular choice of $a, b, c, u, v, w$. ■

**Corollary 7.7** *Theorems 7.1 and 7.2 still hold if* $X$ *and* $Z$ *are replaced by* ${}^eX$ *and* ${}^eZ := ({}^eX)^O$*, or by* ${}^{ee}X$ *and* ${}^{ee}Z := ({}^{ee}X)^O$*.*

Of course from ${}^e\mathcal{S}_6$ we may obtain two further partial spreads ${}^e\mathcal{S}_6'$, ${}^e\mathcal{S}_6''$ such that the three partial spreads ${}^e\mathcal{S}_6, {}^e\mathcal{S}_6', {}^e\mathcal{S}_6''$ form a $\langle C\rangle$-triplet of partial spreads of class VIe.1. A corresponding statement holds for the three partial spreads ${}^{ee}\mathcal{S}_6, {}^{ee}\mathcal{S}_6', {}^{ee}\mathcal{S}_6''$, all of class VIe.1, obtained in a similar way, but using (7.9) instead of (7.7).

## 7.4 Partial spreads $\mathcal{S}_6$ of class VIh.1

Again let $\{a, b, c, u, v\}$ be a basis for $V_5$. Setting $w = u + v$, consider the partial spread ${}^h\mathcal{S}_6 = \{\mu_1, \ldots, \mu_6\}$, where, in our abbreviated notation,

$$\begin{aligned}&\mu_1 = \{bc, abw, acw\}, \quad \mu_2 = \{av, abu, bw\}, \quad \mu_3 = \{abcv, v, abc\},\\ &\mu_4 = \{bu, ab, au\}, \quad \mu_5 = \{cu, w, cv\}, \quad \mu_6 = \{acu, bv, abcw\}.\end{aligned} \tag{7.10}$$

Then ${}^h\mathcal{S}_6$ is of type $\mathsf{L}$, the two reguli being $\rho_{123} = \{\mu_1, \mu_2, \mu_3\}$ and $\rho_{145} = \{\mu_1, \mu_4, \mu_5\}$. Hence, see [7, Table B.2], ${}^h\mathcal{S}_6$ is of class VIh.1 or VIh.2. Upon appeal to [7, Section 4.2] we see that ${}^h\mathcal{S}_6$ is of class VIh.1 and that ${}^h\mathcal{S}_6$ has no symmetry: $\mathcal{G}_{{}^h\mathcal{S}_6} = \langle I\rangle$. (Alternatively the assignment to class VIh.1 follows from Table 2 in section 2.3, since we find that the 4-flat ${}^hX := \mathcal{E}({}^h\mathcal{C}_6)$ is of even type $(2, 29)$.

Upon applying the involution $J$ in (7.4) to ${}^h\mathcal{S}_6$ we obtain a further partial spread ${}^{hh}\mathcal{S}_6$ of class VIh.1:

$$\begin{aligned}&\mu_1 = \{bc, w, bcw\}, \quad \mu_2 = \{cv, acu, aw\}, \quad \mu_3 = \{bv, acv, abc\},\\ &\mu_4 = \{cu, ab, abcu\}, \quad \mu_5 = \{bu, abw, av\}, \quad \mu_6 = \{abu, abcv, cw\}.\end{aligned} \tag{7.11}$$

We now consider the 4-flats ${}^hX = \mathcal{E}({}^h\mathcal{C}_6), {}^{hh}X = \mathcal{E}({}^{hh}\mathcal{C}_6)$, where ${}^h\mathcal{C}_6, {}^{hh}\mathcal{C}_6$ are the Grassmann images of the partial spreads ${}^h\mathcal{S}_6, {}^{hh}\mathcal{S}_6$ , each of class VIh.1, in (7.10), (7.11).

**Theorem 7.8** *The 4-flats* ${}^hX = \mathcal{E}({}^h\mathcal{C}_6), {}^{hh}X = \mathcal{E}({}^{hh}\mathcal{C}_6)$ *arising from the partial spreads* ${}^h\mathcal{S}_6, {}^{hh}\mathcal{S}_6$ *of class VIh.1 coincide, each coinciding with the 4-flat* $X = \mathcal{E}(\mathcal{C}_6)$ *arising from the partial spread* $\mathcal{S}_6$ *of class VIf.1 given in (7.2).*

**Proof.** Checked using Magma, for a particular choice of $a, b, c, u, v, w$. $\blacksquare$

**Corollary 7.9** *Theorems 7.1 and 7.2 still hold if $X$ and $Z$ are replaced by ${}^hX$ and ${}^hZ := ({}^hX)^O$, or by ${}^{hh}X$ and ${}^{hh}Z := ({}^{hh}X)^O$.*

Of course from ${}^h\mathcal{S}_6$ we may obtain two further partial spreads ${}^h\mathcal{S}_6'$, ${}^h\mathcal{S}_6''$ such that the three partial spreads ${}^h\mathcal{S}_6, {}^h\mathcal{S}_6', {}^h\mathcal{S}_6''$ form a $\langle C\rangle$-triplet of partial spreads of class VIh.1. A corresponding statement holds for the three partial spreads ${}^{hh}\mathcal{S}_6, {}^{hh}\mathcal{S}_6', {}^{hh}\mathcal{S}_6''$, all of class VIe.1, obtained in a similar way, but using (7.11) instead of (7.10).

In partial summary, if the "associate 3-cycle" $X, X^{\#}, X^{\#\#}, X^{\#\#\#} = X$ of even 4-flats of type $(2, 29)$ arises, via $X = \mathcal{E}(\mathcal{C}_6)$ from a partial spread $\mathcal{S}_6$ of class Vf.1, then it also arises from two partial spreads of class VIe.1 and from two partial spreads of class VIh.1.

# 8 Flats arising from a regulus-free $\mathcal{S}_8$

Consider a direct sum decomposition $V_5 = V_3 \oplus V_2$ of $V_5$, so that projectively we have $\mathrm{PG}(4,2) = \langle \alpha, \mu \rangle$, where $\alpha = \mathbb{P}V_3$ is a plane and $\mu = \mathbb{P}V_2$ is a disjoint line. Let us fix upon a particular subgroup $\langle A_3 \rangle \cong Z_7$ of $\mathrm{GL}(V_3)$, with generator $A_3$ having characteristic polynomial $x^3 + x + 1$. We also choose $C_3 \in \mathrm{GL}(V_3)$ in the normalizer of $A_3$ to satisfy $C_3 A_3 (C_3)^{-1} = (A_3)^2$, and set

$$A := A_3 \oplus I_2, \; C := C_3 \oplus C_2, \qquad F_{21} := \langle A, C \rangle, \tag{8.1}$$

where $C_2$ is an element of $\mathrm{GL}(V_2)$ of order 3. Thus the subgroup $F_{21} \cong Z_7 \rtimes Z_3$ of $\mathrm{GL}(5,2)$ stabilizes both $\mu$ and $\alpha$. Given a basis $\{u, v\}$ for $V_2$, let us fix $C_2$ to be that element of $\mathrm{GL}(V_2)$ which effects $u \mapsto v \mapsto v + u$. Next, for $a \in V_3$, we define the element $J_a$ of $\mathrm{GL}(5,2)$ by

$$J_a x = x, \; x \in V_3, \qquad J_a u = u + A^{-1} a, \quad J_a v = v + a. \tag{8.2}$$

Since $J_a J_b = J_{a+b}$, note that $\mathcal{J} := \{J_a\}_{a \in V_3}$ is a subgroup $\cong (Z_2)^3$ of $\mathrm{GL}(5,2)$ whose non-identity elements are involutions. Since $A J_a A^{-1} = J_{Aa}$ and $C J_a C^{-1} = J_{CA^{-1}a}$ note that $\mathcal{G} := \langle \mathcal{J}, F_{21} \rangle$ is a subgroup of $\mathrm{GL}(5,2)$ of structure

$$\mathcal{G} = \mathcal{J} \rtimes F_{21} \cong 2^3 : (7:3). \tag{8.3}$$

**Theorem 8.1** *([16, Theorem 2.2]) If*

$$\mathcal{S}_8 = \{\mu_a\}_{a \in V_3}, \qquad \textit{where } \mu_a := J_a(\mu), \tag{8.4}$$

*(and so $\mu_0 = \mu$), then $\mathcal{S}_8$ is a regulus-free partial spread in* $\mathrm{PG}(4,2)$*. Moreover $\mathcal{G}_{\mathcal{S}_8} = \mathcal{G} = \mathcal{J} \rtimes F_{21}$, with the generators $J_b, A, C$ of $\mathcal{G}$ acting as follows on the lines of $\mathcal{S}_8$:*

$$J_b(\mu_a) = \mu_{a+b}, \qquad A(\mu_a) = \mu_{Aa}, \qquad C(\mu_a) = \mu_{CA^{-1}a}. \qquad \blacksquare$$

All regulus-free partial spreads $\mathcal{S}_8$ in $\mathrm{PG}(4,2)$ belong to the class VIIIa.1. Further information about such partial spreads can be found in Appendix C. The plane $\alpha$ is referred to as the *invariant plane* of $\mathcal{S}_8$; it consists of the seven points of $\mathrm{PG}(4,2)$ left uncovered by the eight lines of $\mathcal{S}_8$, and it is the only plane in $\mathrm{PG}(4,2)$ which is stabilized by $\mathcal{G}_{\mathcal{S}_8}$. The Grassmann images in $\mathcal{G}_{1,4,2}$ of the seven lines $\lambda \subset \alpha$ form an internal plane $P(\alpha) \subset \mathrm{PG}(9,2)$, and the Grassmann image in $\mathcal{G}_{2,4,2}$ of the plane $\alpha$ is an internal point $p$ of $\mathrm{PG}(9,2)^* = \mathbb{P}(\wedge^3 V_5)$. Let $\mathrm{orb}(2\gamma)^*$ denote the orbit of external planes in $\mathrm{PG}(9,2)^*$ which is the analogue of the orbit $\mathrm{orb}(2\gamma)$ of external planes in $\mathrm{PG}(9,2)$. A representative $P^*$ of $\mathrm{orb}(2\gamma)$ is given in the next theorem.

**Theorem 8.2** *If, in the foregoing set-up, we define $P^* \subset \mathrm{PG}(9,2)$ by*

$$P^* := \{t_l\}_{l \in P(\alpha)}, \qquad \textit{where } t_l := l \wedge u + T_A l \wedge v, \tag{8.5}$$

*then $P^* \in \mathrm{orb}(2\gamma)^*$ and $\mathcal{G}_{P^*} = \mathcal{G}$.*

**Proof.** Because $P(\alpha)$ is a plane, and since $T_A$ is a linear mapping, $P^*$ is a plane. Moreover $P^*$ is external to $\mathcal{G}_{2,4,2}$. But since $\hat{T}_A t_l = t_{T_A l}$, the stabilizer

$\mathcal{G}_{P^*}$ contains a subgroup $\cong Z_7$, and so $P^* \in \mathrm{orb}(2\gamma)^*$. Finally, we see that $\mathcal{G}_{P^*} = \mathcal{G}$ after checking the relations

$$\text{(i) } \hat{T}_C t_l = t_{l'}, \quad \text{where } l' = T_{A^2C} l; \quad \text{(ii) } \hat{T}_{J_a} t_l = t_l. \tag{8.6}$$

In the course of checking (i) we use, in addition to $CAC^{-1} = A^2$, the fact that $T_{A_3} \in \mathrm{GL}(\wedge^2 V_3)$ has characteristic polynomial $x^3 + x^2 + 1$. That $P^*$ is fixed pointwise under the action of $\mathcal{J}$, see (ii), is seen to hold upon noting that

$$l \wedge A^{-1}a = T_A l \wedge a, \tag{8.7}$$

this last holding since $A^{-1}a \in \lambda$ if and only if $a \in A(\lambda)$. ■

As well as the external plane $P^*$ we will also be interested in the solid $\langle p, P^* \rangle \subset \mathrm{PG}(9,2)^*$, which is of type $(1,14)$. The stabilizer $\mathcal{G}_{\langle p,P^*\rangle}$ of the solid $\langle p, P^* \rangle$ is larger than $\mathcal{G}_{P^*} = \mathcal{G}$. For if $J_{a,a'} \in \mathrm{GL}(5,2)$, for $a, a' \in V_3$, is defined by

$$J_{a,a'}x = x, \ x \in V_3, \qquad J_{a,a'}u = u + a, \quad J_{a,a'}v = v + a',$$

then we see immediately that $\hat{T}_{J_{a,a'}}$ fixes $p$ and sends $t_l$ to $\kappa p^* + t_l \in \langle p, P^* \rangle$, for some $\kappa \in \mathrm{GF}(2)$. Defining $\mathcal{J}_{\mathrm{big}} := \{J_{a,a'}\}_{a,a' \in \alpha} \cong (Z_2)^6$, the stabilizer of $\langle p, P^* \rangle$ is in fact

$$\mathcal{G}_{\mathrm{big}} = \mathcal{J}_{\mathrm{big}} \rtimes \langle A, C \rangle \cong 2^6 : (7:3). \tag{8.8}$$

Under the action of $\mathcal{J}_{\mathrm{big}}$ the plane $P^*$ lies on orbit of eight planes inside $\langle p, P^* \rangle$, namely the eight planes which do not pass through the point $p$, all eight belonging to $\mathrm{orb}(2\gamma)^*$.

Let $\mathcal{C}_8 = \{m_a\}_{a \in V_3} \subset \mathcal{G}_{1,4,2}$ be the 8-cap arising from the regulus-free partial spread $\mathcal{S}_8$ in theorem 8.1. In the rest of this section we wish to study the flats $\mathcal{E}(\mathcal{C}_8)$ and $\langle \mathcal{C}_8 \rangle$. As shown in [16, Theorem 3.2], $\mathcal{E}(\mathcal{C}_8)$ is a 5-flat of type $(7,56)$, meeting $\mathcal{G}_{1,4,2}$ in the plane $P(\alpha)$, and $\langle \mathcal{C}_8 \rangle$ is a 6-flat of type $(15,112)$, meeting $\mathcal{G}_{1,4,2}$ in the 15-set $\mathcal{C}_8 \cup P(\alpha)$.

**Theorem 8.3** *(i) If $X = \langle \mathcal{C}_8 \rangle$ then $X^O = P^*$, as given in eq. (8.5).*
*(ii) If $X = \mathcal{E}(\mathcal{C}_8)$ then $X^O = \langle p, P^* \rangle$.*

**Proof.** (i) Since $X$ is a 6-flat, $X^{\mathrm{O}}$ is a plane. So to prove the result we need to check that $t_l \wedge m_a = 0$ for each $l \in P(\alpha)$, $a \in V_3$. Since $m_a = (u + A^{-1}a) \wedge (v + a)$, we have $t_l \wedge m_a = (l \wedge A^{-1}a + T_A l \wedge a) \wedge u \wedge v$, which $= 0$, see (8.7).

(ii) Since $X$ is a 5-flat contained in $\langle \mathcal{C}_8 \rangle$, then $X^{\mathrm{O}}$ is a 3-flat which contains the plane $\langle \mathcal{C}_8 \rangle^{\mathrm{O}} = P^*$. Hence it suffices to check that $p \in \mathcal{E}(\mathcal{C}_8)^{\mathrm{O}}$. This is so: for since the line $\mu_a$ and $\mu_b$ are skew to the plane $\alpha$, we have $p \wedge (m_a + m_b) = e + e = 0$. ■

**Corollary 8.4** *(i) $\mathcal{G}_{\langle \mathcal{C}_8 \rangle} = \mathcal{G}_{P^*} = \mathcal{G}$; (ii) $\mathcal{G}_{\mathcal{E}(\mathcal{C}_8)} = \mathcal{G}_{\langle p,P^* \rangle} = \mathcal{G}_{big}$.*

We now wish to determine the associates of (a) $\langle \mathcal{C}_8 \rangle$ and (b) $\mathcal{E}(\mathcal{C}_8)$. Concerning (a), observe that the stabilizer $\mathcal{G}_{\langle \mathcal{C}_8 \rangle}$ has four internal orbits, namely $\mathcal{C}_8, P(\alpha), \Omega_{56}$ and $\Omega_{84}$. Here the 56 elements of $\Omega_{56}$ are the images of the 56 transversals of $\mathcal{S}_8$, and the 84 elements of $\Omega_{84}$ are the images of the 84 lines of $\mathrm{PG}(4,2)$ which meet $\alpha$ in a point. We claim that $\langle \mathcal{C}_8 \rangle^{\#}$ contains no points of $\mathcal{G}_{1,4,2}$. Certainly $\mathcal{C}_8 \cap \langle \mathcal{C}_8 \rangle^{\#} = \emptyset$, since $Q^{\ddagger}(m_a, m_b, m_c, m_d, m_e) = 1$ for any 5-set $\{m_a, m_b, m_c, m_d, m_e\} \subset \mathcal{C}_8$. (Recall that $\mathcal{C}_5 = \{m_a, m_b, m_c, m_d, m_e\}$

is of class Vd.1, whence $\langle \mathcal{C}_5 \rangle$ is even.) Secondly for a line $\lambda \subset \alpha$ one can choose $\mu_a, \mu_b, \mu_c, \mu_d \in \mathcal{S}_8$ such that $\{\mu_a, \mu_b, \mu_c, \mu_d, \lambda\}$ is of class Vd.1, whence $Q^\ddagger(m_a, m_b, m_c, m_d, l) = 1$. Next consider a transversal $\tau$ of $\mathcal{S}_8$; then $\{\tau, \mu_a, \mu_e\}$ is a regulus for a uniquely determined pair $\mu_a, \mu_e$ of lines of $\mathcal{S}_8$. If the remaining three lines of $\mathcal{S}_8$ which are skew to $\tau$ are $\mu_b, \mu_c, \mu_d$ then $\{\mu_a, \mu_b, \mu_c, \mu_d, \tau\}$ is of class Vd.1, whence $Q^\ddagger(m_a, m_b, m_c, m_d, t) = 1$. Finally if $\lambda$ meets $\alpha$ in a point then one finds that one can choose $\mu_a, \mu_b, \mu_c, \mu_d \in \mathcal{S}_8$ such that $\{\mu_a, \mu_b, \mu_c, \mu_d, \lambda\}$ is of class Vf.1, and so again $Q^\ddagger(m_a, m_b, m_c, m_d, l) = 1$.

**Theorem 8.5** *(i) If* $X = \langle \mathcal{C}_8 \rangle$ *then* $X^\# = \emptyset$.
*(ii) If* $X = \mathcal{E}(\mathcal{C}_8)$ *then* $X^\# = X$.

**Proof.** (i) On the assumption that $X^\#$ is non-empty, then, by our preamble, $X^\#$ is an external flat which is stabilized by the subgroup $\mathcal{G} = \mathcal{J} \rtimes \langle A, C \rangle$. Now $\mathcal{G}$ has an invariant plane $\alpha$, *but leaves no line of* PG$(4,2)$ *invariant.* But the only external flats which are stabilized by a group of structure $2^3 : (7 : 3)$ are the planes of orb$(2\gamma)$, and these have stabilizers conjugate to the group $\mathcal{G}' = \mathcal{J}' \rtimes \langle A, C \rangle$ defined in [16, Section 2.2], and this group $\mathcal{G}'$ has an invariant line, but leaves no plane of PG$(4,2)$ invariant. Hence $X^\#$ is empty.

(ii) As was shown in [16, Theorem 3.2], the 5-flat $\mathcal{E}(\mathcal{C}_8)$ meets $\mathcal{G}_{1,4,2}$ in the internal plane $P(\alpha)$. So every 4-flat of $\mathcal{E}(\mathcal{C}_8)$ is odd, either meeting $\mathcal{G}_{1,4,2}$ in $P(\alpha)$ or else in a line of $P(\alpha)$. Hence $X := \mathcal{E}(\mathcal{C}_8)$ is contained in $Y := X^\#$. Consequently, from theorem 8.3(ii), $Y^{\mathrm{O}} \subseteq X^{\mathrm{O}} = \langle p, P^* \rangle$. Since $\mathcal{G}_X = \mathcal{G}_{\mathrm{big}}$, see corollary 8.4, there are three possibilities for $Y^{\mathrm{O}}$, namely (a) $Y^{\mathrm{O}} = \langle p, P^* \rangle$, and so $Y = X$; (b) $Y^{\mathrm{O}} = \{p\}$, and so $Y = P(\alpha)$; (c) $Y^{\mathrm{O}} = \emptyset$, and so $Y = \mathrm{PG}(9,2)$. Possibility (b) entails that for each $\mathcal{C}_5 \subset \mathcal{C}_8$ we have $\mathcal{E}(\mathcal{C}_5)^\# = P(\alpha)$, in contradiction to the result in theorem 3.13. Similarly possibility (c) is ruled out. Hence (a) holds: $X^\# = X$. ■

**Theorem 8.6** *If* $X = \mathcal{E}(\mathcal{C}_6)$ *where* $\mathcal{S}_6$ *is of class VIa.1, then* $X^\# = \mathcal{E}(\mathcal{C}_8)$ *where* $\mathcal{S}_8$ *denotes the unique extension of* $\mathcal{S}_6$*. to a partial spread of class VIIIa.1.*

**Proof.** It follows from theorem 8.5 that $\mathcal{E}(\mathcal{C}_6)^\#$ contains $\mathcal{E}(\mathcal{C}_8)^\# = \mathcal{E}(\mathcal{C}_8)$. In the other direction, $\mathcal{E}(\mathcal{C}_6)^\# \subseteq \mathcal{E}(\mathcal{C}_5)^\#$ for each $\mathcal{C}_5 \subset \mathcal{C}_6$, and by using theorem 3.13 to compute one of these $\mathcal{E}(\mathcal{C}_5)^\#$ we find that it lies on a $\mathcal{G}_{\mathcal{S}_6} (\cong Z_6)$-orbit inside $\langle p, P^* \rangle^{\mathrm{O}} = \mathcal{E}(\mathcal{C}_8)$. Hence $\mathcal{E}(\mathcal{C}_6)^\# = \mathcal{E}(\mathcal{C}_8)$. ■

# 9 Some speculations concerning even 4-flats

Because we are exploring new territory there is a temptation to speculate prematurely! For example, provoked by the result in eq. (6.1), we initially wondered whether the following held in general: *if* $Y$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *such that* $G_{Y^{\#}} = G_Y$, *then* $Y^{\#\#} = Y$. But while this holds in the case $Y := X^{\#}$, with $X$ as in (6.1), it is false for $Y = X$ in (7.1). Nevertheless (albeit very tentatively!) we put forward the following conjectures/speculations concerning even 4-flats in $\mathrm{PG}(9,2)$. All we can say in support of these is that they hold up for those even 4-flats treated in the present paper.

**Conjecture 9.1** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then its associate* $X^{\#}$ *is an even 4-flat.*

**Conjecture 9.2** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then its annihilator* $X^{O}$ *is an even 4-flat in* $\mathrm{PG}(9,2)^*$.

**Conjecture 9.3** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *of type* $(n_1, n_2)$ *then its annihilator* $X^{O}$ *is a 4-flat in* $\mathrm{PG}(9,2)^*$ *of type* $(n_1, n_2)$.

**Conjecture 9.4** *If* $X$ *is an even 4-flat in* $\mathrm{PG}(9,2)$ *then* $(X^{\#})^{O} = (X^{O})^{\#}$.

For any flat $X$ we can consider the '#-sequence' $(X_0, X_1, X_2, \dots)$, where $X_0 = X$ and $X_{r+1} = (X_r)^{\#}$. Let us assume now that conjecture 9.1 holds. Then if $X$ is an even 4-flat its sequence is such that each member $X_r$ is also an even 4-flat. (Moreover, corollary 4.2, $X_r$ is disjoint from $X_{r+1}$.) For any even 4-flat $X$ we can therefore define an ordered pair $(r, s)$ of integers $r, s$, with $0 \le r < s$, such that the members of the finite sequence $(X_0, X_1, \dots X_s)$ are distinct and such that $(X_s)^{\#} = X_r$. From (i) theorem 4.8, and from the results in (ii) section 6 and (iii) section 7, we have examples of even 4-flats $X$ for which (i) $(r, s) = (0, 1)$, (ii) $(r, s) = (1, 2)$ and (iii) $(r, s) = (0, 2)$. However at this stage of our investigation we deem it premature to conjecture that no other values of $(r, s)$ are possible!

# A Appendix: bivectors, trivectors and duality

In this paper we started out from a vector space $V_5 = V(5,2)$, and its projective space $\mathrm{PG}(4,2) = \mathbb{P}V_5$, and dealt chiefly with the concomitant space $V_{10} := \wedge^2 V_5$ of bivectors, and the latter's projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$. In this appendix we provide some details concerning relevant duality material.

First take note that each of the following three vector spaces can serve as the dual $V_{10}^*$ of the space $V_{10}$:
(i) the space $L(V_{10})$ of linear forms on $V_{10}$, consisting of all linear maps $f : V_{10} \mapsto \mathrm{GF}(2)$;
(ii) the space $\wedge^2 V_5^* := \wedge^2(V_5^*)$ of dual bivectors;
(iii) the space $\wedge^3 V_5$ of trivectors.

The choice of the space (ii) as dual to $\wedge^2 V_5$ is just the $r = 2$ instance of the standard choice of $\wedge^r V_5^*$ as dual to $\wedge^r V_5$. This last choice is valid because that bilinear pairing $< \cdot|\cdot >: \wedge^r V_5^* \times \wedge^r V_5 \to \mathrm{GF}(2)$ which satisfies

$$< f_1 \wedge \ldots \wedge f_r | v_1 \wedge \ldots \wedge v_r > = \det[f_i(v_j)]_{i,j\in\{1,2,\ldots,r\}}, \tag{A.1}$$

for all $f_i \in V_5^*$, $v_j \in V_5$, is nondegenerate. Concerning the choice of the space (iii) as dual to $\wedge^2 V_5$, take note that, *because we are working over* GF(2), each of the 1-dimensional spaces $\wedge^5 V_5^*$ and $\wedge^5 V_5$ has a *unique* basis vector, say $e^*$ and $e$. Consequently we can *uniquely* define a natural nondegenerate bilinear pairing $\lceil \cdot, \cdot \rfloor$ of $\wedge^3 V_5$ with $\wedge^2 V_5$ by

$$t \wedge b = \lceil t, b \rfloor\, e, \quad t \in \wedge^3 V_5,\ b \in \wedge^2 V_5, \tag{A.2}$$

that is by $\lceil t, b \rfloor =< e^* | t \wedge b >$ .

The three spaces (i), (ii) and (iii) are of course isomorphic, with $f \in L(V_{10})$, $b^* \in \wedge^2 V_5^*$, $t \in \wedge^3 V_5$ being isomorphic images whenever they satisfy

$$f(b) =< b^* | b > = \lceil t, b \rfloor, \qquad \text{for all } b \in \wedge^2 V_5. \tag{A.3}$$

Concerning the isomorphic spaces $\wedge^2 V_5^*$ and $\wedge^3 V_5$, the isomorphisms are given by the *unique* Poincaré isomorphisms

$$\perp : \wedge^2 V_5^* \to \wedge^3 V_5 \qquad \text{and} \qquad \perp' : \wedge^3 V_5 \to \wedge^2 V_5^* \tag{A.4}$$

defined, see for example [17, Section 9.6.2], via the properties

$$< t^* | \perp b^* > =< t^* \wedge b^* | e > \qquad \text{and} \qquad <\perp' t | b > =< e^* | t \wedge b >, \tag{A.5}$$

these holding for all $t^* \in \wedge^3 V_5^*$, $b^* \in \wedge^2 V_5^*$, $t \in \wedge^3 V_5$, $b \in \wedge^2 V_5$. Consequently

$$\lceil t, b \rfloor =< b^* | b > \quad \text{whenever} \quad t = \perp b^*. \tag{A.6}$$

The isomorphisms $\perp$ and $\perp'$ are mutual inverses and they satisfy

$$\perp (e^i \wedge e^j) = e_k \wedge e_l \wedge e_m \qquad \text{and} \qquad \perp' (e_k \wedge e_l \wedge e_m) = e^i \wedge e^j, \tag{A.7}$$

where $\{e^1, e^2, e^3, e^4, e^5\}$, $\{e_1, e_2, e_3, e_4, e_5\}$ are any pair of dual bases for $V_5^*$, $V_5$, and where $ijklm$ is any permutation of 12345; see e.g. [17, eq. (9.6.20)].

Each of the elements $f \in L(V_{10})$, $b^* \in \wedge^2 V_5^*$, $t \in \wedge^3 V_5$ in (A.3) gives rise to an associated alternating bilinear form on $V_5$, say $f(\cdot,\cdot)$, $b^*(\cdot,\cdot)$, $t(\cdot,\cdot)$, given by

$$f(v_1, v_2) = f(v_1 \wedge v_2), \quad b^*(v_1, v_2) =< b^* | v_1 \wedge v_2 >, \quad t(v_1, v_2) = \lceil t, v_1 \wedge v_2 \rfloor. \tag{A.8}$$

Observe therefore that the bilinear form $t(\cdot,\cdot)$ is such that

$$t \wedge v_1 \wedge v_2 = t(v_1, v_2)e, \tag{A.9}$$

and that if $t =\perp b^*$ then $t(v_1, v_2) = b^*(v_1, v_2)$.

Given a particular $t \in \wedge^3 V_5$ each flat $\alpha \subset \mathrm{PG}(4,2)$ determines a 'polar' flat $\alpha^t$ defined by

$$\alpha^t = \{v \in \mathrm{PG}(4,2) | t(x,v) = 0 \quad \text{for all } x \in \alpha\}. \tag{A.10}$$

A flat $\alpha \subset \mathrm{PG}(4,2)$ is termed *isotropic* if $\alpha \subset \alpha^t$, and *selfpolar* if $\alpha = \alpha^t$. In the case when $t \in \mathrm{Rk}_2^*$ is the Grassmann image of a plane $\pi \subset \mathrm{PG}(4,2)$ it follows from the next lemma that *a line $\lambda$ is isotropic if and only if $\lambda$ meets $\pi$.*

**Lemma A.1** *Let $l \in \mathcal{G}_{1,4,2} \subset \mathrm{PG}(9,2)$ and $t \in \mathcal{G}_{2,4,2} \subset \mathrm{PG}(9,2)^*$ be the Grassmann images of a line $\lambda$ and a plane $\pi$ in $\mathrm{PG}(4,2)$. Then*

$$t \wedge l = 0 \qquad \textit{if and only if} \qquad \pi \textit{ meets } \lambda. \tag{A.11}$$

**Proof.** The element $t \wedge l \in \wedge^5 V_5$ is $\neq 0$ if and only if $\langle \pi, \lambda \rangle = \mathrm{PG}(4,2)$. ■

Consider now the case when $t \in \mathrm{Rk}_4^*$. In this case, if a hyperplane $\mathbb{P}V_4 = \sigma \subset \mathrm{PG}(4,2)$ does not contain the kernel $k = \ker t$, see after (2.5), then the restriction of $t(\cdot,\cdot)$ to $V_4$ is a nondegenerate symplectic form; consequently such hyperplanes are equipped with a symplectic polarity. From this, bearing in mind that the polars of all flats contain $k$, it follows, in the case $t \in \mathrm{Rk}_4^*$, that

$$\text{if } \alpha \subset \mathrm{PG}(4,2) \text{ is an } r\text{-flat then } \begin{cases} \alpha^t \text{ is a } (3-r)\text{-flat if } k \notin \alpha \\ \alpha^t \text{ is a } (4-r)\text{-flat if } k \in \alpha \end{cases}. \tag{A.12}$$

Consequently if $\alpha$ is selfpolar then, in the case $t \in \mathrm{Rk}_4^*$, $\alpha$ is necessarily a plane which contains $k$.

**Lemma A.2** *Suppose that the trivector $t$ is in $\mathrm{Rk}_4^*$. Then the 35 planes which contain $k = \ker t$ consist of 15 selfpolar planes together with 20 nonselfpolar planes, the latter comprising 10 polar pairs $\{\alpha, \alpha^t\}$. Moreover there exist precisely 75 isotropic lines, namely the 15 lines $\mathrm{st}(k)$ which pass through $k$ together with the $15 \times 4 = 60$ lines which do not pass through $k$ but which lie in one of the 15 selfpolar planes.*

**Proof.** Recall that if a $\mathrm{PG}(3,2)$ is equipped with a symplectic polarity then its 35 lines consist of 15 selfpolar lines together with 20 nonselfpolar lines which comprise 10 polar pairs $\{\lambda, \lambda^\perp\}$. The lemma now follows quickly from our preamble. ■

# B Appendix: $Z_5$-orbits

## B.1 Elements of order 5: generalities

Let $B \in \mathrm{GL}(n+1,2)$ be of order 5, generating a subgroup $\langle B\rangle \cong Z_5$. Note that we must have $n+1>3$, since $5 \nmid |\,\mathrm{GL}(3,2)|$. Moreover, since we work in characteristic 2, and $2 \nmid |Z_5|$, Maschke's theorem applies: the vector space $V_{n+1}=V(n+1,2)$ decomposes into a direct sum of subspaces upon which $B$ acts irreducibly.

*Suppose now that* $3 \le n \le 6$. It follows that we have the direct sum decomposition

$$B = B_4 \oplus I_{n-3}, \qquad \text{acting upon} \qquad V_{n+1} = V_4 \oplus V_{n-3}, \tag{B.1}$$

with $B_4$ an element of $\mathrm{GL}(V_4)$ of order 5 and $I_{n-3}$ the identity in $\mathrm{GL}(V_{n-3})$. Note that the minimal polynomial of $B$ is $\mu_B = t^5+1 = (t+1)f$ where $f$ is the irreducible polynomial $f := t^4+t^3+t^2+t+1$, and so we have $V_4 = \ker(B^4+B^3+B^2+B+I)$ and $V_{n-3} = \ker(B+I)$. The next lemma lists some consequent properties of elements of $\mathrm{GL}(n+1,2)$, $3 \le n \le 6$, of order 5.

**Lemma B.1** *Let* $B \in \mathrm{GL}(n+1,2)$ *be of order 5 and act on* $\mathrm{PG}(n,2)$, $3 \le n \le 6$. *Then:*
*(i)* $n=3$*:* $\mathrm{PG}(3,2)$ *is a union of three 5-cycles of* $B$*, each of which forms a hyperbasis (i.e. base + unit point) for* $V_4$*.*
*(ii)* $n=4$*:* $\mathrm{PG}(4,2)$ *is a union of six 5-cycles of* $B$*, together with one fixed point; three of the 5-cycles are bases for* $V_5$*, and the other three 5-cycles lie in the common even hyperplane of the first three 5-cycles.*
*(iii)* $n=5$*:* $\mathrm{PG}(5,2)$ *is a union of twelve 5-cycles, together with a line of fixed points;*
*(iv)* $n=6$*:* $\mathrm{PG}(6,2)$ *is a union of twenty-four 5-cycles, together with a plane of fixed points.*
*In each of the cases* $n=3,4,5,6$ *there is a single class of elements of order 5 in* $\mathrm{GL}(n+1,2)$*.*

**Proof.** Concerning (i), applying $B_4$ to any nonzero $v \in V_4$ generates a 5-cycle whose five members $(B_4)^i v$, $i=0,1,2,3,4$, sum to zero, since $f(B_4)=0$, and so form a hyperbasis (no other relations holding between the five because $B_4$ acts irreducibly). It follows that $\mathrm{GL}(4,2)$ contains just one class of elements of order 5. The rest now follows upon using the direct sum decompositions (B.1). ■

## B.2 Elements of order 5 in $\mathrm{GL}(5,2)$

**Lemma B.2** *Each element* $B \in \mathrm{GL}(5,2)$ *of order 5 defines a unique antiflag* $(w,\sigma)$ *in* $\mathrm{PG}(4,2)$ *which is fixed by* $B : Bw = w$ *and* $B(\sigma)=\sigma$*.*

**Proof.** See part (ii) of lemma B.1. In the $B$-invariant decomposition $V_5 = V_4 \oplus V_1$ let $\sigma = \mathbb{P}V_4$ and $\{w\} = V_1$. ■

Consequently, before considering further an element $B = B_4 \oplus I_1 \in \mathrm{GL}(5,2)$ of order 5, let us deal first with an element $B_4 \in \mathrm{GL}(4,2)$ of order 5.

### B.2.1 $Z_5$-orbits of points and lines in $\mathrm{PG}(3,2)=\mathbb{P}(V_4)$

An element $B_4 \in \mathrm{GL}(4,2)$ of order 5 has for its centralizer a subgroup $\langle S_4\rangle \cong Z_{15}$ of $\mathrm{GL}(4,2)$ generated by a Singer element $S_4 \in \mathrm{GL}(4,2)$; see [6, Table 3]. Without loss of generality we may suppose that $S_4$ satisfies $(S_4)^4 = S_4 + I_4$ and that $B_4 = (S_4)^6$. Upon defining $P_4 = (S_4)^{10}$ note that we then have

$$B_4P_4 = S_4 = P_4B_4, \qquad (B_4)^5 = I_4,\ (P_4)^3 = I_4,\ (S_4)^{15} = I_4. \tag{B.2}$$

Let $Z^a = \{a_1, a_2, a_3, a_4, a_5\}$, where $a_{i+1} = (B_4)^i a_1$, be any one of the three orbits of $\langle B_4\rangle \cong Z_5$ upon $\mathrm{PG}(3,2) = \mathbb{P}(V_4)$. Upon defining $b_i = P_4a_i$ and $c_i = (P_4)^2a_i$ the other two $Z_5$-orbits are then $Z^b = \{b_1, b_2, b_3, b_4, b_5\}$ and $Z^c = \{c_1, c_2, c_3, c_4, c_5\}$, where $b_{i+1} = (B_4)^i b_1$ and $c_{i+1} = (B_4)^i c_1$. Since $(P_4)^2 + P_4 + I_4 = 0$, observe that

$$\Sigma := \{\kappa_1, ..., \kappa_5\}, \qquad \text{where } \kappa_i = \{a_i, b_i, c_i\}, \tag{B.3}$$

is a spread for $\mathrm{PG}(3,2)$.

**Lemma B.3** *Given* $B_4 \in \mathrm{GL}(4,2)$ *of order 5 let the three orbits of* $\langle B_4\rangle \cong Z_5$ *in* $\mathrm{PG}(3,2)$ *be* $Z^x = \{x_1, x_2, x_3, x_4, x_5\}$, $x = a, b, c$, *as just defined. Then the* 35 *lines of* $\mathrm{PG}(3,2)$ *form seven* $Z_5$*-orbits, one of which is the spread* $\Sigma$, *see (B.3), and the other six are the following* 'pentagrams' $\Pi_{xxy}, \Pi_{xxz}$ *:*

$$\begin{aligned} \Pi_{xxy} &= \{\langle x_i, x_{i+1}\rangle\}, \quad \textit{where } \langle x_i, x_{i+1}\rangle = \{x_i, x_{i+1}, y_{i+3}\},\\ \Pi_{xxz} &= \{\langle x_i, x_{i+2}\rangle\}, \quad \textit{where } \langle x_i, x_{i+2}\rangle = \{x_i, x_{i+2}, z_{i+1}\}. \end{aligned} \tag{B.4}$$

*Here, and below, the index i labelling the five elements of a* $Z_5$*-orbit runs through the five values* $1, 2, 3, 4, 5 \pmod 5$, *and the triple xyz runs through the three values abc, bca, cab.*

**Proof.** We need to show that (i) $x_i + x_{i+1} = y_{i+3}$ and (ii) $x_i + x_{i+2} = z_{i+1}$. Now (i) asserts that $I_4 + B_4 = P_4(B_4)^3$, that is $I_4 + (S_4)^6 = (S_4)^{28}$, equivalently $(S_4)^2 + (S_4)^8 = I_4$. This last indeed holds, since it is the square of the relation $S_4 + (S_4)^4 = I_4$. Similarly (ii) is seen to hold. ■

### B.2.2 $Z_5$-orbits of points and lines in $\mathrm{PG}(4,2)=\mathbb{P}(V_5)$

If $B \in \mathrm{GL}(5,2)$ is of order 5 then, see lemmas B.1, B.2, $B$ is of the form $B = B_4 \oplus I_1$ with respect to a unique direct sum decomposition $V_5 = V_4 \oplus V_1$, where $V_4 = \ker(B^4+B^3+B^2+B+I)$ and $V_1 = \ker(B+I)$. Setting $\sigma = \mathbb{P}V_4 = \mathrm{PG}(3,2)$ and $\{w\} = V_1$ then $(w, \sigma)$ is the invariant antiflag of $B$ in the latter's natural action upon $\mathrm{PG}(4,2) = \mathbb{P}(V_5)$. If $P_4$ and $S_4$ are as in section B.2.1, and if we define $P := P_4 \oplus I_1$ and $S := S_4 \oplus I_1$, then $P,\ S \in \mathrm{GL}(5,2)$, having $(w, \sigma)$ as invariant antiflag, will satisfy

$$BP = S = PB, \qquad B = S^6,\ P = S^{10},\ (B)^5 = I,\ (P)^3 = I,\ (S)^{15} = I. \tag{B.5}$$

**$Z_5$-orbits of points** Let the three orbits of $\langle B_4\rangle$ in $\sigma = \mathrm{PG}(3,2)$ be $Z^x = \{x_1, x_2, x_3, x_4, x_5\}$, $x = a, b, c$, as in section B.2.1. We display these $Z_5$-orbits in the rows of the array

$$\begin{array}{c|ccccc} & \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 \\ \hline Z^a & a_1 & a_2 & a_3 & a_4 & a_5 \\ Z^b & b_1 & b_2 & b_3 & b_4 & b_5 \\ Z^c & c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \quad . \tag{B.6}$$

Recall that the columns in this array form the spread $\Sigma$ for $\sigma$, see (B.3). In addition to the singleton orbit $\{w\}$ there are three further orbits of $\langle B\rangle \cong Z_5$ on PG$(4,2)$, namely $Z^{x'} = \{x'_1, x'_2, x'_3, x'_4, x'_5\}$, where $x'_i = x_i + u$, and $x$ takes the values $a, b, c$ :

$$\begin{array}{cccccc} Z^{a'} & a'_1 & a'_2 & a'_3 & a'_4 & a'_5 \\ Z^{b'} & b'_1 & b'_2 & b'_3 & b'_4 & b'_5 \\ Z^{c'} & c'_1 & c'_2 & c'_3 & c'_4 & c'_5 \end{array} \quad . \tag{B.7}$$

As in the proof of lemma B.1, we have $\Sigma_{i=1}^5 x_i = 0,\ x = a, b, c$, each orbit $Z^x$ being a hyperbasis for $V_4$. Consequently each orbit $Z^{x'},\ x = a, b, c$, is a basis for $V_5$, the unit point being $w$ in each case: $\Sigma_{i=1}^5 x'_i = w$. Observe that $S$ satisfies

$$Sw = w \quad \text{and} \quad S : a_i \mapsto b_{i+1} \mapsto c_{i+2} \mapsto a_{i+3} \mapsto \ldots \tag{B.8}$$

If we adopt $Z^{a'} = \{a'_1, a'_2, a'_3, a'_4, a'_5\}$ as basis for $V_5$, and denote it $\{1, 2, 3, 4, 5\}$, then, in abbreviated notation (with $ij := a'_i + a'_j,\ u = 12345 = w,\ 1u = 2345$, etc.), the arrays (B.6), (B.7) are:

$$\begin{array}{cccccc} Z^a & 1u & 2u & 3u & 4u & 5u \\ Z^b & 34 & 45 & 51 & 12 & 23 \\ Z^c & 52 & 13 & 24 & 35 & 41 \end{array} \quad \text{and} \quad \begin{array}{cccccc} Z^{a'} & 1 & 2 & 3 & 4 & 5 \\ Z^{b'} & 512 & 123 & 234 & 345 & 451 \\ Z^{c'} & 134 & 245 & 351 & 412 & 523 \end{array} \quad . \tag{B.9}$$

Since $B$ effects the permutation (12345) of this basis, that element $Q \in \text{GL}(5,2)$ of order 4 which effects the permutation (1)(2354) will satisfy $QBQ^{-1} = B^2$. Now $P$ fixes $u$ and sends $a_i$ to $b_i$, so it sends $a'_i$ to $b'_i$. Hence, from the $Z^{b'}$-row of (B.9), the effects of $P$ and of $S = BP$ upon the basis are as follows:

$$\begin{aligned} &B\colon 1,2,3,4,5 \mapsto 2,3,4,5,1; \quad P\colon 1,2,3,4,5 \mapsto 512,123,234,345,451; \\ &S\colon 1,2,3,4,5 \mapsto 123,234,345,451,512; \quad Q\colon 1,2,3,4,5 \mapsto 1,3,5,2,4. \end{aligned} \tag{B.10}$$

Since $Q$ also satisfies $QPQ^{-1} = P^2$, and hence $QSQ^{-1} = S^2$, note that $Q$ lies in the normalizer $N_{\text{GL}(5,2)}(\langle S\rangle)$ of $\langle S\rangle \cong Z_{15}$. In fact the complete normalizer of $\langle S\rangle$ is

$$N_{\text{GL}(5,2)}(\langle S\rangle) = \langle S, Q\rangle \cong Z_{15} \rtimes Z_4. \tag{B.11}$$

In a previous document, when dealing with a particular subgroup $\langle A\rangle \cong Z_{31}$ of GL$(5,2)$, we have adopted a basis $\{1,2,3,4,5\}$ such that $A$ sends $1,2,3,4,5$ to $2,3,4,5,13$. One particular subgroup $\langle B\rangle \cong Z_5$ in the normalizer $N(\langle A\rangle) \cong Z_{31} \rtimes Z_5$ of $\langle A\rangle$ has a generator $B$ which fixes 1, is "squaring" ($BAB^{-1} = A^2$), and which sends $1,2,3,4,5$ to $1,3,5,24,134$. For this different $B$ the invariant antiflag $(w,\sigma)$ has $w = 1$ and $\sigma = \langle 14, 2, 3, 5\rangle$, and the six orbits comparable to (B.9) are as follows:

$$\begin{array}{cccccc} Z^a & 2 & 3 & 5 & 134 & 3u \\ Z^b & 2u & 235 & 145 & 23 & 35 \\ Z^c & u & 25 & 14 & 124 & 5u \end{array} \quad \text{and} \quad \begin{array}{cccccc} Z^{a'} & 12 & 13 & 15 & 34 & 245 \\ Z^{b'} & 345 & 4u & 45 & 123 & 135 \\ Z^{c'} & 1u & 125 & 4 & 24 & 234 \end{array} \quad . \tag{B.12}$$

For the present $B$ we may immediately read off from the arrays (B.12) the analogues of $P, S, Q$ in (B.10):

$$\begin{aligned} &B\colon 1,2,3,4,5 \mapsto 1,3,5,24,134; \quad P\colon 1,2,3,4,5 \mapsto 1,2u,235,15,145; \\ &S\colon 1,2,3,4,5 \mapsto 1,235,145,34,23; \quad Q\colon 1,2,3,4,5 \mapsto 1,2,5,135,3u. \end{aligned} \tag{B.13}$$

**$Z_5$-orbits of lines.** The 155 lines of PG(4, 2) fall into 31 $Z_5$-orbits as we now describe. First of all the lines through $w$ form the three $Z_5$-orbits $\Xi^x$, where

$$\Xi^x = \{\{w, x_i, x'_i\}\}, \quad x = a, b, c. \tag{B.14}$$

Now each of the 35 lines $\{v_1, v_2, v_3\}$ in $\sigma$ gives rise to three further lines of $\mathrm{PG}(4,2)\diagdown\{w\}$, namely $\{v_1, v'_2, v'_3\}$, $\{v'_1, v_2, v'_3\}$ and $\{v'_1, v'_2, v_3\}$, where $v'_i = w + v_i$. In this manner we arrive at all $155 (= 15 + 35 + 3 \times 35)$ lines of PG(4, 2). In detail first observe that the orbit $\Sigma \subset \sigma$ gives rise to the three $Z_5$-orbits

$$\Sigma_{xy'z'} = \{\{x_i, y'_i, z'_i\} |\, i = 1, 2, 3, 4, 5\}, \tag{B.15}$$

each being a partial spread. (Here as previously $xyz$ runs through the three values $abc, bca, cab$.) Next the orbit $\Pi_{xxy} \subset \sigma$ gives rise to the three $Z_5$-orbits $\Sigma_{xx'y'}$, $\Sigma_{x'xy'}$ and $\Pi_{x'x'y}$, where each of

$$\Sigma_{xx'y'} = \{\{x_i, x'_{i+1}, y'_{i+3}\}\}, \quad \text{and } \Sigma_{x'xy'} = \{\{x'_i, x_{i+1}, y'_{i+3}\}\} \tag{B.16}$$

is a partial spread, and where the five lines of

$$\Pi_{x'x'y} = \{\{x'_i, x'_{i+1}, y_{i+3}\}\} \tag{B.17}$$

form a pentagram. Similarly the orbit $\Pi_{xxz} \subset \sigma$ gives rise to the three $Z_5$-orbits $\Sigma_{xx'z'}$, $\Sigma_{x'xz'}$ and $\Pi_{x'x'z}$, where each of

$$\Sigma_{xx'z'} = \{\{x_i, x'_{i+2}, z'_{i+1}\}\}, \quad \text{and } \Sigma_{x'xz'} = \{\{x'_i, x_{i+2}, z'_{i+1}\}\} \tag{B.18}$$

is a partial spread, and where the five lines of

$$\Pi_{x'x'z} = \{\{x'_i, x'_{i+2}, z_{i+1}\}\} \tag{B.19}$$

form a pentagram.

**Summary B.4** *The* 31 *$Z_5$-orbits of lines in* PG(4, 2) *comprise the three orbits (B.14), consisting of lines through $w$, the* $16 (= 1+3+6+6)$ *orbits (B.3), (B.15), (B.16) and (B.18), consisting of partial spreads, and the* $12 (= 6 + 3 + 3)$ *orbits (B.4), (B.17) and (B.19), consisting of pentagrams.*

The 16 partial spreads (B.3), (B.15), (B.16), (B.18) are of course cyclic, with each of their stabilizers containing the same subgroup $\langle B \rangle \cong Z_5$ which cycles through their five lines. We now appeal to the complete classification [7, Appendix B] of all partial spreads in PG(4, 2); see also Appendix D in the present paper. From [7, Table B.2] we note that those partial spreads of size 5 which are cyclic fall into just three classes (= GL(5, 2)-orbits), namely Va.1, Ve.1 and Vj.1, with stabilizers of the respective structures Sym(5), $Z_5$ and $2^4 : \Gamma\mathrm{L}(2, 4)$, and respective profiles $(3,3,3)^5$, $(2,3,4)^5$ and $(7,7,7)^5$. Class Va.1 consists of spreads on the various parabolic quadrics in $\mathrm{PG}(4,2)$ and class Vj.1 consists of the spreads for the hyperplanes of $\mathrm{PG}(4,2)$. As noted in [7, Section3.4.2] the 15-set $\psi$ which supports a partial spread $\mathcal{S} \in$ class Ve.1 supports precisely one other partial spread $\mathcal{S}^*$; moreover the *'twins'* $\mathcal{S}, \mathcal{S}^*$ share the *same* stabilizer $\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{\mathcal{S}^*} \cong Z_5$, and the stabilizer $\mathcal{G}_\psi$ is isomorphic to $D_{10}$. In contrast the underlying 15-set for a partial spread of class Ve.1, that is a parabolic quadric $\mathcal{P}_4$, supports all told a family of six partial spreads of class Va.1, the six stabilizers being distinct, comprising a class of Sym(5)-subgroups of the stabilizer $\mathcal{G}_{\mathcal{P}_4} \cong \mathrm{Sym}(6)$.

**Theorem B.5** *In the natural action of* $\mathrm{GL}(5,2)$ *upon* $\mathrm{PG}(4,2)$ *suppose that a subgroup* $\langle B\rangle \cong Z_5$ *has* $(w,\sigma)$ *for its invariant antiflag. Then there exist precisely sixteen partial spreads of size 5 in* $\mathrm{PG}(4,2)$ *which are stabilized by* $\langle B\rangle$. *Under the action of* $\mathrm{GL}(5,2)$ *these sixteen partial spreads are of three kinds:*
*(i) one is a spread for* $\sigma$, *and so of class Vj.1;*
*(ii) three are of class Va.1 and are spreads on three distinct parabolic quadrics in* $\mathrm{PG}(4,2)$*, each quadric having nucleus* $w$;
*(iii) twelve are of class Ve.1, and comprise six sets* $\{\mathcal{S}, \mathcal{S}^*\}$ *of* 'twins', *where* $\mathcal{S}, \mathcal{S}^*$ *both cover the same* 15*-set of points in* $\mathrm{PG}(4,2)$.

**Proof.** (ii) The 15-set $Z^x \cup Z^{y'} \cup Z^{z'}$ which supports the partial spread $\Sigma_{xy'z'}$ is easily seen to have profile $(3,3,3)^5$, and hence, from the preamble to the theorem, the three partial spreads (B.15) are of class Va.1, and are spreads on three parabolic quadrics, the latter being distinct since the three underlying 15-sets are distinct. The nuclei of these parabolic quadrics must be fixed by $\langle B\rangle$, and hence each nucleus is $w$.

(iii) Each of the partial spreads $\Sigma_{aa'b'}, \Sigma_{a'ab'}$ is supported by the same 15-set $\psi_{aa'b'} = Z^a \cup Z^{a'} \cup Z^{b'}$ and each is stabilized by the *same* $Z_5$ subgroup, namely $\langle B\rangle$. It follows from this (or from noting that each has profile $(2,3,4)^5$) that they are of class Ve.1. ■

**Remark B.6** *The 15-set* $\psi_{aa'b'}$ *can be displayed as the array*

$$\psi_{aa'b'} = \begin{pmatrix} a_1 & b'_4 & a'_2 & - & - \\ - & a_2 & b'_5 & a'_3 & - \\ - & - & a_3 & b'_1 & a'_4 \\ a'_5 & - & - & a_4 & b'_2 \\ b'_3 & a'_1 & - & - & a_5 \end{pmatrix} = \begin{pmatrix} 1u & 345 & 2 & - & - \\ - & 2u & 451 & 3 & - \\ - & - & 3u & 512 & 4 \\ 5 & - & - & 4u & 123 \\ 234 & 1 & - & - & 5u \end{pmatrix}, \qquad \text{(B.20)}$$

*whose five rows constitute the five lines of the partial spread* $\mathcal{S} := \Sigma_{aa'b'}$ *and whose columns constitute the five lines of its twin* $\mathcal{S}^* := \Sigma_{a'ab'}$. *Moreover* $\mathcal{G}_\psi = \langle B, Q^2\rangle \cong Z_5 \rtimes Z_2 \cong D_{10}$, *where, see after (B.9),* $Q^2$, *which sends* $1,2,3,4,5$ *to* $1,5,4,3,2$, *satisfies* $Q^2BQ^{-2} = B^{-1}$.

## B.3 $T_B$-orbits upon $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$

Consider $T_B = \wedge^2 B$, where $B \in \mathrm{GL}(5,2)$ has order 5, acting upon $V_{10} = \wedge^2 V_5$. From (B.1), we have $B = B_4 \oplus I_1$, acting upon $V_5 = V_4 \oplus V_1$,whence, see [6, eq. (4.12)], $T_B = T_{B_4} \oplus B_4$. Here $T_{B_4} = \wedge^2 B_4$, acting upon $V_6 = \wedge^2 V_4$, has a $T_{B_4}$-invariant direct sum decomposition $V_6 = V_4 \oplus V_2$, upon which $T_{B_4}$ acts as

$B_4 \oplus I_2$.. So $V_{10}$ has $T_B$-invariant decompositions of the kind

$$V_{10} = V_4 \oplus V_6 = V_8 \oplus V_2,$$
$$\text{where } V_8 = V_4 \oplus V_4 = \ker(T_B^4 + T_B^3 + T_B^2 + T_B + I) = \operatorname{im}(T_B + I). \qquad (B.21)$$

Note that $V_2 = \ker(T_B + I)$, the fixed-point-set of $T_B$, is uniquely determined by $B$, but that there are many choices for the two $V_4$. Of these choices, two are uniquely determined by $B$. Firstly one $V_4$ arises from the unique $T_B$-invariant Latin solid $\mathrm{St}(u) = \mathbb{P}(V_4)$. Secondly $T_B$ stabilizes ...

Let us set down in the next lemma aspects of the foregoing in projective terms.

**Lemma B.7** *Each element* $B \in \mathrm{GL}(5,2)$ *of order 5 defines a unique antiflag* $(u,\sigma)$ *in* $\mathrm{PG}(4,2)$ *which is fixed by* $B : Bu = u$ *and* $B(\sigma) = \sigma$. *In its induced action* $T_B$ *upon* $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$, *there is a unique* $T_B$*-invariant Latin solid, namely* $\mathrm{St}(u)$. *Moreover there are precisely three fixed points, these forming a special line* $L$ *which lies in* $\mathcal{W}(\sigma)$.

**Proof.** Recall that $B = I_1 \oplus B_4$ acting upon $V_5 = V_1 \oplus V_4$. This last $B$-invariant decomposition $V_5 = V_1 \oplus V_4$ translates into the $B$-invariant antiflag $(u,\sigma)$, where $\prec u \succ = V_1$ and $\sigma = \mathbb{P}(V_4)$. In the decomposition $\wedge^2(\prec u \succ \oplus V_4) = (\prec u \succ \otimes V_4) \oplus \wedge^2 V_4$ , the $T_B$-invariant subspace $\prec u \succ \otimes V_4$ is projectively the $T_B$-invariant Latin solid $\mathrm{St}(u) = \{u \wedge x \,|\, x \in \sigma\}$. In the further decomposition $\wedge^2 V_4 = V_4 \oplus V_2$, the $T_B$-invariant $V_2$ is projectively a line $L$ of fixed points which lies in $\mathbb{P}(\wedge^2 V_4) = \Pi(\sigma)$. Now, by lemma B.1(i), no line of $\sigma = \mathbb{P}(V_4)$ is fixed by $B_4$, so all three points of $L$ are external: $L \subset \mathcal{W}(\sigma)$. ■

**Remark B.8** *If* $B \in \mathrm{GL}(5,2)$ *is defined to permute the basis elements* $\{e_1, e_2, e_3, e_4, e_5\}$ *in the manner* $(e_1e_2e_3e_4e_5)$, *then* $B$ *has order 5. The invariant antiflag in this case is clearly* $(u, \sigma_{even})$ *where* $u = \sum_i e_i$ *and where* $\sigma_{even}$ *is the even hyperplane of the basis* $\{e_1, e_2, e_3, e_4, e_5\}$ *with* $\sigma_{even}$ *comprising the ten points* $e_i + e_j$ *and the five points* $u + e_i$. *Observe that* $\wedge^2 B$ *fixes the three points*

$$b = \textstyle\sum_i e_i \wedge e_{i+1}, \quad b' = \sum_i e_i \wedge e_{i+2}, \quad b'' = b + b' = \sum_{1\le i<j\le 5} e_i \wedge e_j \qquad (B.22)$$

*of* $\mathrm{PG}(9,2)$, *where in the first two summations* $i$ *runs from* 1 *to* 5 $(\mathrm{mod}\, 5)$. *So* $L = \{b, b', b''\}$ *is the special line of fixed points of lemma B.7 for the given* $B$. *Of course, by lemma B.1, any* $B \in \mathrm{GL}(5,2)$ *of order 5 can be expressed as in this remark relative to a suitable choice of basis.*

**Remark B.9** *Given a subgroup* $\langle B \rangle \cong Z_5$ *of* $\mathrm{GL}(5,2)$ *there exists a unique 7-flat* $N$ *in* $\mathrm{PG}(9,2)$ *which is invariant under* $T_B$, *namely* $N = \mathbb{P}V_8$, *where* $V_8$ *is as in (B.21).*

## B.4 Example of a hyperplane $H \in \mathcal{H}_{868}$

Given a basis $\{e_1, e_2, e_3, e_4, e_5\}$ for $V_5$ let $t(\cdot,\cdot)$ be the particular alternating bilinear form on $V_5$ which satisfies

$$t(e_i, e_j) = 1 \quad \text{for each } i \neq j. \qquad (B.23)$$

Observe that the form has 1-dimensional kernel $\prec u \succ$ where $u = \Sigma_{i=1}^5 e_i$. The dual bivector $b^*$, equivalently the trivector $t = \perp b^*$, which give rise to the form $t(\cdot,\cdot)$ are, from (A.9) and (A.7), seen to be

$$t = \Sigma_{1\leq k<l<m\leq 5} e_k \wedge e_l \wedge e_m, \qquad b^* = \Sigma_{1\leq i<j\leq 5} e^i \wedge e^j. \tag{B.24}$$

Consider the hyperplane $H = \langle t\rangle^{\mathrm{O}}$. Because $t \in \mathrm{Rk}_4^*$ the hyperplane $H$ is in the orbit $\mathcal{H}_{868}$, and we wish to find out more about $H$. In particular we would like to display in some useful way the 75 isotropic lines for the form $t(\cdot,\cdot)$, and hence their Grassmann images, the 75 points of $H \cap \mathcal{G}_{1,4,2}$. One approach is to choose a subgroup $\langle B\rangle \cong Z_5$ of $\mathrm{GL}(5,2)$ which leaves $t$ invariant, and then to make use of the results in Appendix B.2.2 describing the $Z_5$-orbits of lines in $\mathrm{PG}(4,2)$. To this end let us choose for $B$ that element of $\mathrm{GL}(5,2)$ which effects the cyclic permutation $(e_1e_2e_3e_4e_5)$ of the basis, and observe that the invariant antiflag of $B$ is $(u, \sigma_{\mathrm{even}})$. Here $\sigma_{\mathrm{even}}$ denotes the hyperplane which consists of the 10 points $e_i + e_j$ and the 5 points $u + e_i$. Clearly, from (B.23) or from (B.24), $B$ leaves $t$ invariant.

**Remark B.10** *The choice* $\langle B\rangle$ *is just one out of* 576 *available choices for a* $Z_5$ *subgroup which leaves t invariant. To see this note that the invariance group* $\mathcal{G}_t$ *of the form* $t(\cdot,\cdot)$ *has the structure* $2^4 : \mathrm{Sp}(4,2) \cong 2^4 : \mathrm{Sym}(6)$. *(There are* 16 $\mathrm{Sp}(4,2)$*-subgroups of* $\mathcal{G}_t$*, each stabilizing one of the* 16 *hyperplanes* $\mathbb{P}V_4 = \sigma$ *of* $\mathrm{PG}(4,2)$ *which do not contain* $u = \ker t$*, the restriction of* $t(\cdot,\cdot)$ *to any such* $V_4$ *being a nondegenerate symplectic form. Also* $\mathcal{G}_t$ *contains a normal subgroup* $\cong (Z_2)^4$*, whose* 15 *transvections all fix* $u$*, each transvection fixing pointwise one of the* 15 *hyperplanes passing through* $u$*.) Since* $\mathrm{Sp}(4,2)(\cong \mathrm{Sym}(6))$ *has* 36 *distinct subgroups* $\cong Z_5$*, it follows that* $\mathcal{G}_t$ *has* $16 \times 36 = 576$ *distinct subgroups* $\cong Z_5$.

The 155 lines in $\mathrm{PG}(4,2)$ fall into 31 $Z_5$-orbits, see Summary B.4, and in the next theorem we determine which 15 orbits provide us with the known 75 isotropic lines for the form $t(\cdot,\cdot)$. In this theorem, and its proof, we adopt the notation surrounding eqs. (B.6), (B.7) and (B.9), taking $Z^{a'} = \{a'_1, a'_2, a'_3, a'_4, a'_5\}$ to be our current basis $\{e_1, e_2, e_3, e_4, e_5\}$.

**Theorem B.11** *For the preceding subgroup* $\langle B\rangle \cong Z_5$ *of* $\mathrm{GL}(5,2)$ *the* 75 *isotropic lines for the form* $t(\cdot,\cdot)$ *in (B.23) fall into 15* $Z_5$*-orbits as follows:*
*(i) the three orbits* $\Xi^a, \Xi^b, \Xi^c$ *of (B.14);*
*(ii) the three orbits* $\Sigma, \Pi_{cca}, \Pi_{bba}$*, see (B.3), (B.4);*
*(iii) the two orbits* $\Pi_{c'c'a}$ *and* $\Pi_{b'b'a}$*, see (B.17) and (B.19);*
*(iv) the three partial spreads* $\Sigma_{ab'c'}, \Sigma_{bc'a'}, \Sigma_{ca'b'}$ *of class Va.1, see (B.15);*
*(v) the four partial spreads* $\Sigma_{cc'a'}, \Sigma_{c'ca'}, \Sigma_{bb'a'}, \Sigma_{b'ba'}$ *of class Ve.1, see (B.16), (B.18), consisting of two sets,* $\{\Sigma_{cc'a'}, \Sigma_{c'ca'}\}$ *and* $\{\Sigma_{bb'a'}, \Sigma_{b'ba'}\}$*, of twins.*

**Proof.** Concerning (i), these three orbits account for the 15 lines $\mathrm{st}(u)$ which pass through $u$, these lines being isotropic as in lemma A.2. The remaining assertions in the theorem are easily checked. For example a representative of the partial spread $\Sigma_{cc'a'}$ in (B.16) is $\lambda = \{c_1, c'_2, a'_4\}$, that is, see (B.9), $\lambda = \{52, 245, 4\}$; but, from (B.23), $t(52, 4) = 1 + 1 = 0$, and so $\lambda$ is isotropic. ■

# C Appendix: classes IVa.1, Vd.1 inside VIIIa.1

## C.1 Partial spreads of class VIIIa.1

### C.1.1 Notation

Let $V_5 = V_3 \oplus V_2$, and so $\mathrm{PG}(4,2) = \mathbb{P}V_5 = \langle \alpha, \mu \rangle$, with $\alpha = \mathbb{P}V_3$ a plane and $\mu = \mathbb{P}V_2$ a disjoint line. Choose a basis $\{e_1, \ \dots \ , e_5\} = \{1,2,3,4,5\}$ with $\alpha = \langle 1,2,3 \rangle$ and $\mu = \langle 4,5 \rangle$. Let $A \in \mathrm{GL}(5,2)$ send $1,2,3$ to $2,3,12$ and fix $4,5$. So $A^3x = Ax + x$ for $x \in V_3$. Put $a_1 = 1$ and $a_i = A^{i-1}a_1$, so $\alpha = \{a_i\}_{i=1,2,\dots,7}$. The 7 lines of $\alpha$ are thus $\{\lambda_a\}_{a \in \alpha}$, where $\lambda_a = \{A^{-1}a, a, A^2a\}$. Equally the 7 lines are $\{\lambda_1, \ \dots \ \lambda_7\}$, where

$$\lambda_i := \lambda_{a_i} = \{a_{i-1}, a_i, a_{i+2}\}, \qquad i \in \{1,2,\dots,7\} \ (\mathrm{mod}\, 7). \tag{C.1}$$

We also write

$$\mu = \{u, v, w\} \qquad \text{where } u = 5, \ v = 4, \ w = 45. \tag{C.2}$$

Define $C \in \mathrm{GL}(5,2)$ such the

$$Ca_i = a_{2i+2}, \ i \in \{1,2,\dots,7\} \ (\mathrm{mod}\, 7), \qquad \text{and} \qquad Cu = v, Cv = w. \tag{C.3}$$

Thus $C$ has order 3 and effects the permutation $(a_5)(a_1a_4a_3)(a_6a_7a_2)$ of the points of $\alpha$ and the permutation $(uvw)$ of the points of $\mu$. Observe that $C$ satisfies $CAC^{-1} = A^2$ and is of cycle type $3^{10}1^1$ on $\mathrm{PG}(4,2)$, with $a_5$ the unique fixed point. Note also that the subgroup

$$F_{21} := \langle A, C \rangle \cong Z_7 \rtimes Z_3$$

of $\mathrm{GL}(5,2)$ stabilizes both $\mu$ and $\alpha$.

For $a \in V_3$ we define elements $J_a \in \mathrm{GL}(5,2)$ by

$$J_a x = x, \ x \in V_3, \qquad J_a u = u + A^{-1}a, \quad J_a v = v + a, \quad ; \tag{C.4}$$

Upon observing that $J_aJ_b = J_{a+b}$ take note that $\mathcal{J} = \{J_a | a \in V_3\}$ is a subgroup $\cong (Z_2)^3$ of $\mathrm{GL}(5,2)$ whose 7 non-identity elements are involutions. Moreover, upon setting $\mathcal{G} := \langle \mathcal{J}, F_{21} \rangle$, we have

$$\mathcal{G} = \mathcal{J} \rtimes F_{21} \cong 2^3 : (7:3). \tag{C.5}$$

That $\mathcal{J}$ is normal in $\mathcal{G}$ follows from the (readily confirmed) relations

$$AJ_aA^{-1} = J_{Aa}, \qquad CJ_aC^{-1} = J_{CA^{-1}a}. \tag{C.6}$$

### C.1.2 Construction of a regulus-free partial spread $\mathcal{S}_8$

Under the action of $\mathcal{J}$ the line $\mu = \mu_0$ lies on an orbit

$$\mathcal{S}_8 = \{\mu_a\}_{a \in V_3}, \qquad \mu_a := J_a(\mu), \tag{C.7}$$

of length 8. Then, see [16], $\mathcal{S}_8$ is a regulus-free partial spread, and so is of class VIIIa.1. The line $\mu_a$ of $\mathcal{S}_8$ is

$$\mu_a = \{u + A^{-1}a, \quad v + a, \quad w + A^2a\}, \qquad a \in V_3. \tag{C.8}$$

It will prove helpful to make use of the following abbreviations:

$$u_i := u + a_i, \quad v_i := v + a_i, \quad w_i := w + a_i. \tag{C.9}$$

If we also write $\mu_i = \mu_{a_i}$, for $i = 1, 2, \dots, 7$, then the 8 lines of the regulus-free partial spread $\mathcal{S}_8 = \{\mu_0, \mu_1, \dots, \mu_7\}$ are as in the following explicit array:

$$\begin{array}{cccc}
\mu_0 & u & v & w \\
\mu_1 & u_7 & v_1 & w_3 \\
\mu_2 & u_1 & v_2 & w_4 \\
\mu_3 & u_2 & v_3 & w_5 \\
\mu_4 & u_3 & v_4 & w_6 \\
\mu_5 & u_4 & v_5 & w_7 \\
\mu_6 & u_5 & v_6 & w_1 \\
\mu_7 & u_6 & v_7 & w_2
\end{array} . \tag{C.10}$$

The stabilizer $\mathcal{G}_{\mathcal{S}_8}$ of $\mathcal{S}_8$ is $\mathcal{G} = \mathcal{J} \rtimes F_{21}$, with $J_b$, $A$, $C$ acting as follows:

$$J_b(\mu_a) = \mu_{a+b}, \qquad A(\mu_a) = \mu_{Aa}, \qquad C(\mu_a) = \mu_{CA^{-1}a} \tag{C.11}$$

(the last two of these following immediately from the relations (C.6)). Thus $\mathcal{J}$ acts transitively on $\mathcal{S}_8$ while $A$ effects the permutation $(\mu_0)(\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7)$ and $C$ effects the permutation $(\mu_0)(\mu_7)(\mu_1\mu_2\mu_4)(\mu_3\mu_6\mu_5)$.

**The 31 solids in** PG$(4, 2)$. The 31 points of PG$(4, 2)$ fall into two $\mathcal{G}$-orbits, namely the 24 points covered by the 8 lines of $\mathcal{S}_8$ and the remaining 7 points, comprising the plane $\alpha$. The 31 solids also fall in to two $\mathcal{G}$-orbits. One orbit consist of the 3 solids which contain $\alpha$, namely

$$\sigma_u := \langle \alpha, u \rangle, \quad \sigma_v := \langle \alpha, v \rangle, \quad \sigma_w := \langle \alpha, w \rangle. \tag{C.12}$$

The other orbit consists of the 28 solids

$$\sigma_{ab} := \langle \mu_a, \mu_b \rangle, \qquad \{a, b\} \subset \{0, 1, \dots, 7\}. \tag{C.13}$$

(Since $\mathcal{S}_8$ is regulus-free the $\binom{8}{2} = 28$ choices for the pair $\{a, b\}$ yield distinct solids.)

**The 155 lines in** PG$(4, 2)$. Consider the 155 lines of PG$(4, 2)$. The 7 lines of $\alpha$ form one $\mathcal{G}$-orbit, say $\omega_7$, and the 8 lines $\mu_a$ form another orbit $\omega_8 (= \mathcal{S}_8)$. Now any $\mathcal{S}_3 = \{\mu_a, \mu_b, \mu_c\} \subset \mathcal{S}_8$ is a non-regulus and so possesses a unique transversal, say $\tau_{abc}$. These $\binom{8}{3} = 56$ transversals comprise another $\mathcal{G}$-orbit $\omega_{56}$. The remaining $(155 - 7 - 8 - 56 =)84$ lines are those which meet $\alpha$ in a point (12 through each point of $\alpha$), and these form a single $\mathcal{G}$-orbit $\omega_{84}$.

### C.1.3 High symmetry

The next theorem highlights the high symmetry enjoyed by the regulus-free partial spreads $\mathcal{S}_8$ of size 8.

**Theorem C.1** *(i) In* $\mathrm{PG}(4,2)$ *the partial spreads of size* 8 *which are regulus-free constitute a single class VIIIa.1. If* $\mathcal{S}_8 = \{\mu_1, \dots, \mu_8\}$ *is of class VIIIa.1 then its stabilizer group* $\mathcal{G}_{\mathcal{S}_8}$ *has structure* $2^3 : (7:3)$ *and is 2-transitive on the eight lines* $\mu_i \in \mathcal{S}_8$*. The seven points of* $\mathrm{PG}(4,2)$ *not on any of the lines of* $\mathcal{S}_8$ *constitute a plane* $\alpha = \alpha(\mathcal{S}_8)$*. (Consequently* $\mathcal{G}_{\mathcal{S}_8}$ *leaves* $\alpha$ *invariant.)*

*(ii) For any line* $\lambda$ *of the invariant plane* $\alpha$ *of* $\mathcal{S}_8$ *the partial spread* $\mathcal{S}_9^{(\lambda)} = \mathcal{S}_8 \cup \{\lambda\}$ *is of class IXa.1 and its stabilizer is a subgroup* $\cong \mathrm{Alt}(4) \times Z_2$ *of* $\mathcal{G}_{\mathcal{S}_8}$*.*

*(iii) If* $\mathcal{S}_7^{(i)} = \mathcal{S}_8 \setminus \{\mu_i\}$*, then* $\mathcal{S}_7^{(i)}$ *is of class VIIa.1 and its stabilizer is a subgroup* $\cong Z_7 \rtimes Z_3$ *of* $\mathcal{G}_{\mathcal{S}_8}$*, with the* $Z_7$ *subgroup cycling through the seven lines of* $\mathcal{S}_7^{(i)}$*. Up to equivalence every regulus-free* $\mathcal{S}_7$ *is of this kind.*

*(iv) If* $\mathcal{S}_6^{(ij)} = \mathcal{S}_8 \setminus \{\mu_i, \mu_j\}, i \neq j,$ *then* $\mathcal{S}_6^{(ij)}$ *is of class VIa.1 and its stabilizer is a* $Z_6$ *subgroup of* $\mathcal{G}_{\mathcal{S}_8}$ *which cycles through the six lines of* $\mathcal{S}_6^{(ij)}$*. (Another class of regulus-free partial spreads exists, namely class VIb.1, with stabilizer* $\cong Z_2$*.)*

*(v) If* $\mathcal{S}_5^{(ijk)} = \mathcal{S}_8 \setminus \{\mu_i, \mu_j, \mu_k\}, i \neq j \neq k \neq i,$ *then* $\mathcal{S}_5^{(ijk)}$ *is of class Vd.1, with stabilizer* $\cong Z_3$*.*

*(vi) If* $\mathcal{S}_5$ *is of class Vd.1 then it has a unique extension to an* $\mathcal{S}_8$ *of class VIIIa.1. (Consequently an* $\mathcal{S}_6$ *of class VIa.1 has a unique extension to an* $\mathcal{S}_8$ *of class VIIIa.1, and similarly for an* $\mathcal{S}_7$ *of class VIIa.1.)*

*(vii) If a partial spread* $\mathcal{S}$*, of size* $> 1$*, is stabilized by a* $Z_7$ *subgroup of* $\mathrm{GL}(5,2)$ *then* $\mathcal{S}$ *is either of class VIIa.1 or of class VIIIa.1.*

**Proof.** Consult [7, Section 6 and Table B.2]. Concerning (vi) the 16-set $\psi(\mathcal{S}_5)^c \subset \mathrm{PG}(4,2)$ uncovered by the lines of $\mathcal{S}_5$ contains, see [7, Table B.1], a unique plane $\alpha$; the 9-set $(\psi(\mathcal{S}_5) \cup \alpha)^c$ supports a uniquely determined non-regulus $\mathcal{S}_3$ which extends $\mathcal{S}_5$ to a regulus-free $\mathcal{S}_8$ which has $\alpha$ has its invariant plane. ■

Concerning the high symmetry, see also theorem C.8 below.

## C.2 Some decompositions of $\mathcal{S}_8$

There exist $\binom{8}{3} = 56$ decompositions of $\mathcal{S}_8$ of the kind $\mathcal{S}_8 = \mathcal{S}_5 \cup \mathcal{S}_3$, and for all of these decompositions $\mathcal{S}_5$ is of class Vd.1 and $\mathcal{S}_3$ is of class IIIa.1 (a non-regulus).

**Theorem C.2** *If* $\mathcal{S}_5$ *is of class Vd.1, with* $\mathcal{G}_{\mathcal{S}_5} = \langle C \rangle \cong Z_3$*, then, for a suitable numbering, we have* $\mathcal{S}_5 = \{\mu_1, \dots, \mu_5\}$ *where:*
*(i)* $C$ *stabilizes* $\mu_4$ *and* $\mu_5$*, and effects the cyclic permutation* $(\mu_1\mu_2\mu_3)$*;*
*(ii)* $\mathcal{S}_5 \setminus \{\mu_5\}$ *is of class IVa.1, while* $\mathcal{S}_5 \setminus \{\mu_i\}$ *is of class IVb.1 for* $i = 1, 2, 3, 4$*.*

**Proof.** Consult [7, Sections 3.3, 3.4 ]. ■

From part (ii) of the theorem it follows that the $\binom{8}{4} = 70$ partial spreads $\mathcal{S}_4$ inside $\mathcal{S}_8$ comprise 14 of class IVa.1 and 56 of class IVb.1. We now explain how the property in part (ii), applied to the partial spreads $\mathcal{S}_5 \subset \mathcal{S}_8$, ties in neatly with some design theory. For $\mathsf{B} \subset \{0,1,2,3,4,5,6,7\}$, $|\mathsf{B}| = 4$, set $\mathcal{S}_4^{\mathsf{B}} = \{\mu_a\}_{a \in \mathsf{B}}$ and denote by $\mathcal{D}$ those fourteen 4-sets $\mathsf{B}$ such that $\mathcal{S}_4^{\mathsf{B}}$ is of class IVa.1.

**Lemma C.3** $\mathcal{D}$ *is a 3-(8,4,1) design.*

**Proof.** Let $\mathcal{D}'$ denote the fourteen complements $\mathsf{B}^c$ of the 4-sets $\mathsf{B} \in \mathcal{D}$. By property (*) any 5-set $\subset \{0,1,2,3,4,5,6,7\}$ contains precisely one element $\mathsf{B} \in \mathcal{D}$, whence any 3-set is contained in precisely one element $\mathsf{B}^c \in \mathcal{D}'$, that is $\mathcal{D}'$ is a 3-(8,4,1) design. But such a design (which is unique up to isomorphism) has the property that the complement of a block is also a block. So $\mathcal{D} = \mathcal{D}'$. ■

Since a familiar model for a 3-(8,4,1) design is provided by the 14 affine planes of the affine geometry $\mathrm{AG}(3,2)$, the next lemma is hardly a surprise.

**Lemma C.4** *If a partial spread* $\mathcal{S}_4 \subset \mathcal{S}_8$ *is of the form* $\mathcal{S}_4 = \{\mu_0, \mu_a, \mu_b, \mu_c\}$ *where* $a, b, c$ *are collinear points of* $\alpha$ *then both* $\mathcal{S}_4$ *and its 'complement'* $\mathcal{S}_4' = \mathcal{S}_8 \setminus \mathcal{S}_4$ *are of class IVa.1. All fourteen* $\mathcal{S}_4 \subset \mathcal{S}_8$ *which are of class IVa.1 are thus accounted for, by way of 7 decompositions* $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}_4'$ *of the kind just described. (The remaining* 28 *of the* 35 *decompositions* $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}_4'$ *consequently have both* $\mathcal{S}_4$ *and* $\mathcal{S}_4'$ *of class IVb.1.)*

**Proof.** For $\mathcal{S}_8$ as in (C.10) then $\mathcal{S}_4 = \{\mu_0, \mu_1, \mu_2, \mu_4\}$ and $\mathcal{S}_4' = \{\mu_7, \mu_3, \mu_6, \mu_5\}$ are as described in the lemma, since $a_1, a_2$ and $a_4$ are collinear points of $\alpha$. The underlying 12-sets $\psi_4$ and $\psi_4'$ of $\mathcal{S}_4$ and $\mathcal{S}_4'$ can be expressed as the $4 \times 4$ arrays

$$\psi_4 = \begin{pmatrix} - & u & v & w \\ u_7 & - & w_3 & v_1 \\ v_2 & w_4 & - & u_1 \\ w_6 & v_4 & u_3 & - \end{pmatrix}, \qquad \psi_4' = \begin{pmatrix} - & u_6 & v_7 & w_2 \\ u_2 & - & w_5 & v_3 \\ v_6 & w_1 & - & u_5 \\ w_7 & v_5 & u_4 & - \end{pmatrix}, \qquad \text{(C.14)}$$

whose rows are the lines $\mu_0, \mu_1, \mu_2, \mu_4$ of $\mathcal{S}_4$ and $\mu_7, \mu_3, \mu_6, \mu_5$ of $\mathcal{S}_4'$ and whose columns are the transversals $\mu_0^*, \mu_1^*, \mu_2^*, \mu_4^*$ of $\mathcal{S}_4$ and $\mu_7^*, \mu_3^*, \mu_6^*, \mu_5^*$ of $\mathcal{S}_4'$. This double-four structure of $\psi_4$ and $\psi_4'$ shows that both $\mathcal{S}_4$ and $\mathcal{S}_4'$ are of class IVa.1. ■

If $\mathcal{S}_8$ as in (C.10) then those fourteen 4-sets $\mathsf{B}$ such that $\mathcal{S}_4^{\mathsf{B}}$ is of class IVa.1 are the blocks of the following 3-(8,4,1) design $\mathcal{D}$

$$\text{The design } \mathcal{D}\text{:} \quad \begin{array}{ccccccc} 0124 & 0235 & 0346 & 0457 & 0561 & 0672 & 0713 \\ 7365 & 1476 & 2517 & 3621 & 4732 & 5143 & 6254 \end{array} . \qquad \text{(C.15)}$$

For we have just verified the blocks in the first column, and the remaining blocks follow from the $Z_7$ symmetry provided by $\langle A \rangle$, which effects the permutation $(a_1 a_2 a_3 a_4 a_5 a_6 a_7)$.

### C.2.1 More about the decompositions $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}_4'$

Given a regulus-free $\mathcal{S}_8$, consider those decompositions $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}_4'$ such that $\mathcal{S}_4$ and $\mathcal{S}_4'$ are of class IVa.1. Now, see [7, Section 3.3], the underlying 12-set $\psi$ of any $\mathcal{S}_4$ of class IVa.1 supports precisely two partial spreads $\mathcal{S}_4$ and $\mathcal{S}_4^*$, these being given by the rows and columns of a double-four array of the kind

$$\psi = \begin{pmatrix} - & x_{12} & x_{13} & x_{14} \\ x_{21} & - & x_{23} & x_{24} \\ x_{31} & x_{32} & - & x_{34} \\ x_{41} & x_{42} & x_{43} & - \end{pmatrix}. \qquad \text{(C.16)}$$

Moreover this double-four is non-degenerate, meaning that $\langle \psi \rangle$ is the whole of $\mathrm{PG}(4,2)$.

**Lemma C.5** *(See [7, Lemma 3.4]) If* $\psi$ *is a non-degenerate double-four in* $\mathrm{PG}(4,2)$ *then it extends to a unique parabolic quadric* $\mathcal{P}_4$*. In detail, if* $\psi$ *is as in (C.16), and if the points* $y_1, y_2, y_3$ *are defined by* $y_i = x_{i4} + x_{4i} = x_{jk} + x_{kj}$*, for any permutation* $ijk$ *of* $123$*, then* $\mathcal{P}_4 = \psi \cup \lambda$*, where* $\lambda = \{y_1, y_2, y_3\}$*. Moreover, for any 3-set* $\{i,j,k\} \subset \{1,2,3,4\}$*, the nucleus* $n$ *of* $\mathcal{P}_4$ *is given by* $n = x_{ij} + x_{jk} + x_{ki}$*.*

**Remark C.6** *Recall from theorem C.2 that a partial spread* $\mathcal{S}_5$ *of class Vd.1 has a distinguished decomposition* $\mathcal{S}_5 = \mathcal{S}_3 \cup \mathcal{S}_1 \cup \mathcal{S}'_1$ *such that* $\mathcal{S}_3 \cup \mathcal{S}_1$ *is of class IVa.1 and such that the stabilizer* $\mathcal{G}_{\mathcal{S}_5} \cong Z_3$ *stabilizes* $\mathcal{S}_1$ *and* $\mathcal{S}'_1$ *and is transitive on* $\mathcal{S}_3$*. It follows that in a decomposition* $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}'_4$ *with* $\mathcal{S}_4$ *and* $\mathcal{S}'_4$ *of class IVa.1 there is a natural bijection* $\mathcal{S}_4 \leftrightarrow \mathcal{S}'_4$*, with* $\lambda' \in \mathcal{S}'_4$ *being paired with that* $\lambda \in \mathcal{S}_4$ *such that* $\mathcal{S}_3 \cup \{\lambda\} \cup \{\lambda'\}$ *is the distinguished decomposition of* $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda'\}$*. In (C.15) the blocks are ordered so as to exhibit this bijection. Thus from the first column of (C.15) one reads off that the natural bijection* $\mathcal{S}_4^{0124} \leftrightarrow \mathcal{S}_4^{7365}$ *is* $\mu_0 \leftrightarrow \mu_7$*,* $\mu_1 \leftrightarrow \mu_3$*,* $\mu_2 \leftrightarrow \mu_6$*,* $\mu_4 \leftrightarrow \mu_5$*. Note that this bijection is effected by the involution* $J_{a_7} \in \mathcal{J}$*.*

**Theorem C.7** *Let* $\mathcal{S}_8$ *be regulus-free partial spread of eight lines in* $\mathrm{PG}(4,2)$*, and let* $\alpha$ *denote the plane of points left uncovered by* $\mathcal{S}_8$*. Consider a decomposition* $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}'_4$ *such that* $\mathcal{S}_4$*,* $\mathcal{S}'_4$ *are of class IVa.1, and let* $\psi$*,* $\psi'$ *be the associated double-fours. Let* $\lambda$*,* $\lambda'$ *be the unique (lemma C.5) lines such that* $\psi \cup \lambda$*,* $\psi' \cup \lambda'$ *are parabolic quadrics* $\mathcal{P}_4$*,* $\mathcal{P}'_4$*, respectively, and let* $n$*,* $n'$ *be the nuclei of* $\mathcal{P}_4$*,* $\mathcal{P}'_4$*. Then* $\lambda = \lambda'$ *and* $n = n'$*. Moreover each of the seven lines* $\lambda_a \subset \alpha$ *serves as* $\lambda = \lambda'$ *for one of the seven decompositions* $\mathcal{S}_4 \cup \mathcal{S}'_4$ *of the IVa.1 kind.*

**Proof.** For $\psi$, $\psi'$ as in (C.14) lemma C.5 gives $\lambda = \lambda' = \{a_7, a_2, a_6\} = \lambda_7$ and $n = n' = a_5$. ■

**Theorem C.8** *Let* $\mathcal{S}_8 = \mathcal{S}_4 \cup \mathcal{S}'_4$ *be a decomposition of a regulus-free partial spread such that* $\mathcal{S}_4$*,* $\mathcal{S}'_4$ *are of class IVa.1. Then the stabilizer* $\mathcal{G}_{\{\mathcal{S}_4, \mathcal{S}'_4\}}$ *of such a decomposition has the structure* $\mathrm{Alt}(4) \times Z_2$*.*

**Proof.** Half of the elements of the stabilizer $\mathcal{G}_{\mathcal{S}_4} \cong \mathrm{Sym}\,4$, see [7, Section 3.3.1], send $\mathcal{S}'_4$ to $(\mathcal{S}'_4)^*$, see before eq. (C.16). So that subgroup of $\mathcal{G}_{\{\mathcal{S}_4, \mathcal{S}'_4\}}$ which stabilizes $\mathcal{S}_4$, $\mathcal{S}'_4$ separately is isomorphic to Alt 4. But that element $J \in \mathcal{J}$ which effects the natural bijection $\mathcal{S}_4 \leftrightarrow \mathcal{S}'_4$ in remark C.6 lies in the centre of $\mathcal{G}_{\{\mathcal{S}_4, \mathcal{S}'_4\}}$. Hence $\mathcal{G}_{\{\mathcal{S}_4, \mathcal{S}'_4\}} \cong \mathrm{Alt}(4) \times Z_2$. ■

Concerning the last part of the proof, if $\mathcal{S}_8$ is as in (C.10) and $\mathcal{S}_4 = \{\mu_0, \mu_1, \mu_2, \mu_4\}$ and $\mathcal{S}'_4 = \{\mu_7, \mu_3, \mu_6, \mu_5\}$, then $\mathcal{G}_{\{\mathcal{S}_4, \mathcal{S}'_4\}} = (\langle J_{a_1}, J_{a_2} \rangle \rtimes \langle C \rangle) \times \langle J_{a_7} \rangle$.

**Remark C.9** *The seven* $Z_2$ *subgroups of* $\mathcal{J}$ *are in correspondence with the seven decompositions* $\mathcal{S}_4 \cup \mathcal{S}'_4$ *of* $\mathcal{S}_8$ *such that* $\mathcal{S}_4$*,* $\mathcal{S}'_4$ *are of class IVa.1.*

# D Appendix: classification of partial spreads in PG(4, 2)

The following tables, reproduced from [7], provide details of the 64 GL(5, 2)-orbits of partial spreads which exist in PG(4, 2). In the second column $N_r$ is the number of reguli contained in the partial spread $\mathcal{S}_r$. In the final column it should be noted that **the references are to sections, lemmas and theorems of [7], and not to ones in the present paper.**

Table D.1a: the classes of partial spreads $\mathcal{S}_r$, $1 \leq r \leq 6$

| Class($\mathcal{S}_r$) | $N_r$ | Type | $\mathcal{G}(\mathcal{S}_r)$ | profile($\mathcal{S}_r$) | Notes |
|---|---|---|---|---|---|
| Ia.1 | 0 | O | 64512 | (111) | $2^6$:($L_2(2) \times L_3(2)$) |
| IIa.1 | 0 | O | 1152 | $(111)^2$ | $\{2^4$:($L_2(2) \times L_2(2)$)$\}$.2 |
| IIIa.1 | 0 | O | Sym(4) × $Z_2$ | $(112)^3$ | §3.2, non-regulus, cyclic |
| IIIb.1 | 1 | I | 576 | $(222)^3$ | §3.1, regulus, cyclic |
| IVa.1 | 0 | O | Sym(4) | $(222)^4$ | §3.3.1, cyclic |
| IVb.1 | 0 | O | Sym(3) | $(123)^3(222)$ | §3.3.2 |
| IVc.1 | 1 | I | Sym(3) × $Z_2$ | $(233)^3(114)$ | §3.1, §3.5, eq. (3.28) |
| IVd.1 | 4 | $\binom{4}{3}$ | 1152 | $(444)^4$ | §3.1, cyclic |
| Va.1 | 0 | O | Sym(5) | $(333)^5$ | §3.4, §A.3.2, cyclic |
| Vb.1 | 0 | O | $D_8$ | $(333)^4(135)$ | §3.4 |
| Vc.1 | 0 | O | Sym(3) × $Z_2$ | $(333)^5$ | §3.4 |
| Vd.1 | 0 | O | $Z_3$ | $(234)^3(333)^2$ | §3.4 |
| Ve.1 | 0 | O | $Z_5$ | $(234)^5$ | §3.4, cyclic |
| Vf.1 | 1 | I | $Z_2$ | $(344)^2(335).(234)^2$ | §3.5 |
| Vg.1 | 1 | I | $Z_2$ | $(344)^3.(135)(225)$ | §3.5 |
| Vh.1 | 2 | L | $D_8$ | $(355).(335)^4$ | §3.5, lemma 4.1 |
| Vi.1 | 4 | $\binom{4}{3}$ | Sym(4) × $Z_2$ | $(555)^4.(117)$ | §3.1 |
| Vj.1 | 10 | $\binom{5}{3}$ | $2^4{:}\Gamma\,\mathrm{L}(2,4)$ | $(777)^5$ | §3.1, **maximal**, cyclic |
| VIa.1 | 0 | O | $Z_6$ | $(445)^6$ | §6.2, cyclic |
| VIb.1 | 0 | O | $Z_2$ | $(355)^2(445)^4$ | §6.2 |
| VIc.1 | 1 | I | Sym(3) | $(555)^3.(355)^3$ | §7.3.1 |
| VIc.2 | 1 | I | Sym(3) | $(555)^3.(355)^3$ | §7.3.1 |
| VId.1 | 1 | I | Sym(3) | $(555)^3.(445)^3$ | §7.3.1 |
| VId.2 | 1 | I | Sym(3) | $(555)^3.(445)^3$ | §7.3.1 |
| VIe.1 | 1 | I | 1 | $(456)^2(555).(346)^2(445)$ | §7.3.1 |
| VIe.2 | 1 | I | 1 | $(456)^2(555).(346)^2(445)$ | §7.3.1 |
| VIf.1 | 1 | I | $Z_2$ | $(555)^3.(355)^2(337)$ | §7.3.1 |
| VIg.1 | 2 | II | $Z_4$ | $2 \times (456)^2(447)$ | §7.2.1 |
| VIh.1 | 2 | L | 1 | $(566).(456)^2(447)(555).(355)$ | §4.2, lemma 4.2 |
| VIh.2 | 2 | L | $Z_2$ | $(557).(456)^4.(355)$ | §4.2, lemma 4.2 |
| VIi.1 | 3 | Δ | Sym(3) | $(557)^3.(555)^3$ | §4.2, lemma 4.3 |
| VIj.1 | 4 | $\binom{4}{3}$ | $D_8$ | $(667)^4.(337)^2$ | §7.1, lemma 7.2 |

Table D.1b: the classes of partial spreads $\mathcal{S}_r,\ r > 6$

| Class($\mathcal{S}_r$) | $N_r$ | Type | $\mathcal{G}(\mathcal{S}_r)$ | profile($\mathcal{S}_r$) | Notes |
|---|---|---|---|---|---|
| VIIa.1 | 0 | $\mathsf{O}$ | 7:3 | $(666)^7$ | §6.1, cyclic |
| VIIb.1 | 1 | $\mathsf{I}$ | $Z_2$ | $(668)(677)^2.(567)^4$ | §7.3.2 |
| VIIb.2 | 1 | $\mathsf{I}$ | $Z_2$ | $(677)^3.(558)(567)^2(666)$ | §7.3.2 |
| VIIb.3 | 1 | $\mathsf{I}$ | $Z_2$ | $(677)^3.(558)(567)^2(666)$ | §7.3.2 |
| VIIc.1 | 2 | $\mathsf{II}$ | $Z_4$ | $\{2 \times (668)(677)^2\}.(558)$ | §7.2.2, lemma 7.3 |
| VIIc.2 | 2 | $\mathsf{II}$ | $Z_4$ | $\{2 \times (668)(677)^2\}.(558)$ | §7.2.2, lemma 7.3 |
| VIIc.3 | 2 | $\mathsf{L}$ | $Z_2$ | $(778).(668)^2(677)^2.(567)^2$ | §4.4 |
| VIIc.4 | 2 | $\mathsf{L}$ | $Z_2$ | $(778).(668)^2(677)^2.(567)^2$ | §4.4 |
| VIIc.5 | 2 | $\mathsf{L}$ | 1 | $(778).(668)^2(677)^2.(567)^2$ | §4.4 |
| VIIc.6 | 2 | $\mathsf{L}$ | 1 | $(778).(668)^2(677)^2.(567)^2$ | §4.4 |
| VIId.1 | 2 | $\mathsf{II}$ | 1 | $(668)^6.(666)$ | §7.2.2, lemma 7.3 |
| VIIe.1 | 3 | $\Delta$ | Sym(3) | $(778)^3.(677)^3.(666)$ | §4.3, theorem 4.5 |
| VIIe.2 | 3 | $\Delta$ | $Z_3$ | $(778)^3.(677)^3.(666)$ | §4.3, theorem 4.5 |
| VIIe.3 | 3 | $\Delta$ | $Z_2$ | $(778)^3.(677)^3.(666)$ | §4.3, theorem 4.5 |
| VIIe.4 | 3 | $\mathsf{Y}$ | $Z_6$ | $(888).(677)^6$ | §4.4 |
| VIIe.5 | 3 | $\mathsf{F}$ | $Z_2$ | $(778)^2.(668)(677)^4$ | §4.4 |
| VIIf.1 | 4 | $\binom{4}{3}_{(4,0)}$ | Sym(4) | $(888)^4.(558)^3$ | §7.1, **maximal** |
| VIIf.2 | 4 | $\binom{4}{3}_{(0,4)}$ | Sym(4) | $(888)^4.(558)^3$ | §7.1, **maximal** |
| VIIf.3 | 4 | $\binom{4}{3}_{(2,2)}$ | Sym(3) × $Z_2$ | $(888)^4.(558)^3$ | §7.1, **maximal** |
| | | | | | |
| VIIIa.1 | 0 | $\mathsf{O}$ | $2^3{:}\,\mathrm{F}_{21}$ | $(888)^8$ | §6.1, transitive |
| VIIIb.1 | 2 | $\mathsf{II}$ | $Z_2$ | $(88\mathtt{t})^6.(888)^2$ | §5.2 |
| VIIIc.1 | 3 | $\Delta$ | Sym(3) | $(99\mathtt{t})^3.(899)^3.(888)^2$ | §4.3, thm 4.5, §5.2 |
| VIIIc.2 | 3 | $\Delta$ | $Z_3$ | $(99\mathtt{t})^3.(899)^3.(888)^2$ | §4.3, thm 4.5, §5.2 |
| VIIIc.3 | 3 | $\Delta$ | $Z_2$ | $(99\mathtt{t})^3.(899)^3.(888)^2$ | §4.3, thm 4.5, §5.2 |
| VIIIc.4 | 3 | $\mathsf{Y}$ | $Z_3$ | $(\mathtt{ttt}).(899)^6.(888)$ | §5.2 |
| VIIIc.5 | 3 | $\mathsf{I}^{\rho}\mathsf{L}$ | $Z_2$ | $(99\mathtt{t}).(899)^2(88\mathtt{t})^2 \mid (899)^3$ | §5.2 |
| VIIIc.6 | 3 | $\mathsf{I}^{\kappa}\mathsf{L}$ | $Z_2$ | $(99\mathtt{t}).(899)^2(88\mathtt{t})^2 \mid (899)^3$ | §5.2 |
| VIIIc.7 | 3 | $\mathsf{F}$ | 1 | $(99\mathtt{t})^2.(888)(899)^4.(888)$ | §5.2 |
| | | | | | |
| IXa.1 | 4 | $\mathsf{X}$ | Alt(4) × $Z_2$ | $(13,13,13).(11,11,11)^8$ | §5.1, thms 5.1, 6.2 **maximal** |
| IXa.2 | 4 | $\mathsf{I}^{\rho}\Delta$ | Sym(3) | $(11,11,13)^3.(11,11,11)^6$ | §5.1, thm 5.1, **maximal** |
| IXa.3 | 4 | $\mathsf{I}^{\kappa}\Delta$ | Sym(3) | $(11,11,13)^3.(11,11,11)^6$ | §5.1, thm 5.1, **maximal** |
| IXa.4 | 4 | $\mathsf{E}$ | $Z_6$ | $(11,11,13)^3.(11,11,11)^6$ | §5.1, thm 5.1, **maximal** |

*Profiles* are as described in.[7, Section 1.1]. In the tables a profile $(1,2,3)^3(2,2,2)$ is written $(123)^3(222)$. We use `t` as an abbreviation for 10. Dots are used to separate groups of lines of the same valency, with the valencies occurring in descending order. Thus for an $\mathcal{S}_6$ of type $\mathsf{L}$ a profile $(557).(456)^4.(355)$ conveys the information that the line of valency 2 has profile $(5,5,7)$, each of the four lines of valency 1 has profile $(4,5,6)$ and the line of valency 0 has profile $(3,5,5)$. The entry $2 \times (456)^2(447)$ for the profile of class VIg.1, of type $\mathsf{II}$, indicates that both reguli contribute $(4,5,6)^2(4,4,7)$ to the overall profile. In the cases VIIIc.5, c.6 of an $\mathcal{S}_8$ of type $\mathsf{IL}$ the 7 lines which belong to a single regulus have a natural $4+3$ split, with the 3 forming the stand-alone regulus. A vertical line $|$ is used to separate the profile of the 4 from that of the 3.

*Reguli patterns and type.* Given a partial spread $\mathcal{S}_r$ in PG(4, 2), let $R_{ijk}$, for distinct $i, j, k$, carry the meaning that the triple of lines $\{\mu_i, \mu_j, \mu_k\} \subset \mathcal{S}_r$

is a regulus. The only *reguli patterns* which arise are those listed in the second column of the following Table D.2, which is reproduced from [7]. The regulus *type,* see the third column of Table D.2 (and also of Tables D.1a, D.1b), informs us of the pattern in a conveniently abbreviated form.

Table D.2 The possible reguli patterns

| $N$ | Regulus Pattern | Type | Types for $\mathcal{S}_r \setminus \{\mu\}$ |
|---|---|---|---|
| 0 | regulus-free | O | O |
| 1 | $R_{123}$ | I | I (if $r > 3$), O |
| 2 | $R_{123}, R_{456}$ | II | II (if $r > 6$), I |
| 2 | $R_{123}, R_{345}$ | L | L (if $r > 5$), I, O |
| 3 | $R_{123}, R_{456}, R_{678}$ | IL | L, II, I |
| 3 | $R_{123}, R_{145}, R_{167}$ | Y | Y (if $r > 7$), L, O |
| 3 | $R_{123}, R_{145}, R_{267}$ | F | F (if $r > 7$), II, L, I |
| 3 | $R_{123}, R_{345}, R_{561}$ | Δ | Δ (if $r > 6$), L, I |
| 4 | $R_{123}, R_{345}, R_{561}, R_{789}$ | IΔ | Δ, IL, II |
| 4 | $R_{123}, R_{145}, R_{167}, R_{189}$ | X | Y, O |
| 4 | $R_{123}, R_{145}, R_{267}, R_{389}$ | E | F, II |
| 4 | $R_{123}, R_{124}, R_{134}, R_{234}$ | $\binom{4}{3}$ | $\binom{4}{3}$ (if $r > 4$), I |
| 10 | $R_{ijk},\ 1 \leq i < j < k \leq 5$ | $\binom{5}{3}$ | $\binom{4}{3}$ ($r = 5$) |

# E Appendix: classification of external flats

A complete classification of flats external to the Grassmannian $\mathcal{G}_{1,4,2}$ was achieved in [14], [15]:

**Theorem E.1** *Under the action of* $\mathrm{GL}(5,2)$ *there are precisely ten orbits of flats in* $\mathrm{PG}(9,2)$ *which are external to the Grassmannian* $\mathcal{G}_{1,4,2}$*, namely those listed in the table E.1 below.*

Concerning column 4, $N_0(X_k)$ is the number of extensions of the $k$-flat $X_k$ to an external $(k+1)$-flat. Concerning column 5, $M_{k+2}(X_k)$ is the number of $(k+2)$-caps $\mathcal{C}_{k+2} \subset \mathcal{G}_{1,4,2}$ such that $\mathcal{E}(\mathcal{C}_{k+2}) = X_k$. For example, a solid $D \in \mathrm{orb}(3\beta)$ can be expressed in the form $D = \mathcal{E}(\mathcal{C}_5)$ for precisely three choices of partial spread $\mathcal{S}_5$.

Table E.1: $|\mathcal{G}_{X_k}|$, $N_0(X_k)$ and $M_{k+2}(X_k)$

| $k$ | orbit of $X_k$ | $\lvert\mathcal{G}_{X_k}\rvert$ | $N_0(X_k)$ | $M_{k+2}(X_k)$ |
|---|---|---|---|---|
| 0 | $\mathrm{Rk}_4$ | 11520 | 366 | $M_2 = 10$ |
| 1 | $\mathrm{orb}(1\alpha)$ | 96 | 128 | $M_3 = 2$ |
|  | $\mathrm{orb}(1\beta)$ | 5760 | 120 | $M_3 = 10$ |
| 2 | $\mathrm{orb}(2\alpha)$ | 6 | 29 | $M_4 = 1$ |
|  | $\mathrm{orb}(2\beta)$ | 48 | 24 | $M_4 = 4$ |
|  | $\mathrm{orb}(2\gamma)$ | 168 | 28 | $M_4 = 0$ |
| 3 | $\mathrm{orb}(3\alpha)$ | 5 | 2 | $M_5 = 1$ |
|  | $\mathrm{orb}(3\beta)$ | 6 | 0 | $M_5 = 3$ |
| 4 | $\mathrm{orb}(4+)$ | 155 | 0 | $M_6 = 0$ |
|  | $\mathrm{orb}(4-)$ | 155 | 0 | $M_6 = 0$ |

*Notes.* (i) Flats belonging to the orbits $\mathrm{orb}(1\beta)$, $\mathrm{orb}(2\beta)$ and $\mathrm{orb}(3\beta)$ are *special* flats. Unlike the $k$-flats, $k > 0$, belonging to the other orbits, these special flats lie inside $\mathcal{W}(\sigma)$, see eq. (6.22), for some solid $\sigma \subset \mathrm{PG}(4,2)$.
(ii) Planes $P \in \mathrm{orb}(2\gamma)$, which (unlike those belonging to $\mathrm{orb}(2\alpha)$ and $\mathrm{orb}(2\beta)$) can not be expressed in the form $P = \mathcal{E}(\mathcal{C}_4)$, are treated further in [16]. There they are shown to be in bijective correspondence with *conclaves* (see [21]) of planes in $\mathrm{PG}(4,2)$. (In $\mathrm{PG}(4,2)$ a conclave of eight planes is the dual of a regulus-free partial spread of eight lines.)
(iii) See [11] and [14, Section 6.2] for applications of the classification to coding theory, and also for consideration of generalizations to $\mathrm{PG}(n,q)$, as in Cooperstein [3] and Cossidente & Siciliano [4].

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