SOME ASPECTS OF THE LINEAR GROUPS $\mathrm{GL}(n, q)$

NEIL A. GORDON*, TREVOR M. JARVIS†, AND RON SHAW‡

Abstract. The first half of the paper summarizes results relevant to the action of the group $\mathrm{GL}(V_n)$ upon the n-dimensional vector space $V_n = V(n, \mathbb{F})$, especially in the case $\mathrm{GL}(V_n) = \mathrm{GL}(n, q)$ when the base field $\mathbb{F}$ is the finite Galois field $\mathrm{GF}(q)$. In particular it is noted that a representative for an arbitrary class of $\mathrm{GL}(n, q)$ can be simply constructed using solely the $2n - 1$ members of a fixed set $\{S_1, \ldots, S_n;\ J_2, \ldots, J_n\}$ of linear mappings. Here $S_k \in \mathrm{GL}(k, q)$ has order $q^k - 1$ and $J_k \in \mathrm{GL}(k, q)$ is unipotent of index k.

In the second half we specialize to the case $q = 2$, and deal with the action of $\mathrm{GL}(n, 2)$ upon the projective space $\mathrm{PG}(n-1, 2)$. Using elementary methods, all classes of the linear groups $\mathrm{GL}(n, 2)$, $n < 7$, are studied in considerable detail. It is intended that the tabulated results should be immediately accessible to finite geometers, and to all others who have need of these groups. In the case $n = 4$ attention is paid to the maximal subgroup $\Gamma\mathrm{L}(2, 4)$. In the case $n = 6$ the maximal subgroups $\Gamma\mathrm{L}(2, 8)$ and $\Gamma\mathrm{L}(3, 4)$ are treated, as are class aspects of the tensor product structure $V_6 = V_2 \otimes V_3$, and of the exterior product structure $V_6 = \wedge^2 V_4$.

Key words. Finite Linear Groups, Classes, Irreducible Polynomials, Singer Elements, $\mathrm{GL}(n, 2)$

1. Introduction and plan. We use $V_n = V(n, \mathbb{F})$ to denote an n-dimensional vector space over a field $\mathbb{F}$. In section 3 onwards the field is finite: $\mathbb{F} = \mathrm{GF}(q)$, $q = p^h$, where p is a prime, and $V_n = V(n, q)$; consequently the general linear group $\mathrm{GL}(V_n)$ is a finite group, denoted $\mathrm{GL}(n, q)$. Moreover $q = 2$ from section 4 onwards. If one is investigating an area of finite geometry, or design or coding theory, where the base field is $\mathrm{GF}(q)$ then one is quite likely to require particular facts concerning the elements of one of the finite groups $\mathrm{GL}(n, q)$. *The main aim of the present paper is to make readily accessible (even to the non-expert) such facts for the finite group* $\mathrm{GL}(n, 2)$, *acting upon* $\mathrm{PG}(n - 1, 2)$ *in the cases* $n < 7$.

In the case of the groups $\mathrm{GL}(4, 2)$, $\mathrm{GL}(5, 2)$ and $\mathrm{GL}(6, 2)$ the chief facts are displayed in appendix A, see tables 3, 4 and 5a,b . In these tables the rows refer to the different conjugacy classes of the group, the number of distinct classes of $\mathrm{GL}(n, 2)$ being 14, 27, 60 according as $n = 4, 5, 6$. The columns in these tables convey information concerning such things as power maps, characteristic and minimal polynomials, fixed points, cycle type and centralizers. For a description of this information see section 5.1.

It seems to us that the information about fixed points and cycle types for elements of $\mathrm{GL}(n, 2)$ acting upon $\mathrm{PG}(n-1, 2)$ is especially useful. Such information was certainly needed at several stages in the course of the classification, see [5], of all the partial spreads in $\mathrm{PG}(4, 2)$. For example, if a partial spread $\mathcal{S}_r$ of r lines of $\mathrm{PG}(4, 2)$ is *cyclic* — that is if there exists $A \in \mathrm{GL}(5, 2)$ of order r such that $\mathcal{S}_r$ is of the form $\{\lambda, A(\lambda),\ \ldots\ A^{r-1}(\lambda)\}$ for some line λ of $\mathrm{PG}(4, 2)$ — then it is clearly necessary that A, in its action upon $\mathrm{PG}(4, 2)$, should have at least three cycles of length r. Upon glancing at table 4 in appendix A we immediately deduce that no cyclic $\mathcal{S}_8$ exists, and that in attempting to construct a cyclic $\mathcal{S}_6$ one must use an element $A \in \mathrm{GL}(5, 2)$ of class 6B and not of class 6A. Using such an element A one quickly checks that a cyclic $\mathcal{S}_6$ in fact exists, see [14, equation (3.2)]; it is allocated to the $\mathrm{GL}(5, 2)$-orbit VIa.1 in [5, Table B.2].

*Department of Computer Science, University of Hull, Hull HU6 7RX n.a.gordon@hull.ac.uk
†Department of Mathematics, University of Hull, Hull HU6 7RX t.m.jarvis@hull.ac.uk
‡Department of Mathematics, University of Hull, Hull HU6 7RX r.shaw@hull.ac.uk

As a second example, consider the problem of classifying the *r-dimensional normalized linear sections*, denoted $\mathrm{NLS}_r(n,q)$'s, of $\mathrm{GL}(n,q)$. (Such a section is, by definition, an r-dimensional subspace of the n^2-dimensional vector space $\mathrm{End}(n,q)$ which contains the identity element $I_n \in \mathrm{GL}(n,q)$ and is such that every non-zero element lies in $\mathrm{GL}(n,q)$.) This classification problem was posed in [4], but was solved, for $n > 2$, only in the cases of $\mathrm{GL}(3,2)$ and $\mathrm{GL}(4,2)$. It is easy to see, [4, Lemma 2.1], that each element A of an $\mathrm{NLS}_r(n,q)$, other than the scalar multiples of I_n, must be fixed-point-free upon the points of $\mathrm{PG}(n-1,q)$. Thus, see table 3, the elements of an $\mathrm{NLS}_r(4,2)$ must be drawn solely from the classes 1A, 3A, 5A, 6A and 15A,B of $\mathrm{GL}(4,2)$ and, see table 4, the elements of an $\mathrm{NLS}_r(5,2)$ must be drawn solely from the classes 1A, 21A,B and 31A-31F of $\mathrm{GL}(5,2)$. See section 6.2 below for a few more details.

In section 7 we also provide information, see tables 6, 7, concerning certain maximal subgroups of $\mathrm{GL}(n,2)$, namely the maximal subgroups $\Gamma\,\mathrm{L}(2,4)$ of $\mathrm{GL}(4,2)$, and $\Gamma\,\mathrm{L}(3,4)$, $\Gamma\,\mathrm{L}(2,8)$ of $\mathrm{GL}(6,2)$. In the cases $n=4$ and $n=6$ we further provide, see theorems 5.2, 5.3 and 5.4, class information surrounding the tensor product and wedge product structures $V_4 = V_2 \otimes V_2$, $V_6 = V_2 \otimes V_3$ and $V_6 = V_4 \wedge V_4$.

Before we specialize to the case of a finite field it helps to have to hand, see the next section 2, certain notions and results valid over any field. When displaying canonical forms, we find it convenient to adopt the slightly unorthodox tensor product choice in eq. (2.8) for our indecomposable blocks. This choice is not only matrix-free but, as spelled out in section 3.4, leads to considerable economy when treating the classes of $\mathrm{GL}(n,q)$, in that a representative for an arbitrary class of $\mathrm{GL}(n,q)$ can be simply constructed using solely the $2n-1$ members of a fixed set $\{S_1, \ldots, S_n;\ J_2, \ldots, J_n\}$ of linear mappings.

In section 3 we consider various aspects of $\mathrm{GL}(n,q)$ valid for a general Galois field $\mathrm{GF}(q)$. In particular we consider (i) Singer elements of $\mathrm{GL}(n,q)$ and irreducibility, and (ii) unipotent elements of $\mathrm{GL}(n,q)$, including, in certain cases, their centralizer orders. For the number of conjugacy classes of $\mathrm{GL}(n,q)$, and for material of a more advanced nature, see [3], [9], [15].

Much of the material in sections 2 and 3 is well-known, although for several of our results we found it difficult to pin down specific references. Nevertheless we believe the reader will welcome having all this material to hand in order to make the paper self-contained, and to streamline the GF(2) considerations which occupy us from section 4 onwards.

2. Mappings of finite-dimensional vector spaces.

2.1. Notation. For the moment we work over a general base field $\mathbb{F}$, and for $V_n = V(n,\mathbb{F})$ we consider the n^2-dimensional vector space $\mathrm{End}(V_n) = \mathrm{L}(V_n, V_n)$ consisting of all linear mappings of V_n into V_n. Take note that $\mathrm{End}(V_n)$ is in fact an associative algebra over $\mathbb{F}$, of dimension n^2.

Given $A \in \mathrm{End}(V_n)$, $B \in \mathrm{End}(V_k)$ we use $[A,B]$ to denote the set of linear maps $T : V_n \to V_k$ which *intertwine* A and B (in that order!):

$$[A,B] = \{T \in \mathrm{L}(V_n, V_k) \mid TA = BT\}.$$

The *intertwining space* $[A,B]$ is a vector subspace of the nk-dimensional vector space $\mathrm{L}(V_n, V_k)$. Two elements $A, B \in \mathrm{End}(V_n)$ are said to be *similar* whenever $[A,B]$ contains a nonsingular mapping T, in which case we write $A \sim B$. The *commutant* $[A]$ of $A \in \mathrm{End}(V_n)$ is defined to be $[A] := [A,A]$, and we denote by $\mathfrak{A}_A$ the subset of

$\mathrm{End}(V_n)$ consisting of all polynomials in A over $\mathbb{F}$. Observe that we have the *subalgebra* inclusions

$$\mathfrak{A}_A \subseteq [A] \subseteq \mathrm{End}(V_n). \tag{2.1}$$

If the (monic) minimal polynomial $\mu_A \in \mathbb{F}[t]$ of $A \in \mathrm{End}(V_n)$ has degree m then the subalgebra $\mathfrak{A}_A$ has dimension m, with $\{I, A, \dots, A^{m-1}\}$ a basis. Of course we have $m \leq n = \dim V_n = \deg \chi_A$, where $\chi_A \in \mathbb{F}[t]$ is the characteristic polynomial of A. (Recall that μ_A divides χ_A; moreover μ_A and χ_A share the same irreducible factors, although in general with different multiplicities.)

If $A, B \in \mathrm{GL}(V_n)$ then $A \sim B$ if and only if they belong to the same conjugacy class of $\mathrm{GL}(V_n)$. The *centralizer* $\{X \in \mathrm{GL}(V_n) \mid XA = AX\}$ in $\mathrm{GL}(V_n)$ of an element $A \in \mathrm{GL}(V_n)$ is denoted $C(A)$. Note that we have the *subgroup* inclusions

$$\langle A \rangle \subseteq C(A) \subseteq \mathrm{GL}(V_n), \tag{2.2}$$

where $\langle A \rangle \cong Z_r$, $r =$ order of A.

A subspace $V_r \subseteq V_n$ is *invariant* under the action of $A \in \mathrm{End}(V_n)$ if $Av \in V_r$ for all $v \in V_r$. Given $e \in V$ the subspace $W = \mathfrak{A}_A e$, which is spanned by the vectors $A^s e$, $s \geq 0$, is an example of an A-invariant subspace; such an invariant subspace is termed the *cyclic subspace* for A *generated* by the vector e. If $\mathfrak{A}_A e = V_n$ then the vector e is termed a *cyclic vector* for A.

If there exists a non-zero proper subspace $V_r \subset V_n$ which is invariant under the action of $A \in \mathrm{End}(V_n)$ then A is said to be *reducible.* If V_n admits a non-trivial direct sum decomposition $V_n = V_r \oplus V_s$ where both V_r and V_s are A-invariant, then $A \in \mathrm{End}(V_n)$ is said to be *decomposable,* and we write $A = A_r \oplus A_s$ where A_r and A_s are the restrictions of A to the subspaces V_r and V_s. Here, and often below, we use lower indices attached to a linear mapping to indicate the dimension; thus $A_n, B_n, J_n, S_n, \dots \in \mathrm{End}(V_n)$. The identity mapping is denoted I_n. (However Z_n and D_n will denote the cyclic and dihedral groups of the indicated order.)

Consider an element $N_n \in \mathrm{End}(V_n)$ which is nilpotent of index n, satisfying $(N_n)^n = 0$ and $(N_n)^{n-1} \neq 0$. The minimal and characteristic polynomials of $N = N_n$ are $\mu_N = \chi_N = t^n$. Choosing $e_1 \in V_n$ such that $N^{n-1}e_1 \neq 0$, then e_1 is a cyclic vector for N: if $e_i = N^{i-1}e_1$, $i = 2, 3, \dots, n$, then $\{e_1, \dots, e_n\}$ is a basis for V_n upon which the effect of N_n is $e_1 \mapsto e_2 \mapsto \ \dots \ \mapsto e_n \mapsto 0$. Defining $J_n = I_n + N_n$, then $J_n \in \mathrm{GL}(V_n)$. In matrix terms, relative to $\{e_1, \dots, e_n\}$ as basis, J_n is a Jordan matrix of size n with 1's down the main diagonal:

$$J_n = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} = I_n + N_n, \quad \text{where} \quad N_n = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{2.3}$$

In future sections we will reserve the notations N_n and $J_n = I_n + J_n$ for elements of $\mathrm{End}(V_n)$ of the preceding kind. If $N = N_n$ and $J = I + N$, observe that

$$\mathfrak{A}_J = \mathfrak{A}_N = \prec \ I, N, \dots, N^{n-1} \succ . \tag{2.4}$$

(We use $\prec \ u, v, \dots, w \succ$ to denote the linear span, over the agreed field $\mathbb{F}$, of elements $u, v, \dots, w$ of a vector space.)

2.2. Useful general results. In the present section we list a number of general results for linear mappings of V_n which hold for a general base field $\mathbb{F}$. Most of these are well-known, in particular the two decomposition results, theorems 2.1 and 2.2, which are stated without proof. (However we will adopt an unorthodox choice for the indecomposable "blocks" which occur in the classical canonical form for an element $A \in \mathrm{GL}(V_n)$, see eq. (2.8)). Presumably theorem 2.3 also states known material, and so the proof is omitted; however details are provided in the research report [11].

THEOREM 2.1. (Cyclic decomposition theorem) *If $A \in \mathrm{End}(V_n)$ then there exists a direct sum decomposition $V_n = W_1 \oplus \ldots \oplus W_l$ such that*

(i) each W_i is a non-zero cyclic subspace for A;

(ii) the minimal polynomials μ_i of the restriction maps $A|_{W_i}$ satisfy the division properties $\mu_1|\mu_2 \ldots |\mu_l$;

(iii) moreover the polynomials (but not the subspaces W_i, if $l > 1$) are uniquely determined by (i) and (ii);

(iv) $\mu_A = \mu_l$ and $\chi_A = \mu_1\mu_2 \ldots \mu_l$.

In the next theorem let $\mu = \pi_1^{r_1} \ldots \pi_k^{r_k}$, $r_i \geq 1$, be the primary decomposition of the minimal polynomial $\mu = \mu_A$ of $A \in \mathrm{End}(V_n)$, the monic polynomials $\pi_1, \ldots, \pi_k$ being irreducible and distinct. Concerning the last half of the theorem note that, because the polynomials $\mu_i = \mu/{\pi_i}^{r_i}$, $i = 1, \ldots, k$, are relatively prime, there exist polynomials $\psi_1, \ldots, \psi_k$ such that $\sum_i \mu_i\psi_i = 1$. The final assertions (2.6) follow, because any $X \in [A]$ commutes with each of the polynomials $E^{(i)} = \mu_i(A)\psi_i(A)$.

THEOREM 2.2. (Primary decomposition theorem) *In the foregoing let $A^{(i)}$ denote the restriction of $A \in \mathrm{End}(V_n)$ to the A-invariant subspace $V^{(i)} = \ker \pi_i(A)^{r_i}$. Then we have the direct sum decompositions*

$$V_n = \oplus_{i=1}^k V^{(i)}, \qquad A = \oplus_{i=1}^k A^{(i)}, \tag{2.5}$$

the minimal polynomial of $A^{(i)}$ being ${\pi_i}^{r_i}$.

If $I_n = \sum_i E^{(i)}$ is the decomposition of the identity associated with the direct sum decompositions (2.5), then each of the mutually annihilating projection operators $E^{(1)}, \ldots, E^{(k)}$ is a polynomial in $A : E^{(i)} = \mu_i(A)\psi_i(A)$. Also

$$[A] = \oplus_i [A^{(i)}] \quad \textit{and, if } A \in \mathrm{GL}(V_n), \textit{ then} \quad C(A) \cong C(A^{(1)}) \times \ldots \times C(A^{(k)}). \tag{2.6}$$

THEOREM 2.3. *If $A \in \mathrm{End}(V_n)$ then*

(i) $\mathfrak{A}_A$ is a field $\iff \mu_A$ is irreducible;

(ii) A is irreducible $\iff \chi_A$ is irreducible;

(iii) A is cyclic on $V \iff \chi_A = \mu_A \iff [A] = \mathfrak{A}_A$;

(iv) A is indecomposable if and only if, for some $u \geq 1$, $\chi_A = \mu_A = \pi^u$ where π is monic and irreducible.

REMARK 2.4. *See eq. (3.9) for the $\mathbb{F} = \mathrm{GF}(q)$ version of part (i). The three properties in part (iii) of the theorem hold in particular for any irreducible element $A \in \mathrm{End}(V)$. Concerning part (iv), since $\ker \pi(A)$ is an invariant subspace if $\mu_A = \pi^u$, note that A is indecomposable but reducible if and only if $\chi_A = \mu_A = \pi^u$ for some $u > 1$.*

A mapping $A^{(i)}$ in theorem 2.2 has $\mu_{A^{(i)}} = \pi_i^{r_i}$ and $\chi_{A^{(i)}} = \pi_i^{s_i}$ with $r_i \leq s_i$. Whenever r_i is less than s_i we obtain, by applying theorem 2.1, a decomposition of $A^{(i)}$ as a direct sum of *classical canonical "blocks"*, each block B being such that $\chi_B = \mu_B = \pi^u$, for some $u \leq r_i$, and therefore indecomposable, by theorem 2.3(iv). Such a decomposition applies to each summand $A^{(i)}$ in (2.5), and we thereby end up with a general $A \in \mathrm{End}(V_n)$ being expressed in *classical canonical form*, as a direct sum of indecomposable blocks.

In matrix terms an indecomposable block B with $\chi_B = \mu_B = \pi^u$ is commonly taken to have the $u \times u$ block matrix form

$$B_{\text{classical}} = \begin{pmatrix} C_d & & & & \\ M_d & C_d & & & \\ & M_d & C_d & & \\ & & \ddots & \ddots & \\ & & & M_d & C_d \end{pmatrix}, \tag{2.7}$$

where C_d is the $d \times d$ companion matrix of the irreducible polynomial π of degree d and where every element $(M_d)_{ij}$ of the $d \times d$ matrix M_d is 0 except that $(M_d)_{1d} = 1$. For the purposes of displaying canonical forms *for elements of* $\mathrm{GL}(V_n)$ we prefer instead to use

$$B = J_u \otimes C_d, \tag{2.8}$$

where $J_u = I_u + N_u \in \mathrm{GL}(V_u)$ is as in section 2.1 and C_d is an irreducible element of $\mathrm{GL}(V_d)$. Besides being economically expressed, our choice (2.8) has the virtue of being matrix-free; moreover it is also helpful, see lemma 3.9, in the calculation of centralizers. By the next lemma our choice of indecomposable block is a valid one.

LEMMA 2.5. *If $B = J_u \otimes C_d \in \mathrm{End}(V_u \otimes V_d)$ is as in eq. (2.8), with C_d irreducible and $\chi_{C_d} = \pi$, then $\mu_B = \chi_B = \pi^u$.*

Proof. See [11]. □

REMARK 2.6. *Our choice (2.8) for B can be viewed in matrix terms as that matrix, B_C say, which is obtained from $B_{classical}$ in (2.7) by replacing M_d by C_d. Another possible choice of indecomposable block for the canonical form of an element of $\mathrm{GL}(V_n)$ is the matrix, B_I say, which is obtained from $B_{classical}$ in (2.7) by replacing M_d by I_d; cf. [7, p.406]. That B_I is similar to B_C may be seen quite explicitly:*

$$\mathrm{diag}(C_d^{u-1}, \ldots, C_d, I) B_C = B_I \, \mathrm{diag}(C_d^{u-1}, \ldots, C_d, I).$$

We now prove several lemmas which will be of use in the computation of centralizers of elements of $\mathrm{GL}(V_n)$. In the first lemma we consider a decomposable element $A \in \mathrm{End}(V_n) = \mathrm{End}(V_r \oplus V_s)$ of the form $A = A_r \oplus A_s$, with $A_r \in \mathrm{End}(V_r)$ and $A_s \in \mathrm{End}(V_s)$. A general element $X \in \mathrm{End}(V_n)$ is of the form $(x, y) \mapsto (Rx + Sy, Tx + Uy)$, for $R \in \mathrm{End}(V_r), S \in \mathrm{L}(V_s, V_r), T \in \mathrm{L}(V_r, V_s)$, and $U \in \mathrm{End}(V_s)$. In such a setting we will often adopt the block notation

$$A = \begin{pmatrix} A_r & 0 \\ 0 & A_s \end{pmatrix}, \quad X = \begin{pmatrix} R & S \\ T & U \end{pmatrix}. \tag{2.9}$$

LEMMA 2.7. *Suppose that $A \in \mathrm{End}(V_n)$ possesses the decomposition $A = A_r \oplus A_s$ with respect to a direct sum decomposition $V_n = V_r \oplus V_s$. Then, for $X \in \mathrm{End}(V_n)$ as in (2.9),*

(i) $X \in [A]$ if and only if $R \in [A_r], S \in [A_s, A_r], T \in [A_r, A_s]$ and $U \in [A_s]$;

(ii) if $[A_s, A_r] = \{0\}$ and $[A_r, A_s] = \{0\}$, then $[A_r \oplus A_s] = [A_r] \oplus [A_s]$; if further $A \in \mathrm{GL}(V_n)$, then $C(A_r \oplus A_s) \cong C(A_r) \times C(A_s)$.

Proof. The proof of (i) is immediate, and (ii) follows from (i). □

LEMMA 2.8. *Suppose that* $A_r \in \operatorname{End}(V_r)$ *is fixed-point-free on* $V_r \setminus \{0\}$. *Then* $[A_r \oplus I_s] \cong [A_r] \oplus \operatorname{End}(V_s)$. *If further* $A_r \in \mathrm{GL}(V_r)$, *then* $C(A_r \oplus I_s) \cong C(A_r) \times \mathrm{GL}(V_s)$, *and in particular* $C(A_r \oplus I_1) \cong C(A_r) \times \mathbb{F}^\times$.

Proof. If A_r is fixed-point-free on $V_r \setminus \{0\}$ then one easily sees that $[I_s, A_r] = \{0\}$ and $[A_r, I_s] = \{0\}$, and so the lemma follows from lemma 2.7(ii). □

LEMMA 2.9. *If* $A = A_r \otimes I_k \in \operatorname{End}(V_r \otimes V_k)$ *then* $[A] = [A_r] \otimes \operatorname{End}(V_k)$.

Proof. Straightforward: see, for example, [10, theorem 8.6.2]. □

Caution. If $A \in \mathrm{GL}(V_r)$ it does *not* follow that $C(A_r \otimes I_k) = C(A_r) \times \mathrm{GL}(V_k)$. Cf. eq. (5.2).

In the sequel it will prove useful to have to hand certain properties of the mappings J_n, as in the next two lemmas.

LEMMA 2.10. *Let* $J_n = I_n + N_n \in \mathrm{GL}(V_n)$ *be as in eqs. (2.3), (2.4). Then* $[J_n] = \mathfrak{A}_N = \prec\ I, N, \ldots, N^{n-1} \succ$ *and*

$$C(J_n) = \{a_0 I + a_1 N + \ldots + a_{n-1} N^{n-1} \mid a_i \in \mathbb{F}, a_0 \neq 0\}. \tag{2.10}$$

Proof. The result for $[J_n]$ follows from theorem 2.3(iii) and eq. (2.4), and the result for $C(J_n)$ is an immediate consequence. □

Viewing N_r and N_s as matrices as in eq. (2.3), both parts of the next lemma may be verified by straightforward matrix computations.

LEMMA 2.11. *If* $r > s > 0$ *then, setting* $t = r - s$,

(i) $S \in [N_s, N_r]$ *if and only if* S *is of the block form* $S = \begin{pmatrix} 0_{t\times s} \\ S' \end{pmatrix}$ *with* $S' \in \mathfrak{A}_{N_s} = \prec\ I, N_s, \ldots, N_s^{s-1} \succ$;

(ii) $T \in [N_r, N_s]$ *if and only if* T *is of the block form* $T = (T' | 0_{s\times t})$ *with* $T' \in \mathfrak{A}_{N_s}$.

We end this section with a very general result: it holds for *semilinear* automorphisms of a vector space V, whose dimension, over an arbitrary field $\mathbb{F}$, may even be *infinite* — provided only that the order $o(A) = a$ of A is finite. Such an element $A \in \Gamma\mathrm{L}(V)$ permutes the vectors of V in A-cycles (= orbits of the group $\langle A \rangle \cong Z_a$) of the form $Z = \{x, Ax, \ldots, A^{l-1}x\}$, whose length $l = |Z|$ is finite, satisfying $A^l z = z$ and $A^i z \neq z$ for $z \in Z$ and $0 < i < l$. Of course $l | a$. By a *long cycle* for A we mean an A-cycle of length $l = a$.

THEOREM 2.12. *([12]) If* $A \in \Gamma\mathrm{L}(V)$ *is of finite order then* A, *acting on the vectors of* V, *possesses at least one long cycle.*

3. Some $\mathrm{GL}(n, q)$ results.

3.1. Introduction. In the present section our base field $\mathbb{F}$ will be $\mathrm{GF}(q)$, with $q = p^h$, p a prime; so $V_n = V(n, q)$, $\operatorname{End}(V_n) = \operatorname{End}(n, q)$ and $\mathrm{GL}(V_n) = \mathrm{GL}(n, q)$. The order of this last group is

$$|\,\mathrm{GL}(n, q)| = q^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n} (q^i - 1). \tag{3.1}$$

We will consider properties of $\mathrm{GL}(n, q)$ which make specific use of the finiteness of the base field, and which also involve irreducible polynomials over $\mathrm{GF}(q)$. So let us first remind ourselves of certain well-known facts in this area, a good reference here being [8].

Recall that, for each prime power p^h, a field $\mathrm{GF}(p^h)$ of order p^h exists and is unique up to isomorphism. We will appeal to this uniqueness in several places below, for example in the proof of lemma 3.1. Next the multiplicative group $\mathrm{GF}(q)^\times$ of the nonzero elements of $\mathrm{GF}(q)$ is isomorphic to Z_{q-1}, and any generator s of the cyclic group $\mathrm{GF}(q)^\times$ is termed a *primitive element* of $\mathrm{GF}(q)$. Given $a \in \mathrm{GF}(q)^\times$ then $a = s^r$, for some r such that $1 \le r \le q-1$, and its order $\mathrm{o}(a)$ is $(q-1)/(r, q-1)$. For each positive divisor e of $q-1$ there are $\phi_{\mathrm{Euler}}(e)$ elements in $\mathrm{GF}(q)^\times$ of order e; in particular there are $\phi_{\mathrm{Euler}}(q-1)$ primitive elements in $\mathrm{GF}(q)$. (Here ϕ_{Euler} is Euler's ϕ-function.)

Since we will be dealing with $V(n,q)$ we are interested in the field $\mathrm{GF}(q^n)$ *considered as a vector space of dimension n over the subfield* $\mathrm{GF}(q)$. Now a field $\mathbb{F}$ satisfying $\mathrm{GF}(q) \subseteq \mathbb{F} \subseteq \mathrm{GF}(q^n)$ is necessarily isomorphic to $\mathrm{GF}(q^d)$ for some divisor d of n. Moreover, for each choice of positive divisor d of n, there is a unique $\mathrm{GF}(q^d)$ subfield of $\mathrm{GF}(q^n)$, namely that given by

$$\mathrm{GF}(q^d) = \{a \in \mathrm{GF}(q^n) \mid a^{q^d} = a\}, \tag{3.2}$$

the elements of $\mathrm{GF}(q^d)^\times$ being precisely those elements $a \in \mathrm{GF}(q^n)^\times$ such that $\mathrm{o}(a)$ divides $q^d - 1$.

If d is any positive divisor of n then each monic irreducible polynomial in $\mathrm{GF}(q)[t]$ *of degree d occurs precisely once in the factorization of the polynomial* $t^{q^n} - t \in \mathrm{GF}(q)[t]$; *moreover there are no other monic irreducible factors.* It follows, on equating degrees, that

$$q^n = \sum\nolimits_{d|n} dN(d,q), \tag{3.3}$$

where $N(d,q)$ denotes the number of monic irreducible polynomials in $\mathrm{GF}(q)[t]$ of degree d. By use of the Möbius inversion formula we obtain from (3.3) the following well-known formula, see [8, theorem 3.25], for $N(n,q)$:

$$N(n,q) = \frac{1}{n} \sum\nolimits_{d|n} \mathrm{Möb}(d) q^{n/d}, \tag{3.4}$$

where $\mathrm{Möb}(d)$ is the Möbius function, commonly denoted $\mu(d)$.

So the number of monic irreducible polynomials in $\mathrm{GF}(q)[t]$ of degree ≤ 6 is:

$$\begin{aligned} &N(1,q) = q,\ N(2,q) = \tfrac{1}{2}(q^2 - q),\ N(3,q) = \tfrac{1}{3}(q^3 - q),\ N(4,q) = \tfrac{1}{4}(q^4 - q^2),\\ &N(5,q) = \tfrac{1}{5}(q^5 - q),\ N(6,q) = \tfrac{1}{6}(q^6 - q^3 - q^2 + q). \end{aligned} \tag{3.5}$$

Let the minimal polynomial, *over the subfield* $\mathrm{GF}(q)$, of a field element $a \in \mathrm{GF}(q^n)$ be denoted by $\mu_a \in \mathrm{GF}(q)[t]$, let $\mathrm{GF}(q)(a)$ denote the simple extension of $\mathrm{GF}(q)$ obtained by adjoining a, and let $d = \deg \mu_a$. Then

$$\text{(i) } \mu_a \text{ is irreducible;} \quad \text{(ii) } d \text{ divides } n; \quad \text{(iii) } \mathrm{GF}(q)(a) \cong \mathrm{GF}(q^d). \tag{3.6}$$

Moreover

$$\mu_a = (t-a)(t-a^q) \dots (t - a^{q^{d-1}}) = \mu_{a^q} = \dots = \mu_{a^{q^{d-1}}}, \tag{3.7}$$

the d roots $\{a, a^q, \dots, a^{q^{d-1}}\}$ of μ_a in $\mathrm{GF}(q^n)$ being distinct. Now $t^{q^n} - t = \Pi_{a \in \mathrm{GF}(q^n)}(t - a)$ and so, by the lead-in to eq. (3.3), it follows that, in $\mathrm{GF}(q)[t]$, *each irreducible polynomial whose degree d divides n is the minimal polynomial μ_a of some* $a \in \mathrm{GF}(q^n)$.

Let s be a primitive element of $\mathrm{GF}(q^n)$, so that a general element $a \in \mathrm{GF}(q^n)^\times \cong Z_{q^n - 1}$ is a power of s. Note that

$$\text{if } a = s^r,\ 1 \le r \le q^n - 1,\ \text{then } \mathrm{o}(a) = (q^n - 1)/(r, q^n - 1). \tag{3.8}$$

In such a context we find it convenient to say that $a = s^r$ is an *ordinary power* of the primitive element $s \in \mathrm{GF}(q^n)$ whenever $\mathrm{GF}(q)(a) \cong \mathrm{GF}(q^n)$. So, by (3.6), a is an ordinary power whenever $\deg \mu_a = n$, or, by (3.2), whenever $\mathrm{o}(a)$ does not divide $q^d - 1$ for some divisor $d < n$ of n. If instead $\mathrm{GF}(q)(a) \cong \mathrm{GF}(q^d)$ for $d < n$, we will say that $a = s^r$ is a $\mathrm{GF}(q^d)$-*subfield power* of s. This last is the case precisely when $\deg \mu_a = d$, or also when d is the smallest positive divisor of n such that $\mathrm{o}(a)$ divides $q^d - 1$. There are $nN(n, q)$ ordinary powers of s, each irreducible polynomial $\in \mathrm{GF}(q)[t]$ of degree n contributing, cf. (3.7), n ordinary powers. Similarly the $\mathrm{GF}(q^d)$-subfield powers of s number $dN(d, q)$. For example, if s is a primitive element of $\mathrm{GF}(2^8)$, and so $\mathrm{o}(s) = 255$, then $s^r, 1 \leq r \leq 255$, is a GF(4)-subfield power of s if and only if $r \in \{85, 170\}$, and is a GF(16)-subfield power if and only if $r = 17k$ for $k \in \{1, 2, \ldots, 14\} \setminus \{5, 10\}$; otherwise $s^r($ if $\neq 1)$ is an ordinary power.

3.2. Singer elements of $\mathrm{GL}(n, q)$ and irreducibility. In the next three subsections we are still very much in familiar territory, and so we presume that most of the material presented must be "well-known". Nevertheless we did not succeed in finding references for many of our specific results. We delay until section 4 some further results relevant to the case $q = 2$ which is our chief concern.

Lemma 3.1. *If $A \in \mathrm{GL}(n, q)$, and $\deg \mu_A = d$, then*

$$\mu_A \textit{ is irreducible} \Longleftrightarrow \mathfrak{A}_A \cong \mathrm{GF}(q^d). \tag{3.9}$$

If μ_A is irreducible then

$$[A] \cong \mathrm{End}(n/d, q^d) \quad \textit{and} \quad C(A) \cong \mathrm{GL}(n/d, q^d). \tag{3.10}$$

Proof. (Note that if μ_A is irreducible then $\chi_A = (\mu_A)^k$ for some integer $k \geq 1$ and so d divides $n (= dk)$.) The mapping $\phi \mapsto \phi(A)$ (which maps t to A) is an algebra epimorphism $\mathrm{GF}(q)[t] \to \mathfrak{A}_A$ whose kernel is the ideal (μ_A) of $\mathrm{GF}(q)[t]$ which is generated by μ_A. So the mapping gives rise to an algebra isomorphism

$$\mathrm{GF}(q)[t]/(\mu_A) \cong \mathfrak{A}_A. \tag{3.11}$$

Now recall the standard result, see e.g. [8, theorems 1.61, 1.86], that $\mathbb{F}[t]/(\pi)$ is a field if and only if π is irreducible, and that if π is irreducible then the field $\mathbb{F}[t]/(\pi)$ is a simple algebraic extension of $\mathbb{F}$ of degree $d = \deg \pi$. So the result (3.9) follows, since a field $\mathfrak{A}_A$ which is a simple algebraic extension (obtained by adjoining A) of $\mathrm{GF}(q)I_n \cong \mathrm{GF}(q)$ of degree d is isomorphic to $\mathrm{GF}(q^d)$.

Given that μ_A is irreducible, and hence $\mathfrak{A}_A \cong \mathrm{GF}(q^d)$, we may view $V(n, q)$ as a vector space $V(k, q^d)$ of dimension k over the field $\mathfrak{A}_A$. In which case observe that $X \in [A]$ precisely if X is $\mathfrak{A}_A$-linear. So (3.10) follows: $[A] = \mathrm{End}(V(k, q^d))$ and $C(A) = \mathrm{GL}(V(k, q^d))$. □

Lemma 3.2. *If $A \in \mathrm{GL}(n, q)$ then*

$$\chi_A \textit{ is irreducible} \Longleftrightarrow \mathfrak{A}_A \cong \mathrm{GF}(q^n). \tag{3.12}$$

If χ_A is irreducible then

$$[A] \cong \mathrm{GF}(q^n) \quad \textit{and} \quad C(A) \cong Z_{q^n - 1}. \tag{3.13}$$

Proof. This is the special case $d = n$ of lemma 3.1. □

Lemma 3.3. *The order $\mathrm{o}(A)$ of any $A \in \mathrm{GL}(n, q)$ satisfies $\mathrm{o}(A) \leq q^n - 1$.*

Proof. If $\deg \mu_A = m$ then $\dim \mathfrak{A}_A = m$ and so $|\mathfrak{A}_A \setminus \{0\}| = q^m - 1$. But $A^r \in \mathfrak{A}_A \setminus \{0\}$, and hence at most $q^m - 1$ of the powers A^r are distinct. Thus $\mathrm{o}(A) \leq q^m - 1$. But (Cayley-Hamilton) $m \leq n$. □

We are thus motivated to consider elements of $\mathrm{GL}(n, q)$ whose order is the upper bound in the lemma, and so we make the following definition.

Definition. An element $S_n \in \mathrm{GL}(n, q)$ is said to be a *Singer element* if $\mathrm{o}(S_n) = q^n - 1$.

LEMMA 3.4. *For each (n, q), $q = p^h$, the group $\mathrm{GL}(n, q)$ possesses Singer elements.*

Proof. Choose $\pi \in \mathrm{GF}(q)[t]$ to be irreducible and of degree n. Choose $A \in \mathrm{GL}(n, q)$ to have matrix, relative to some basis, the companion matrix of π. Then $\chi_A = \pi$ is irreducible and so, by lemma 3.2, $\mathfrak{A}_A \cong \mathrm{GF}(q^n)$. Choosing S to be a primitive element of the field $\mathfrak{A}_A$ then $\mathrm{o}(S) = q^n - 1$. □

THEOREM 3.5. *If S is a Singer element of $\mathrm{GL}(n, q)$ then*

(i) $\mathfrak{A}_S$ is a field $\cong \mathrm{GF}(q^n)$, with S a primitive element;

(ii) S is irreducible and χ_S is irreducible;

(iii) S permutes the $q^n - 1$ nonzero vectors of $V(n, q)$ in a single cycle;

(iv) if $A = S^r \neq I$ then A is fixed-point-free on $V(n, q) \setminus \{0\}$.

Proof. (i) Since $\dim \mathfrak{A}_S \leq n$, for any $S \in \mathrm{GL}(n, q)$ we have $|\mathfrak{A}_S \setminus \{0\}| \leq q^n - 1$. But if $\mathrm{o}(S) = q^n - 1$ then $\mathfrak{A}_S \setminus \{0\}$ contains $q^n - 1$ distinct powers of S. So $\dim \mathfrak{A}_S = n$ and every element of $\mathfrak{A}_S \setminus \{0\}$ is a power of S and therefore invertible. Hence $\mathfrak{A}_S$ is a field $\mathbb{F}$ of order q^n and so is isomorphic to $\mathrm{GF}(q^n)$.

(ii) By lemma 3.2 and theorem 2.3(ii), assertion (ii) follows from (i).

(iii) If $0 \neq e \in V(n, q)$ then the $q^n - 1$ vectors $\langle S \rangle e$ are distinct since, by (i), $\langle S \rangle \cup \{0\}$ is a field. So (iii) holds. (Alternatively, (iii) follows from theorem 2.12.)

(iv) This follows from (iii). □

THEOREM 3.6. *Let S be a fixed choice of Singer element of $\mathrm{GL}(n, q)$.*

(i) If $A \in \langle S \rangle$, and so $A = S^r$, with $1 \leq r \leq q^n - 1$, and if $d = \deg \mu_A$, then

$$\text{(a) } \mu_A \text{ is irreducible;} \qquad \text{(b) } d \text{ divides } n; \qquad \text{(c) } \mathfrak{A}_A \cong \mathrm{GF}(q^d); \tag{3.14}$$

$$\text{(d) } [A] \cong \mathrm{End}(n/d, q^d); \qquad \text{(e) } C(A) \cong \mathrm{GL}(n/d, q^d). \tag{3.15}$$

(ii) For each divisor $d > 1$ of n there are $N(d, q)$ classes of $\mathrm{GL}(n, q)$ for which the minimal polynomial μ is irreducible and of degree d, and there are $N(1, q) - 1 = q - 1$ classes with $\deg \mu = 1$. Each such class has a representative A which is some $\mathrm{GF}(q^d)$-subfield power S^r of S.

(iii) There are $N(n, q)$ classes of irreducible elements $\in \mathrm{GL}(n, q)$, $(n > 1)$, each having a representative A which is some ordinary power of S.

(iv) If $A = S^r$ is as in part (i), then A is completely reducible, being the direct sum of n/d copies of an irreducible element $\in \mathrm{GL}(d, q)$. Moreover the d elements of the family

$$\mathcal{F}_A = \{A, A^q, \ldots, A^{q^{d-1}}\} \tag{3.16}$$

are distinct and lie in the same class of $\mathrm{GL}(n, q)$, and they are not conjugate to any other element of $\langle S \rangle$.

Proof. Since the minimal polynomial of $A = S^r$ as an element of the field $\mathfrak{A}_S$ coincides with its minimal polynomial as an element of $\mathrm{End}(n, q)$, the considerations in the second half of section 3.1 apply to the field $\mathfrak{A}_S \cong \mathrm{GF}(q^n)$, with $A = S^r$ and $\mathfrak{A}_A$ corresponding to the previous $a = s^r$ and $\mathrm{GF}(q)(a)$.

(i) Consequently the results (3.14) follow from those in (3.6). Also (3.15) follows from (3.10).

(ii) If μ is irreducible then $\chi = \mu^k$ for some $k \geq 1$, and so the degree d of μ divides $n(= dk)$. The stated results follow from (i) upon recalling that in $\mathrm{GF}(q)[t]$ there are $N(d, q)$ irreducible monic polynomials of degree d, and, see after eq. (3.7), each irreducible polynomial of degree d dividing n is the minimal polynomial μ_A of some $A \in \mathfrak{A}_S \cong \mathrm{GF}(q^n)$. (In the case $d = 1$ the irreducible polynomial t corresponds to $A = 0 \notin \mathrm{GL}(n, q)$, and so is discarded.)

(iii) Since A is irreducible if and only if χ_A is irreducible, (iii) is the special case $d = n$ of (ii).

(iv) This follows because, see eq. (3.7), the d elements of $\mathcal{F}_A$ share the same minimal polynomial, which is moreover irreducible, and because other elements of $\langle S \rangle$ have a different minimal polynomial. □

3.3. Unipotent elements of $\mathrm{GL}(n, q)$**.** If $A_n \in \mathrm{GL}(n, q)$ is such that $A_n - I_n$ is nilpotent of index $m (\leq n)$ then A_n is said to be *unipotent of index* m; in which case $\chi_{A_n} = (t-1)^n$ and $\mu_{A_n} = (t-1)^m$. Recall our notation J_n for a special kind of unipotent element of $\mathrm{GL}(n, q)$, namely one of index $m = n$. A general unipotent element $A_n \in \mathrm{GL}(n, q)$ is a direct sum of such J_r (including $J_1 = I_1$). In the next lemma we suppose as usual that $q = p^h$ with p prime; we also make use of the subset $\mathbb{N}_{r,p}$ of $\mathbb{N}$ defined by

$$\mathbb{N}_{r,p} = \{m \in \mathbb{N} \mid p^{r-1} < m \leq p^r\}. \tag{3.17}$$

LEMMA 3.7. *(i) An element A of $\mathrm{GL}(n, q)$ has order p^r if and only if A is unipotent of index $m \in \mathbb{N}_{r,p}$.*

(ii) Let k be the smallest integer such that $p^k \geq u$, and let $B = J_u \otimes C_d$ be as in eq. (2.8). Then

$$\mathrm{o}(J_u) = p^k \quad \textit{and} \quad \mathrm{o}(B) = p^k \times \mathrm{o}(C_d). \tag{3.18}$$

Proof. (i) Given that $\mathrm{o}(A) = p^r$ then $(A - I)^{p^r} = A^{p^r} - I = 0$, so that A is unipotent of index $m \leq p^r$. In fact m must lie in $\mathbb{N}_{r,p}$, since assuming $m \leq p^{r-1}$ leads to $0 = (A - I)^{p^{r-1}} = A^{p^{r-1}} - I$, contradicting $\mathrm{o}(A) = p^r$. In the other direction, given that A is unipotent of index $m \in \mathbb{N}_{r,p}$, then $A^{p^{r-1}} - I = (A - I)^{p^{r-1}} \neq 0$ and $A^{p^r} - I = (A - I)^{p^r} = 0$, whence $\mathrm{o}(A) = p^r$.

(ii) Since J_u is unipotent of index u, the result for $\mathrm{o}(J_u)$ follows from part (i). Since C_d is irreducible, it is a (ordinary) power of a Singer element of $\mathrm{GL}(d, q)$, and so $\mathrm{o}(C_d)$ divides $q^d - 1$. So $(p, \mathrm{o}(C_d)) = 1$, and the result for $\mathrm{o}(B)$ follows from that for $\mathrm{o}(J_u)$. □

The next lemma gives the centralizer order for certain unipotent elements of $\mathrm{GL}(n, q)$.

LEMMA 3.8. *(i)* $|C(J_n)| = (q-1)q^{n-1}$.

(ii) If $r > s > 0$, then $|C(J_r \oplus J_s)| = (q-1)^2 q^{r+3s-2}$.

(iii) If $r > 1$ and $s > 0$ then $|C(J_r \oplus I_s)| = (q-1)q^{r+2s-1}|\mathrm{GL}(s, q)|$.

Proof. (i) This follows immediately from eq. (2.10).

(ii) Let the notation be as in eq. (2.9) and lemma 2.7, with $A_r = J_r$ and $A_s = J_s$. Then, by lemmas 2.7 and 2.10, $X \in [J_r \oplus J_s]$ if and only if $R \in \mathfrak{A}_{N_r}$, $U \in \mathfrak{A}_{N_s}$ and S and T are as in lemma 2.11. Since $t > 0$, we then see that $\det X = (\det R)(\det U)$. Hence $X \in C(J_r \oplus J_s)$ if and only if $R \in C(J_r)$, $U \in C(J_s)$ and S and T are as in lemma 2.11. Now $|\mathfrak{A}_{N_s}| = q^s$, and so we have $|C(J_r \oplus J_s)| = q^s.q^s.|C(J_r)|.|C(J_s)| = (q-1)^2 q^{r+3s-2}$, since, from lemma 2.10, $|C(J_r)| = (q-1)q^{r-1}$ and $|C(J_s)| = (q-1)q^{s-1}$.

(iii) Proceeding as in the proof of part (ii), we see that $X \in [J_r \oplus I_s]$ if and only if $R \in \mathfrak{A}_{N_r}$, $U \in \mathrm{End}(V_s)$ and S and T are as follows: the final row of S is arbitrary, but the other rows are all zero, and the first column of T is arbitrary, but the other columns are all zero. It follows that $\det X = (\det R)(\det U)$. Hence $X \in C(J_r \oplus I_s)$ if and only if $R \in C(J_r)$, $U \in \mathrm{GL}(V_s)$ and S and T are as just described. So we have $|C(J_r \oplus I_s)| = q^s.q^s.|C(J_r)|.|\mathrm{GL}(s,q)| = (q-1)q^{r+2s-1}|\mathrm{GL}(s,q)|$. □

Observe that parts (ii) and (iii) still hold in the special case $s = 1$, $r > 1$, each yielding the result

$$|C(J_r \oplus I_1)| = (q-1)^2 q^{r+1}, \quad (r > 1). \tag{3.19}$$

Lemma 3.9. *If, as in eq. (2.8), $B = J_u \otimes C_d$ where $C_d \in \mathrm{GL}(d,q)$ is irreducible, then $|C(B)| = (q^d-1)q^{(u-1)d}$.*

Proof. Since $[C_d] \cong \mathrm{GF}(q^d)$, we may view B as a scalar multiple of an element $J_u \in \mathrm{GL}(u,q^d)$. So the result follows upon replacing q by q^d and n by u in lemma 3.8(i). □

3.4. Consequences for the classes of $\mathrm{GL}(n,q)$. In each dimension $k \le n$, let us choose a Singer element $S_k \in \mathrm{GL}(k,q)$ and, for $2 \le k \le n$, let us also choose a unipotent element $J_k \in \mathrm{GL}(k,q)$ of index k. Our previous results entail the following:

Representatives for each conjugacy class of $\mathrm{GL}(n,q)$ can be simply constructed solely from the $2n-1$ elements $\{S_1, \dots, S_n;\ J_2, \dots J_n\}$.

The construction merely involves: (i) forming ordinary powers of the S_k (ii) forming tensor products of a J_u with ordinary powers of an S_k (iii) taking direct sums of elements of the kinds (i) and (ii).

To see this, all we need to do is to construct a suitable representative for each possible indecomposable block in the classical canonical form for elements $A \in \mathrm{GL}(n,q)$. Firstly, by theorem 3.6(iii), any irreducible block of size d has representative some ordinary power $(S_d)^r$ of S_d. Secondly consider an indecomposable block B in the canonical form for $A \in \mathrm{GL}(n,q)$ which is reducible, and so, see remark 2.4, $\chi_B = \mu_B = \pi^u$ with $u > 1$. Then, by lemma 2.5 and theorem 3.6(iii), B will have a representative of the form $J_u \otimes (S_d)^r$, for some ordinary power $(S_d)^r$ of S_d. The number of different ordinary powers of S_d needed is $N(d,q)$, one for each irreducible polynomial $\in \mathrm{GF}(q)[t]$ of degree d.

(If we were dealing with the canonical form for general elements of $\mathrm{End}(n,q)$ the above list of $2n-1$ elements would need to be extended to include the nilpotent elements $N_2, \dots, N_n$.)

Remark 3.10. *Observe that $S_1 = sI_1$ where s is a fixed choice of primitive element of $\mathrm{GF}(q)$, and so instead of $J_u \otimes (S_1)^r$ we may use aJ_u, with a a general element of $\mathrm{GF}(q)^\times$. For general q the canonical form of an element $A \in \mathrm{GL}(n,q)$ may thus involve direct sums of the kind $cI_1 \oplus c'I_1 \oplus \dots \oplus aJ_u \oplus a'J_{u'} \oplus \dots$ If $q = 2$, then $c = c' = \dots = 1$ and $a = a' = \dots = 1$, and so the preceding expression simplifies to $I_r \oplus J_u \oplus J_{u'} \oplus \dots$. Consequently results such as those of lemma 3.8(ii), (iii) are particularly useful in the $q = 2$ case to be considered in the rest of this paper.*

4. Some $\mathrm{GL}(n,2)$ results. We now deal with the case $q = 2$. (Nevertheless some of the results, for example those of lemmas 4.1 and 4.2, have easy generalizations to general q.) From eq. (3.1) the orders of $\mathrm{GL}(n,2)$ for small values of n are

$n =$	2	3	4	5	6
$\lvert\mathrm{GL}(n,2)\rvert$	2.3 $= 6$	$2^3.3.7$ $= 168$	$2^6.3^2.5.7$ $= 20,160$	$2^{10}.3^2.5.7.31$ $= 9,999,360$	$2^{15}.3^4.5.7^2.31$ $= 20,158,709,760$

. (4.1)

Lemma 4.1. *If $A \in \mathrm{GL}(n,2)$ has odd order then A is completely reducible: $A = \oplus_i A^{(i)}$, where each $A^{(i)}$ acts irreducibly.*

Proof. Since the characteristic of the field does not divide the order of the finite group $\langle A \rangle$, Maschke's theorem applies. □

If $(0 \neq)\, a \in V_{n-1} \subset V_n$, where the hyperplane V_{n-1} has equation $\langle \alpha, x \rangle = 0$ with $(0 \neq)\, \alpha \in V_n^*$, then the *transvection* $J(a,\alpha)$ is defined by

$$J(a,\alpha)x = x + \langle \alpha, x \rangle a, \qquad x \in V_n\,. \tag{4.2}$$

In projective language $J(a,\alpha)$ yields that *elation* which has *centre* the projective point $a \in \mathrm{PG}(n-1,2)$ and *axis* the hyperplane $\alpha \subset \mathrm{PG}(n-1,2)$ (with $a \in \alpha$). As is well-known, the group $\mathrm{GL}(n,2)$ is generated by the transvections. Let V_{n-2} be any complement of $\prec a \succ$ inside the hyperplane V_{n-1} and choose any $b \in V_n \setminus V_{n-1}$ (i.e. such that $\langle \alpha, b \rangle = 1$). Then, relative to the direct sum decomposition $V_n = V_2 \oplus V_{n-2}$, where $V_2 = \prec b, a \succ$, the transvection $J(a,\alpha)$ is cast into the canonical form

$$J(a,\alpha) = J_2 \oplus I_{n-2}, \quad \text{where } J_2 b = b + a,\ J_2 a = a\,. \tag{4.3}$$

The next lemma now follows, after noting that there are $|\,\mathrm{PG}(n-1,2)| = 2^n - 1$ choices for α, and each hyperplane α has $|\,\mathrm{PG}(n-2,2)| = 2^{n-1} - 1$ points a.

Lemma 4.2. *The transvections form a single conjugacy class in* $\mathrm{GL}(n,2)$. *This class has length* $(2^n - 1)(2^{n-1} - 1)$.

Since the minimal polynomial $\mu_A(t)$ and characteristic polynomial $\chi_A(t)$ of $A \in \mathrm{GL}(n,2)$ do not have t as a factor, we will be interested in the *relevant* polynomials $f \in \mathrm{GF}(2)[t]$ of degree $d > 0$, namely those which satisfy $f(0) \neq 0$, or equivalently $f(0) = 1$. There are 2^{d-1} relevant polynomials of degree d. Also, of the irreducible polynomials of degree $d \geq 1$, all are relevant except for the polynomial t of degree $d = 1$. So, from eq. (3.5), the number of relevant irreducible polynomials $\in \mathrm{GF}(2)[t]$ of degree $d = 1,2,3,4,5,6$ is $1,1,2,3,6,9$, respectively. These are given in the next lemma. In this lemma we denote irreducible polynomials of degree d by $f_d, g_d, \ldots$, and we set $\hat{f}_d(t) := t^d f_d(t^{-1})$. (If $A \in \mathrm{GL}(n,2)$ has characteristic polynomial f_n then A^{-1} has characteristic polynomial $\hat{f}_n$.)

Lemma 4.3. *The relevant irreducible polynomials in* $\mathrm{GF}(2)[t]$ *of degree* $d \leq 6$ *are:*

$d \leq 2:$ $\quad f_1 = t+1\ (=\hat{f}_1);\quad f_2 = t^2+t+1\ (=\hat{f}_2);$
$d = 3:$ $\quad f_3 = t^3+t+1,\ \hat{f}_3 = t^3+t^2+1;$
$d = 4:$ $\quad f_4 = t^4+t+1,\ \hat{f}_4 = t^4+t^3+1;\quad g_4 = t^4+t^3+t^2+t+1\ (=\hat{g}_4);$
$d = 5:$ $\quad f_5 = t^5+t^2+1,\ \hat{f}_5 = t^5+t^3+1;\quad g_5 = t^5+t^4+t^3+t+1,$
$\hat{g}_5 = t^5+t^4+t^2+t+1;\quad h_5 = t^5+t^4+t^3+t^2+1,\ \hat{h}_5 = t^5+t^3+t^2+t+1;$
$d = 6:$ $\quad f_6 = t^6+t^5+1,\ \hat{f}_6 = t^6+t+1;\quad g_6 = t^6+t^5+t^2+t+1,$
$\hat{g}_6 = t^6+t^5+t^4+t+1;\quad h_6 = t^6+t^4+t^3+t+1,\ \hat{h}_6 = t^6+t^5+t^3+t^2+1;$
$k_6 = t^6+t^4+t^2+t+1,\ \hat{k}_6 = t^6+t^5+t^4+t^2+1;\quad l_6 = t^6+t^3+1\ (=\hat{l}_6).$

Proof. Either proceed step by step of increasing d: the relevant reducible polynomials of degree d are determined as appropriate products of the relevant irreducible polynomials of degree $< d$, and the remaining relevant polynomials of degree d must then be irreducible. Or see [8, table C]! □

In the next two lemmas we deal with Singer elements of $\mathrm{GL}(5,2)$ and $\mathrm{GL}(6,2)$. Since 31 is prime, all 30 elements $A \neq I$ of a Singer cyclic subgroup $Z_{31} \subset \mathrm{GL}(5,2)$ are Singer elements, and $Z_{31} \setminus \{I\}$ partitions into six families of the kind (3.16). Of course these 6 families correspond to the $N(5,2) = 6$ choices of irreducible polynomials over

GF(2) of degree 5. If $\mathcal{F}_A = \{A, A^2, A^4, A^8, A^{16}\}$ is one family, with minimal polynomial f_5, then, see the next lemma, the other families are $\mathcal{F}_{A^{-1}}, \mathcal{F}_{A^5}, \mathcal{F}_{A^{-5}}, \mathcal{F}_{A^6}, \mathcal{F}_{A^{-6}}$ with minimal polynomials $\hat{f}_5, \hat{g}_5, g_5, h_5, \hat{h}_5$ respectively.

In the case of a subgroup $Z_{63} \subset \mathrm{GL}(6,2)$, there are $\phi_{\mathrm{Euler}}(63) = 36$ Singer elements which fall into six families $\{A, A^2, A^4, A^8, A^{16}, A^{32}\}$ of the kind (3.16). There are also $\phi_{\mathrm{Euler}}(21) = 12$ elements of Z_{63} having order 21 which fall into two families, and there are $\phi_{\mathrm{Euler}}(9) = 6$ elements of order 9 which form a single family of mutually conjugate elements. These $6+2+1 = 9$ families correspond to the $N(6,2) = 9$ choices of irreducible polynomials over GF(2) of degree 6. Furthermore there are $\phi_{\mathrm{Euler}}(7) = 6$ elements of Z_{63} having order 7, and these fall into two families of the kind $\{A, A^2, A^4\}$ appropriate to $d = 3$ (and which correspond to the $N(3,2) = 2$ choices $f_3, \hat{f}_3$ of irreducible polynomials over GF(2) of degree 3). Also there are $\phi_{\mathrm{Euler}}(3) = 2$ elements having order 3 forming a single family $\{A, A^2\}$ appropriate to $d = 2$ (and corresponding to the single choice f_2 of irreducible polynomial over GF(2) of degree 2).

All of the results in the two lemmas can be obtained by carrying out routine computations. For example, in lemma 4.5(iii), if $C = S^3$ then C has order $63/(3,63) = 21$ and satisfies $C^2 + I = S^6 + I = S^5$, and hence $(C^2 + I)^3 = S^{15} = C^5$, that is $k_6(C) = 0$. And, in lemma 4.5(iv), if $D = S^7$ then D has order 9 and $D^3 = S^{21}$, of order 3, is a GF(4)-subfield power of S; so μ_{D^3} has degree 2 and hence $f_2(D^3) = 0$, that is $l_6(D) = 0$. The remaining proofs are omitted, but may be found in [11].

Lemma 4.4. *If the Singer element $S \in \mathrm{GL}(5,2)$ has minimal polynomial f_5 then the minimal polynomials of S^5 and S^6 are respectively $\hat{g}_5$ and h_5.*

Lemma 4.5. *If $S \in \mathrm{GL}(6,2)$ has minimal polynomial f_6 then*

(i) S is a Singer element of $\mathrm{GL}(6,2)$;

(ii) S^{-5} and S^{11} are Singer elements such that $\mu_{S^{-5}} = g_6$ and $\mu_{S^{11}} = h_6$;

(iii) S^3 has order 21 and $\mu_{S^3} = \hat{k}_6$;

(iv) S^7 has order 9 and $\mu_{S^7} = l_6$;

(v) S^9 has order 7 and $\mu_{S^9} = f_3$.

5. The classes of $\mathrm{GL}(n,2)$, $n \leq 6$.

5.1. Outline. In each dimension $k \geq 2$ let us make a fixed choice of Singer element $S_k \in \mathrm{GL}(k,2)$ and of unipotent element $J_k \in \mathrm{GL}(k,2)$ of index k. Then, see section 3.4, for any class of $\mathrm{GL}(n,2)$ a representative A_n can be chosen which is a direct sum of indecomposable elements of the four kinds (i) I_1; (ii) J_k; (iii) $(S_k)^r$; (iv) $J_u \otimes (S_k)^r$, for $N(k,2)$ selected choices of ordinary powers $(S_k)^r$ of S_k. If $n = 4$ or 5, then we need only the one indecomposable element $J_2 \otimes S_2$ of the kind (iv), and if $n = 6$ we only need three more, namely $J_3 \otimes S_2$, $J_2 \otimes S_3$, and $J_2 \otimes S_3^{-1}$. For dimension $n \leq 6$ the distinct possibilities are thus easily enumerated, and one finds that the group $\mathrm{GL}(n,2)$ has $C(n)$ classes where $C(2) = 3$, $C(3) = 6$, $C(4) = 14$, $C(5) = 27$, $C(6) = 60$.

In the rest of this section, and especially in the tables in appendix A, we provide detailed information concerning all of the classes of $\mathrm{GL}(n,2)$, $2 \leq n \leq 6$. For $2 < n < 6$ the names of the classes are as in the Atlas [2]. For each class of $\mathrm{GL}(n,2)$ we give a representative A_n of the form just described, and also the associated characteristic and minimal polynomials χ and μ. For each class we compute the centralizer order $|C(A_n)|$, and hence the length $|\mathrm{GL}(n,2)|/|C(A_n)|$ of the conjugacy class with representative $A_n \in \mathrm{GL}(n,2)$. (Since the class 2A of $\mathrm{GL}(n,2)$ consists of transvections, its length, namely $(2^n - 1)(2^{n-1} - 1)$, is more easily obtained as in lemma 4.2.) To reassure ourselves that we had not overlooked any possibility we carried out the arithmetical check that Σ(lengths) is indeed equal to $|\,\mathrm{GL}(n,2)|$.

In the column headed "F.p.'s" we indicate the fixed points of a representative A_n of a class of $\mathrm{GL}(n,2)$ *in projective language,* that is in terms of the natural action of $\mathrm{GL}(n,2)$ upon the projective space $\mathrm{PG}(n-1,2)$ (which last may be identified with $V_n \setminus \{0\}$). Thus if the vectors fixed by A_n form a V_2, this is reported in the table as the existence of a *line* of fixed points. If there are no fixed points, we indicate this by an entry f.p.f. (= fixed-point-free).

The column headed "Cycle type" refers to the permutational action of $A_n \in \mathrm{GL}(n,2)$ when acting upon $\mathrm{PG}(n-1,2)$. (The cycle type is readily computed and is of considerable use, see for example section 6.1.) By way of illustration, consider the permutational action of the element $A_4 = S_2 \oplus J_2 \in \mathrm{GL}(4,2)$ of class 6B (see table 3 in appendix A) upon the 15 points of $\mathrm{PG}(3,2) = V_4 \setminus \{0\}$. In projective terms a decomposition $V_4 = V_2 \oplus V_2$ determines two skew lines in $\mathrm{PG}(3,2)$, and S_2 acts as a 3-cycle on one line, and J_2 as a 2-cycle on the other line. It follows that A_4 acts as the product of one 6-cycle, two 3-cycles, one 2-cycle and one 1-cycle (fixed point) on $\mathrm{PG}(3,2)$. We will accordingly record the cycle type of A_4 on $\mathrm{PG}(3,2)$ as $6^1\,3^2\,2^1\,1^1$, and use a similar notation for general elements of $\mathrm{GL}(n,2)$.

For an element $A \in \mathrm{GL}(n,2)$ of composite order r it is of interest to know the class of A^p for those primes p which divide r. In many cases the required information is readily obtained from the representative column of the tables. For example class 21B of $\mathrm{GL}(6,2)$ has representative $A = S_2 \otimes S_3^{-1}$, and so $A^3 = I_2 \otimes S_3^4 \sim I_2 \otimes S_3$ is of class 7A; thus $(21\mathrm{B})^3 = 7\mathrm{A}$, and similarly $(21\mathrm{A})^3 = 7\mathrm{B}$. In some cases such *power maps* may be obtained by using the cycle type column. For example, class 8A of $\mathrm{GL}(6,2)$ has representative A with $\mathrm{CT}(A) = 8^4 4^6 2^2 1^3$, whence $\mathrm{CT}(A^2) = 4^8 2^{12} 1^7$, which cycle type is peculiar to class 4B; thus $(8\mathrm{A})^2 = 4\mathrm{B}$. However in the case of $A \in$ class 8B of $\mathrm{GL}(6,2)$ we have $\mathrm{CT}(A) = 8^6 4^3 2^1 1^1$, and so $\mathrm{CT}(A^2) = 4^{12} 2^6 1^3$, but this last cycle type is shared by classes 4C and 4E. The ambiguity is easily resolved: for since $\mu_A = (t+1)^6 = (t^2+1)^3$, it follows that $\mu_{A^2} = (t+1)^3$, which is the minimal polynomial for class 4C, but not for 4D; so $(8\mathrm{B})^2 = 4\mathrm{C}$. In the second column of tables 3-5 we provide the power map information in all cases where the order is composite, the prime divisors of the order being taken in the order $p < p' < p'' < \ldots$. Thus, in table 5b, the entry BAC for class 30B conveys the power map information $(30\mathrm{B})^2 = 15\mathrm{B}$, $(30\mathrm{B})^3 = 10\mathrm{A}$ and $(30\mathrm{B})^5 = 6\mathrm{C}$.

For information on the Singer elements of $\mathrm{GL}(5,2)$ and $\mathrm{GL}(6,2)$, and on their powers, see end of section 4.

In section 7 we also provide information, see tables 6, 7, concerning certain subgroups of $\mathrm{GL}(n,2)$, namely the maximal subgroups $\Gamma\mathrm{L}(2,4)$ of $\mathrm{GL}(4,2)$, and $\Gamma\mathrm{L}(3,4)$, $\Gamma\mathrm{L}(2,8)$ of $\mathrm{GL}(6,2)$. For these groups we made frequent use of MAGMA [1]. In the cases $n=4$ and $n=6$ we further provide, see theorems 5.2, 5.3 and 5.4, information surrounding the tensor product and wedge product structures $V_4 = V_2 \otimes V_2$, $V_6 = V_2 \otimes V_3$ and $V_6 = V_4 \wedge V_4$.

5.2. The classes of $\mathrm{GL}(2,2)$ and $\mathrm{GL}(3,2)$. We have $\mathrm{GL}(2,2) = \{I_2,\, S_2,\, S_2^{-1},\, J_2,\, J_2',\, J_2'\}$ where, in matrix terms,

$$S_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\; S_2^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\; J_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\; J_2' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\; J_2'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The classes of $\mathrm{GL}(2,2)$ are as in table 1, the Singer elements being S_2 and S_2^{-1}. The group $\mathrm{GL}(2,2)$ is isomorphic to the symmetric group $\mathrm{Sym}(3)$ on 3 objects (the 3 points of the projective line PG(1, 2)).

Table 1. Classes of GL(2, 2)								
Class	Representative	χ	μ	F.p.'s	Cycle type	$C(A)$	$\lvert C(A)\rvert$	Length
1A	I_2	$(f_1)^2$	f_1	line	1^3	GL(2, 2)	6	1
2A	$J_2 = I_2 + N_2$	$(f_1)^2$	$(f_1)^2$	point	$2^1\,1^1$	$\langle J_2\rangle \cong Z_2$	2	3
3A	S_2	f_2	f_2	f.p.f.	3^1	$\langle S_2\rangle \cong Z_3$	3	2

Representatives of the 6 classes of GL(3, 2) are now easily found, see table 2. In the final column of this table, lemma is abbreviated to L.

Table 2. Classes of GL(3, 2)									
Class	Reptve.	χ	μ	F.p.'s	Cycles	$C(A)$	$\lvert C(A)\rvert$	Length	Notes
1A	I_3	$(f_1)^3$	f_1	plane	1^7	GL(3, 2)	168	1	
2A	$J_2 \oplus I_1$	$(f_1)^3$	$(f_1)^2$	line	$2^2\,1^3$	D_8	8	21	eq. (5.1), L 4.2
3A	$S_2 \oplus I_1$	$f_1 f_2$	$f_1 f_2$	point	$3^2\,1^1$	Z_3	3	56	L 4.1, L 2.8
4A	J_3	$(f_1)^3$	$(f_1)^3$	point	$4^1\,2^1\,1^1$	Z_4	4	42	L 3.7, L 3.8(i)
7A	S_3	f_3	f_3	f.p.f.	7^1	Z_7	7	24	Singer,
7B	S_3^{-1}	$\hat{f}_3$	$\hat{f}_3$	f.p.f.	7^1	Z_7	7	24	§ 3.2

Concerning $C(J_2 \oplus I_1) \cong D_8$, see lemmas 2.11 and 3.8(iii) (proof):

$$\text{if } A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then } C(A_3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & b \\ c & 0 & 1 \end{pmatrix} : a, b, c \in \mathrm{GF}(2) \right\}, \tag{5.1}$$

the two elements with $b = c = 1$ being those of order 4 in $Z_4 \rtimes Z_2 = D_8$. Also, by lemma 4.2, the length of the class is $7 \times 3 = 21$.

5.3. The classes of GL(4, 2)**.** In accordance with our plan, see section 5.1, we now exhibit a representative A for each of the 14 classes of GL(4, 2); see table 3 in appendix A. The notes in the final column of the table, and in section 5.3.1 below, are chiefly concerned with details concerning the centralizer $C(A)$.

5.3.1. Notes (Table 3). (i) We may take $J_2 \oplus J_2$ in the form $A_4 = J_2 \otimes I_2$, where $J_2 = I_2 + N_2$. It then follows from lemmas 2.9 and 2.10 that $[A_4] = \prec I_2, N_2 \succ \otimes \mathrm{End}(V_2)$. Consequently $X \in C(A_4)$ if and only if X is of the form

$$X = I_2 \otimes A_2 + N_2 \otimes B_2 \qquad \text{for some } A_2 \in \mathrm{GL}(V_2),\ B_2 \in \mathrm{End}(V_2). \tag{5.2}$$

Since $|\,\mathrm{GL}(V_2)| = 6$ and $|\,\mathrm{End}(V_2)| = 16$, it follows that

$$|C(A_4)| = 6 \times 16 = 96. \tag{5.3}$$

(To see that X in (5.2) indeed lies in GL(4, 2), note that it has an inverse, namely $X^{-1} = I_2 \otimes A_2^{-1} + N_2 \otimes A_2^{-1} B_2 A_2^{-1}$. Equivalently, taking A_4 to have the block form $\begin{pmatrix} I_2 & 0 \\ I_2 & I_2 \end{pmatrix}$, then $X \in [A_4]$ if and only if $X = \begin{pmatrix} A_2 & 0 \\ B_2 & A_2 \end{pmatrix}$, and such X will, for any $B_2 \in \mathrm{End}(V_2)$, lie in GL(4, 2) if and only if $A_2 \in \mathrm{GL}(V_2)$.)

(ii) Identifying S_2 with $w \in \mathrm{GF}(4) = \{0, 1, w, w^2\}$, we may view $A_4 = J_2 \otimes S_2$ as the scalar multiple wJ_2 of an element $J_2 \in \mathrm{GL}(2, 4)$; see lemma 3.9, proof. So in effect we seek all $X \in \mathrm{End}(2,4)$ which commute with $A_4 = \begin{pmatrix} w & 0 \\ w & w \end{pmatrix}$. Such X are of the form $\begin{pmatrix} a_0 & 0 \\ a_1 & a_o \end{pmatrix}$, $a_i \in \mathrm{GF}(4)$; cf. lemma 2.10. Consequently

$$C(A_4) \cong \left\{ \begin{pmatrix} a_0 & 0 \\ a_1 & a_o \end{pmatrix} : a_i \in \mathrm{GF}(4),\ a_0 \neq 0 \right\} \quad \text{and so} \quad |C(A_4)| = 12. \tag{5.4}$$

REMARK 5.1. *For both $A = J_2 \otimes S_2$ and $A = J_2 \oplus S_2$ we find that $[A] = \prec I, A, A^2, A^3 \succ$ is of dimension 4. In the case $A = J_2 \otimes S_2$, with minimal polynomial $\mu(t) = t^4 + t^2 + 1 = (t^2 + t + 1)^2$, the only nonzero singular elements of $[A]$ are $I + J$, $I + J'$ and $I + J''$, where $J = A^3$, $J' = A + A^2$ and $J'' = A^4 + A^5 = I + A + A^2 + A^3$ are the 3 involutions in $C(A) = \langle A \rangle \times \langle J' \rangle \cong Z_6 \times Z_2 \cong Z_3 \times (Z_2)^2$. In the case $A = J_2 \oplus S_2$, with $\mu(t) = t^4 + t^3 + t + 1 = (t+1)^2(t^2+t+1)$, the only non-singular elements of $[A]$ are the powers of A.*

5.3.2. Tensor product mappings. We may view V_4 as a tensor product $V_2 \otimes V_2$ and so ask: which classes of $\mathrm{GL}(4,2)$ are represented by tensor product mappings of the form $A = B \otimes C$, for $B, C \in \mathrm{GL}(V_2)$? To answer this we need (leaving aside $I_4 = I_2 \otimes I_2$) to determine the $\mathrm{GL}(4,2)$ class of each of the five tensor product mappings

$$\text{(a) } J_2 \otimes I_2 \quad \text{(b) } S_2 \otimes I_2 \quad \text{(c) } J_2 \otimes S_2 \quad \text{(d) } S_2 \otimes S_2 \quad \text{(e) } J_2 \otimes J_2. \tag{5.5}$$

Now if $A = J_2 \otimes I_2$ then $A \sim J_2 \oplus J_2$ and so $A \in$ class 2B. Similarly if $A = S_2 \otimes I_2$ then $A \sim S_2 \oplus S_2$ and so $A \in$ class 3A. The mapping $J_2 \otimes S_2$ is already in our standard form, see class 6A in table 3. Concerning (d), if $A = S_2 \otimes S_2$ then $A^2 = (I + S_2) \otimes (I + S_2)$, whence $f_2(A) = S_2 \otimes I_2 + I_2 \otimes S_2 \neq 0$, and so A belongs to class 3B, and not class 3A, of table 3. In the case of (e) we may easily check that $J_2 \otimes J_2$, of order 2, has just 3 fixed points, and so belongs to class 2B. Alternatively the fact that $J_2 \otimes J_2$ is conjugate to $J_2 \oplus J_2$ can be seen quite explicitly: for, since $(S_2)^2 J_2 = J_2 S_2$, we have

$$\begin{pmatrix} I_2 & 0 \\ S_2 & I_2 \end{pmatrix} \begin{pmatrix} J_2 & 0 \\ J_2 & J_2 \end{pmatrix} = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ S_2 & I_2 \end{pmatrix}.$$

Summarizing:

THEOREM 5.2. *An element $A \in \mathrm{GL}(4,2)$ can be expressed in the form $A = B \otimes C$, for $B, C \in \mathrm{GL}(V_2)$, if and only if A belongs to one of the following classes in table 3: 1A, 2B, 3A, 3B, 6A.*

5.4. The classes of $\mathrm{GL}(5,2)$. Proceeding with our plan, we now exhibit representatives for each of the 27 classes of $\mathrm{GL}(5,2)$, see table 4 in appendix A. In the table, in order to save space, we abbreviate a paired entry such as $(f_1)^2 f_3$, $(f_1)^2 \hat{f}_3$ by $(f_1)^2 f_3\,\&\,(\hat{\ })$. Except for class 2B, the order of the centralizer follows immediately from the results in the previous sections, as indicated in the final column of the table.

Concerning class 2B, if we take $A_5 = (J_2 \otimes I_2) \oplus I_1$ to have the $(2+2+1) \times (2+2+1)$ block form $\begin{pmatrix} I_2 & 0 & 0 \\ I_2 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then, cf. eq. (5.2), $X \in C(A_5)$ if and only if $X = \begin{pmatrix} A_2 & 0 & 0 \\ B_2 & A_2 & x \\ {}^t y & 0 & 1 \end{pmatrix}$, where $A_2 \in \mathrm{GL}(V_2)$, $B_2 \in \mathrm{End}(V_2)$, $x, y \in (\mathrm{GF}(2))^2$. Hence

$$|C(A_5)| = 6 \times 2^4 \times 2^2 \times 2^2 = 2^9 \times 3 = 1536. \tag{5.6}$$

5.5. The classes of $\mathrm{GL}(6,2)$**.** Information concerning the 60 classes of $\mathrm{GL}(6,2)$ is given in tables 5a and 5b (see appendix A). In the 31A ... F entry x_5 runs through f_5, $\hat{g}_5$, $\hat{h}_5$; for the detailed assignments see table 4.

5.5.1. Notes (Tables 5a,b). (i) For the great majority of classes the order of the centralizer follows immediately from the results in the previous sections, as indicated in the final column of the tables. For example, consider $A = (I_2 \oplus J_2) \oplus S_2 \in$ class 6B. Since, see table 3, $|C(I_2 \oplus J_2)| = 192$, it follows from lemma 2.7(ii) that $|C(A)| = 192 \times 3 = 576$. Similarly for $A = J_2 \oplus (S_2 \oplus S_2) \in$ class 6C we have $|C(A)| = |C(J_2)| \times |C(S_2 \oplus S_2)| = 2 \times 180 = 360$, and for $A = S_2 \oplus (I_1 \oplus J_3) \in$ class 12B we have $|C(A)| = |C(S_2)| \times |C(I_1 \oplus J_3)| = 3 \times 16 = 48$.

For the remaining classes the orders of the centralizers are determined in notes (ii)-(viii) below.

(ii) Let the notation be as in eq. (2.9) and lemma 2.7, with $A = B \oplus C$, where $B = J_2 \oplus J_2$ and $C = I_2$. Then $X \in [A]$ if and only if $R \in [J_2 \oplus J_2]$, $U \in \mathrm{End}(V_2)$, S is of the form $\begin{pmatrix} * & 0 & * & 0 \\ * & 0 & * & 0 \end{pmatrix}$ and ${}^tT$ is of the form $\begin{pmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \end{pmatrix}$. We find that $\det X = (\det R)(\det U)$, and so $X \in C(A)$ if and only if $R \in C(J_2 \oplus J_2)$, $U \in \mathrm{GL}(V_2)$ and S and T are as just described. Hence, on recalling eq. (5.3), $|C(A)| = 2^4.2^4.|C(J_2 \oplus J_2)|.|\mathrm{GL}(2,2)| = 2^8.96.6 = 147,456$.

(iii) We may take $J_2 \oplus J_2 \oplus J_2$ in the form $A_6 = J_2 \otimes I_3$, where $J_2 = I_2 + N_2$. It then follows from lemmas 2.9 and 2.10 that $[A_6] = \prec I_2, N_2 \succ \otimes \mathrm{End}(V_3)$. Consequently $X \in C(A_6)$ if and only if X is of the form

$$X = I_2 \otimes A_3 + N_2 \otimes B_3, \qquad \text{for some } A_3 \in \mathrm{GL}(V_3),\ B_3 \in \mathrm{End}(V_3). \tag{5.7}$$

Since $|\mathrm{GL}(V_3)| = 168$ and $|\mathrm{End}(V_3)| = 2^9$, it follows that

$$|C(A_6)| = 168 \times 512 = 86,016. \tag{5.8}$$

(iv) By lemma 3.1, if $A \in$ class 3A then $C(A) \cong \mathrm{GL}(3,4)$, of order 181,440.

(v) Let the notation be as in eq. (2.9) and lemma 2.7, with $A = B \oplus C$, where $B = J_3 \oplus J_2$ and $C = I_1$. Then $X \in [A]$ if and only if $R \in [J_3 \oplus J_2]$, $U \in \mathrm{End}(V_1)$, S is of the form $(0\ 0\ *\ 0\ *)$ and ${}^tT$ is of the form $(*\ 0\ 0\ *\ 0)$. We find that $\det X = (\det R)(\det U)$, and so $X \in C(A)$ if and only if $R \in C(J_3 \oplus J_2)$, $U = 1$ and S and T are as just described. Hence, on recalling lemma 2.11, $|C(A)| = 2^2.2^2.|C(J_3 \oplus J_2)| = 2^4.2^7 = 2,048$.

(vi) We may take $J_3 \oplus J_3$ in the form $A_6 = J_3 \otimes I_2$, where $J_3 = I_3 + N_3$. It then follows from lemmas 2.9 and 2.10 that $[A_6] = \prec I_3, N_3, (N_3)^2 \succ \otimes \mathrm{End}(V_2)$. Consequently $X \in C(A_6)$ if and only if X is of the form

$$X = I_3 \otimes A_2 + N_2 \otimes B_2 + (N_2)^2 \otimes C_2, \quad \text{for some } A_2 \in \mathrm{GL}(V_2),\ B_2, C_2 \in \mathrm{End}(V_2). \tag{5.9}$$

Since $|\mathrm{GL}(V_2)| = 6$ and $|\mathrm{End}(V_2)| = 2^4$, it follows that

$$|C(A_6)| = 6.2^4.2^4 = 1,536. \tag{5.10}$$

(vii) Since $[S_2] \cong \mathrm{GF}(4)$, the problem of determining $C(A_6)$, where $A_6 = (S_2 \otimes J_2) \oplus S_2 = S_2 \otimes (J_2 \oplus I_1) \in$ class 6A, reduces to that of determining the centralizer inside $\mathrm{GL}(3,4)$ of the matrix $B = J_2 \oplus I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathrm{GF}(4))$. Now X

commutes with B if and only if $X = \begin{pmatrix} z & 0 & 0 \\ a & z & b \\ c & 0 & z' \end{pmatrix}$ for some $z, z', a, b, c \in \mathrm{GF}(4)$. Since $\det X \neq 0$ if and only if both $z \neq 0$ and $z' \neq 0$, it follows that $|C(A_6)| = 3^2.4^3 = 576$.

(viii) By lemma 3.1, if $A \in$ class 7A then $C(A) \cong \mathrm{GL}(2, 8)$, of order 3,528.

5.5.2. Tensor product mappings. We may view V_6 as a tensor product $V_2 \otimes V_3$ and so ask: which classes of $\mathrm{GL}(6, 2)$ are represented by tensor product mappings of the form $A = B \otimes C$, for $B \in \mathrm{GL}(V_2)$, $C \in \mathrm{GL}(V_3)$? Most possibilities are easily dealt with. Thus, since $B \otimes I_3 \sim B \oplus B \oplus B$, it follows that $J_2 \otimes I_3 \in$ class 2C and $S_2 \otimes I_3 \in$ class 3A. Since $I_2 \otimes C \sim C \oplus C$ it follows that $I_2 \otimes J_3 \in$ class 4C, $I_2 \otimes S_3 \in$ class 7A and $I_2 \otimes S_3^{-1} \in$ class 7B.

Also, in the cases where $C = C_2 \oplus I_1$ decomposes, we have $B \otimes C \sim (B \otimes C_2) \oplus B$, and so the class of $B \otimes C$ follows easily from that of $B \otimes C_2$, as determined in section 5.3.2. In this manner we see that

$$I_2 \otimes (J_2 \oplus I_1) \in \text{class 2B}, \quad I_2 \otimes (S_2 \oplus I_1) \in \text{class 3C}, \quad J_2 \otimes (J_2 \oplus I_1) \in \text{class 2C},$$
$$S_2 \otimes (J_2 \oplus I_1) \in \text{class 6A}, \quad J_2 \otimes (S_2 \oplus I_1) \in \text{class 6F}, \quad S_2 \otimes (S_2 \oplus I_1) \in \text{class 3C}.$$

(Recall, see after eq. (5.5), that $S_2 \otimes S_2 \in$ class 3B of $\mathrm{GL}(4, 2)$.)

We have still to consider the cases for which $B \neq I_2$ and C is indecomposable:

$$\text{(a) } J_3 \otimes S_2 \quad \text{(b) } J_2 \otimes S_3^{\pm 1} \quad \text{(c) } S_2 \otimes S_3^{\pm 1} \quad \text{(d) } J_2 \otimes J_3\,. \tag{5.11}$$

Case (a) is our standard form for an indecomposable element with $\chi = \mu = (f_2)^3$, and occurs as class 12A in table 5b. Similarly the cases (b) occur as classes 14A,B in table 5b. In case (c), if $A = S_2 \otimes S_3$ then $A^6 = I_2 \otimes (S_3^2 + I_3)$ and $A^4 = S_2 \otimes (S_3^2 + S_3)$, and so $A^6 + A^4 + A^2 + I = S_2 \otimes S_3 = A$, that is $k_6(A) = 0$. Hence $A \in$ class 21A in table 5b; consequently $A^{-1} \in$ class 21A in table 5b.

Finally consider, case (d), the element $A = J_2 \otimes J_3$. Since A is of order 4, $f_1(A)^4 = 0$. But, using $J_3^3 = J_3^2 + J_3 + I_3$, we see that $(A + I)^3 = (J_2 + I_2) \otimes (J_3^2 + I_3) \neq 0$. Hence $\chi_A = (f_1)^6$ and $\mu_A = (f_1)^4$, and so A belongs to class 4D or 4E in table 5a. *Note therefore that, unlike mappings (a), (b) and (c) of (5.11), $A = J_2 \otimes J_3$ is decomposable.* We now show that A is in fact of class 4E, with $V_2 \otimes V_3$ admitting (several) direct sum decompositions of the kind $V_2 \oplus V_4$ with respect to which A decomposes in the manner $A = J_2 \oplus J_4$. With the choice $J_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ we may take A in the block form $A = \begin{pmatrix} J_3 & 0 \\ J_3 & J_3 \end{pmatrix}$, and so readily see that $A + I = \begin{pmatrix} N_3 & 0 \\ J_3 & N_3 \end{pmatrix}$ has nullity 2. Thus the f.p.f. set of A is a line, and hence A belongs to class 4E (and not 4D with f.p.f. set a plane). To obtain $A = J_2 \otimes J_3$ in the form $A = J_2 \oplus J_4$, consider a product basis $\{e_i \otimes f_j\}$ for $V_2 \otimes V_3$, where $J_2 e_1 = e_1 + e_2$, $J_2 e_2 = e_2$ and $J_3 f_1 = f_1 + f_2$, $J_3 f_2 = f_2 + f_3$, $J_3 f_3 = f_3$. Then we see that $V_2 \otimes V_3$ admits the A-invariant decomposition

$$V_2 \otimes V_3 = \prec x, Ax \succ \oplus \prec y, Ay, A^2 y, A^3 y \succ, \quad \text{where } x = e_1 \otimes f_2, \quad y = e_1 \otimes f_1.$$

Summarizing:

Theorem 5.3. *An element $A \in \mathrm{GL}(6, 2)$ can be expressed in the form $A = B \otimes C$, for $B \in \mathrm{GL}(V_2)$, $C \in \mathrm{GL}(V_3)$, if and only if A belongs to one of the following classes in tables 5a, 5b:*

1A, 2B, 2C, 3A, 3C, 4C, 4E, 6A, 6F, 7A,B, 12A, 14A,B, 21A,B.

5.5.3. Which $A_6 \in \mathrm{GL}(6,2)$ are of the form $\wedge^2 A_4$, $A_4 \in \mathrm{GL}(4,2)$?. Since $6 = \binom{4}{2}$, we may view V_6 as an exterior product $\wedge^2 V_4$ and so ask: which classes of $\mathrm{GL}(V_6)$ are represented by mappings of the form $B = \wedge^2 A_4$ with $A_4 \in \mathrm{GL}(V_4)$? The answer is provided in the theorem below. In order to prove the theorem we need two results (valid over any field $\mathbb{F}$) in exterior algebra. Firstly, given that V has the direct sum decomposition $V = X \oplus Y$, it follows that $\wedge^2 V = \wedge^2 X \oplus \wedge^2 Y \oplus X \otimes Y$, where $X \otimes Y$ is that subspace of $\wedge^2 V$ (having the indicated tensor product structure) which is spanned by elements of the form $x \wedge y$ with $x \in X$, $y \in Y$. Consequently, if $A \in \mathrm{End}(X)$, $C \in \mathrm{End}(Y)$, then

$$\wedge^2(A \oplus C) = \wedge^2 A \oplus \wedge^2 C \oplus A \otimes C. \tag{5.12}$$

Secondly, if $\hat{A}_n \in \mathrm{SL}(V_n^*)$ denotes the contragredient $({}^tA_n)^{-1}$of $A_n \in \mathrm{SL}(V_n)$, recall that a Poincaré isomorphism $\wedge^r V_n \to \wedge^{n-r} V_n^*$ intertwines $\wedge^r A_n$ with $\wedge^{n-r} \hat{A}_n$; see for example [10, Eq. 9.6.24]. Consequently (since a matrix A is similar to its transpose ${}^tA$) we have

$$\wedge^r A_n \sim \wedge^{n-r} A_n^{-1}, \qquad \text{if } \det A_n = 1. \tag{5.13}$$

If now $\mathbb{F} = \mathrm{GF}(2)$ and $A_n \in \mathrm{GL}(n,2)$, then $\det A_n = 1$, and so the last result yields

$$\text{(i) } \wedge^2 A_2 = I_1, \quad \text{(ii) } \wedge^2 A_3 \sim A_3^{-1}, \quad \text{(iii) } \wedge^2 A_4 \sim \wedge^2 A_4^{-1}. \tag{5.14}$$

By using (5.12) in conjunction with (5.14) we readily obtain the results in the next theorem in those cases when A_4 is decomposable. For example if $B = \wedge^2 A_4$ where $A_4 = S_2 \oplus S_2 \in$ class 3A then, from (5.12), $B = I_1 \oplus I_1 \oplus S_2 \otimes S_2$. So, from section 5.3.2, $B = I_2 \oplus I_2 \oplus S_2$ and hence $B \in$ class 3B. Similarly if $A_4 = S_3 \oplus I_1 \in$ class 7A then $B \sim S_3^{-1} \oplus S_3 \otimes I_1 \sim S_3^{-1} \oplus S_3 \in$ class 7E.

The last result can be obtained even more simply as follows. By (5.14), if $B = \wedge^2 A_4$ then B and B^{-1} lie in the same class of $\mathrm{GL}(6,2)$; but for elements of $\mathrm{GL}(6,2)$ of order 7 the only class stable under $B \mapsto B^{-1}$ is class 7E. By the same argument, classes 15A,B of $\mathrm{GL}(4,2)$ both map into class 15C of $\mathrm{GL}(6,2)$; and of course class 5A maps into class 5A.

Theorem 5.4. *Under the injective mapping* $\mathrm{GL}(4,2) \to \mathrm{GL}(6,2) : A_4 \mapsto \wedge^2 A_4$ *the* 14 *classes of* $\mathrm{GL}(4,2)$ *are mapped into* 11 *of the classes of* $\mathrm{GL}(6,2)$ *as indicated in the following table:*

class of A_4	*1A*	*2A*	*2B*	*3A*	*3B*	*4A*	*4B*	*5A*	*6A*	*6B*	*7A,B*	*15A,B*
class of $\wedge^2 A_4$	*1A*	*2B*	*2B*	*3B*	*3C*	*4C*	*4E*	*5A*	*6D*	*6E*	*7E*	*15C*

Proof. After the preamble there is left to consider just two classes, namely 4B, represented by $J_4 \in \mathrm{GL}(4,2)$, and 6A, represented by $J_2 \otimes S_2 \in \mathrm{GL}(4,2)$. In the first case let us choose a basis $\{e_1, e_2, e_3, e_4\}$ such that $J_4 + I_4$ effects $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto 0$. We then find that $B = \wedge^2 J_4$ is of the form $B = B_2 \oplus B_4 \sim J_2 \oplus J_4$ with respect to the B-invariant decomposition

$$\wedge^2 V_4 = \prec u, J_4 u \succ \oplus \prec v, J_4 v, J_4^2 v, J_4^3 v \succ, \quad \text{where } u = e_1 \wedge e_3, \quad v = e_1 \wedge e_2.$$

So $B \in$ class 4E. Secondly, if $B = \wedge^2(J_2 \otimes S_2)$ then $B^2 = \wedge^2(I_2 \otimes S_2^2)$ and $B^3 = \wedge^2(J_2 \otimes I_2)$. So $B^2 \in$ class 3B and $B^3 \in$ class 2B. But, see table 5a, of the six classes 6A, ... ,6E, this last holds only for $B \in$ class 6D. □

Remark 5.5. *The image of* $\mathrm{GL}(4,2)$ *under the group monomorphism* $A \mapsto \wedge^2 A$ *is a subgroup* G *of* $\mathrm{GL}(6,2)$ *which is isomorphic to* $\mathrm{O}_6^+(2)$. *If* A, A' *belong to* *different classes of the group* GL(4, 2)*, then of course their images* B, B' *belong to different classes of the isomorphic group* G*. However* B, B' *may still be conjugate inside the larger group* GL(6, 2)*. Thus, see the theorem, classes 2A,2B of* GL(4, 2) *are both mapped into class 2B of* GL(6, 2)*, classes 7A,7B are both mapped into class 7E, and classes 15A,15B are both mapped into class 15C. Incidentally, since the* 499, 968 *subgroups of* GL(6, 2) *which are isomorphic to* $O_6^+(2)$ *are all conjugate in* GL(6, 2)*, their set union is precisely the union of the eleven* GL(6, 2) *classes 1A,2B,...,15C listed in the second row of the above table.*

6. Further remarks.

6.1. Cycle type. The tables show that the following result holds: given $A, B \in \mathrm{GL}(n, 2)$, $2 \le n \le 6$, then

$$A \sim B \quad \Longleftrightarrow \quad \mathrm{CT}(A) = \mathrm{CT}(B) \text{ and } \mu_A = \mu_B, \tag{6.1}$$

where CT(A) denotes the cycle type of $A \in \mathrm{GL}(n, 2)$ in its natural action on PG($n-1, 2$). Indeed *in many cases the cycle type alone suffices to distinguish between the classes* — which is useful since the cycle type of an element is easily determined. For example, from table 5a, we see that the cycle type alone distinguishes between the six classes of elements in GL(6, 2) which have order 6. In fact, with two exceptions, given only that $\mathrm{CT}(A) = \mathrm{CT}(B)$, where $A, B \in \mathrm{GL}(n, 2)$, $2 \le n \le 6$, it follows that A is conjugate to B^r, for some r. The two exceptions arise for $n = 6$: (i) classes 4C and 4E of GL(6, 2) share the same cycle type $4^{12}2^6 1^3$; (ii) classes 7A,B and 7E of GL(6, 2) share the same cycle type 7^9.

6.2. Fixed-point-free elements and linear sections of GL(n, 2). For certain purposes — see below for an example — it is of importance to know the f.p.f. classes of GL(n, 2). Now, see theorem 3.5(iv), any power $S^r \ne I$ of a Singer element $S \in \mathrm{GL}(n, 2)$ is fixed-point-free on PG($n - 1, 2$). In the case of GL(4, 2), the f.p.f. classes 3A, 5A and 15A,B arise from the Singer elements and their powers, and, see table 3, there is only one further f.p.f. class, namely class 6A. In the case of GL(5, 2), the Singer elements give rise solely to the f.p.f. classes 31A-31F, and, see table 4, there are just two further f.p.f. classes, namely classes 21A and 21B.

A far richer supply of f.p.f. elements is available in the case of GL(6, 2). First of all the Singer elements and their powers provide us with twelve f.p.f. classes, namely classes 3A, 7A,B, 9A, 21A,B and 63A-63F. Secondly, see tables 5a and 5b, there exist eight further classes of f.p.f. elements, namely classes 6A, 7E, 12A, 14A,B, 15C and 15D,E.

In [4] the problem was posed of classifying the *r-dimensional normalized linear sections*, denoted $\mathrm{NLS}_r(n, q)$'s, of GL(n, q). (Such a section is, by definition, an r-dimensional subspace of the n^2-dimensional vector space End(n, q) which contains I_n and is such that every non-zero element lies in GL(n, q).) It is easy to see that $r \le n$, and that $r = n$ is achieved by use of a Singer cyclic subgroup $\cong Z_{q^n-1}$. Also, [4, Lemma 2.1], each non-scalar element A of a $\mathrm{NLS}_r(n, q)$ must be f.p.f. upon the points of PG($n - 1, q$). The classification problem was solved in the case of GL(4, 2), and it was found that, up to an appropriate notion of equivalence, there are just two classes $\mathcal{M}_3, \mathcal{M}_3'$ of maximal $\mathrm{NLS}_3(4, 2)$'s and three classes $\mathcal{M}_4, \mathcal{M}_4', \mathcal{M}_4''$ of $\mathrm{NLS}_4(4, 2)$'s. The existence of the class $\mathcal{M}_3$ is, see [13], related to the existence of "7-clusters" in Alt(7), and any linear section $\mathcal{S} \in \mathcal{M}_3$ has the property that all six elements of $\mathcal{S} \smallsetminus \{0, I\}$ belong to the non-Singer f.p.f. class 6A of GL(4, 2).

Concerning $\mathrm{GL}(5,2)$, observe that $A \in \mathrm{GL}(5,2)$ is of class 21A (class 21B) according as $I + A$ is of class 21B (class 21A). (This follows from $A = S_3 \oplus S_2$, since a corresponding property holds for elements S_3 and $I_3 + S_3$ of the classes 7A and 7B of $\mathrm{GL}(3,2)$.) Consequently all $\mathrm{NLS}_2(5,2)$'s of the kind $\{0, I, A, B\}$ with $\mathrm{o}(A) = \mathrm{o}(B) = 21$ are conjugate in $\mathrm{GL}(5,2)$. Such a $\mathrm{NLS}_2(5,2)$ is not maximal. Indeed in [6] it was found, using MAGMA [1], that for $r = 3, 4$ and 5 there exist maximal $\mathrm{NLS}_r(5,2)$'s whose elements $\neq 0, I$ all belong to the non-Singer f.p.f. classes 21A,B.

The classification problem for maximal $\mathrm{NLS}_r(n,2)$'s is still open for dimension $n \geq 5$. In the case $n = 6$, granted the afore-mentioned rich supply of f.p.f. elements present in $\mathrm{GL}(6,2)$, the existence of very many inequivalent kinds of maximal $\mathrm{NLS}_r(6,2)$'s seems likely.

7. Subgroups arising from field extensions. Choose a direct sum decomposition $V_4 = V_2 \oplus V_2$ and set $W = S_2 \oplus S_2$, $G_{2,4} = C(W)$ and $J = J_2 \oplus J_2$. By lemma 3.1, we may identify $\mathfrak{A}_W$ with the field GF(4), V_4 with a $V(2,4)$ and $G_{2,4}$ with $\mathrm{GL}(2,4)$. Since $JWJ^{-1} = W^2$ it follows that J is a σ-semilinear map of $V(2,4)$ with respect to the automorphism $\sigma : X \mapsto X^2$ of GF(4). Consequently the subgroup $\Gamma_{2,4} = G_{2,4} \cup JG_{2,4} \cong G_{2,4} \rtimes \langle J \rangle$ may be identified with $\Gamma\mathrm{L}(2,4)$, and we have a subgroup chain

$$\mathrm{SL}(2,4) < \mathrm{GL}(2,4) < \Gamma\mathrm{L}(2,4) < \mathrm{GL}(4,2) \tag{7.1}$$

where $\mathrm{SL}(2,4) (\cong \mathrm{Alt}(5))$ is the commutator subgroup $G'_{2,4}$ of $G_{2,4}$. In fact $\Gamma\mathrm{L}(2,4)$ is a maximal subgroup of $\mathrm{GL}(4,2)$. The groups $\mathrm{SL}(2,4)$, $\mathrm{GL}(2,4)$ and $\Gamma\mathrm{L}(2,4)$ have orders 60, 180 and 360, and possess 5, 15 and 12 conjugacy classes, respectively.

Table 6. $\mathrm{SL}(2,4)$, $\mathrm{GL}(2,4)$ and $\Gamma\mathrm{L}(2,4)$ classes inside $\mathrm{GL}(4,2)$

GL(4,2) class	Reptve.	Cycle type	SL(2,4)	GL(2,4)	ΓL(2,4)
1A	I_4	1^{15}	1	1+0	1+0
2B	$J_2 \oplus J_2$	$2^6\,1^3$	1	1+0	1+1
3A	$S_2 \oplus S_2$	3^5	1	1+2	2+0
3B	$S_2 \oplus I_2$	$3^4 1^3$	—	0+2	1+0
4B	J_4	$4^3\,2^1\,1^1$	—	—	0+1
5A	$(S_4)^3$	5^3	2	2+0	1+0
6A	$J_2 \otimes S_2$	$6^2\,3^1$	—	0+2	1+1
15A,B	$S_4^{\pm 1}$	15^1	—	0+2, 0+2	1+0, 1+0
Total number of classes			5	15	12

Information concerning the relation of these classes to those of $\mathrm{GL}(4,2)$ is provided in table 6. In particular the 4 classes 1A, 2B, 3A and 5A of $\mathrm{GL}(4,2)$ may be represented by elements of a $\mathrm{SL}(2,4)$ subgroup, the further 4 classes 3B, 6A and 15A,B may be represented if we use elements of $\mathrm{GL}(2,4) \setminus \mathrm{SL}(2,4)$, and class 4B may be represented by an element of $\Gamma\mathrm{L}(2,4) \setminus \mathrm{GL}(2,4)$. (The notation in the final two columns of the table is explained below, see the discussion of table 7.) The 5 classes 2A, 4A, 6B and 7A,B do not have representatives $\in \Gamma\mathrm{L}(2,4)$.

Similarly, choosing a direct sum decomposition $V_6 = V_2 \oplus V_2 \oplus V_2$ and setting $W = S_2 \oplus S_2 \oplus S_2$, $G_{3,4} = C(W)$ and $J = J_2 \oplus J_2 \oplus J_2$, we may identify $\mathfrak{A}_W$ with the field GF(4), V_6 with a $V(3,4)$ and $G_{3,4}$ with $\mathrm{GL}(3,4)$, and so arrive at a subgroup chain

$$\mathrm{SL}(3,4) < \mathrm{GL}(3,4) < \Gamma\mathrm{L}(3,4) < \mathrm{GL}(6,2). \tag{7.2}$$

Here $\mathrm{SL}(3,4) = G'_{3,4}$ and $\Gamma\mathrm{L}(3,4) = G_{3,4} \cup JG_{3,4} \cong G_{3,4} \rtimes \langle J \rangle$. The groups $\mathrm{SL}(3,4)$; $\mathrm{GL}(3,4)$ and $\Gamma\mathrm{L}(3,4)$ have orders $60,480$, $181,440$ and $362,880$, and possess 28, 60 and 39 classes, respectively.

Finally, choose instead a direct sum decomposition $V_6 = V_3 \oplus V_3$ and set $U = S_3 \oplus S_3$ and $G_{2,8} = C(U)$. By lemma 3.1, we may identify $\mathfrak{A}_U$ with the field GF(8), V_4 with a $V(2,8)$ and $G_{2,8}$ with GL(2,8). Now the normalizer $N(\langle S_3\rangle)$ of $\langle S_3\rangle$ in GL(3,2) has structure $\langle S_3\rangle \rtimes Z_3$. Choosing K_3 in one of the Z_3 subgroups of $N(\langle S_3\rangle)$ to satisfy $K_3 S_3 K_3^{-1} = (S_3)^2$, and setting $K = K_3 \oplus K_3$, then $KUK^{-1} = U^2$. It follows that K is a ψ-semilinear map of $V(2,8)$ with respect to the automorphism $\psi : X \mapsto X^2$ of GF(8). Consequently the subgroup $\Gamma_{2,8} = G_{2,8} \rtimes \langle K\rangle$ may be identified with $\Gamma\,\mathrm{L}(2,8)$, and we have a subgroup chain

$$\mathrm{SL}(2,8) < \mathrm{GL}(2,8) < \Gamma\,\mathrm{L}(2,8) < \mathrm{GL}(6,2) \tag{7.3}$$

where $\mathrm{SL}(2,8) = G'_{2,8}$. The groups SL(2,8), GL(2,8) and $\Gamma\,\mathrm{L}(2,8)$ have orders 504, 3,528 and 10,584, and possess 9, 63 and 29 classes, respectively.

It is known that both $\Gamma\,\mathrm{L}(3,4)$ and $\Gamma\,\mathrm{L}(2,8)$ are maximal subgroups of GL(6,2). (For a list of the maximal subgroups of GL(6,2) consult [16].)

In table 7 we provide information concerning the relation of the classes of the foregoing subgroups of GL(6,2) to those of GL(6,2) itself. In the column headed GL(2,8) an entry $r+s$ against class nX of GL(6,2) means that class nX contains $r+s$ classes of GL(2,8) of which r lie in SL(2,8) and s lie in $\mathrm{GL}(2,8) \smallsetminus \mathrm{SL}(2,8)$. Similarly in the column headed $\Gamma\,\mathrm{L}(2,8)$ an entry $r+s$ against class nX of GL(6,2) means that class nX contains $r+s$ classes of $\Gamma\,\mathrm{L}(2,8)$ of which r lie in GL(2,8) and s lie in $\Gamma\,\mathrm{L}(2,8) \smallsetminus \mathrm{GL}(2,8)$. Similarly for the GL(3,4) and $\Gamma\,\mathrm{L}(3,4)$ columns. To save space, an entry $r+s$, $r+s$ against paired classes nX, Y is abbreviated to $(r+s)^2$.

In arriving at an understanding of table 7 it should be borne in mind that the above automorphisms σ, ψ of GF(4), GF(8) have periods 2, 3, respectively. Hence *the order of any element* $A \in \Gamma\,\mathrm{L}(3,4) \smallsetminus \mathrm{GL}(3,4)$ *is necessarily even, and that of* $A \in \Gamma\,\mathrm{L}(2,8) \smallsetminus \mathrm{GL}(2,8)$ *is a multiple of* 3. In fact, from the table, we have

$$\mathrm{o}(A) \in \begin{cases} \{2,4,6,8,14\} & \text{if } A \in \Gamma\,\mathrm{L}(3,4) \smallsetminus \mathrm{GL}(3,4) \\ \{3,6,9\} & \text{if } A \in \Gamma\,\mathrm{L}(2,8) \smallsetminus \mathrm{GL}(2,8)\,. \end{cases}$$

Also, *two distinct classes of* GL(3,4) *may well fuse to form a single class of* $\Gamma\,\mathrm{L}(3,4)$, *and three distinct classes of* GL(2,8) *may fuse to form a single class of* $\Gamma\,\mathrm{L}(2,8)$. (Three classes of SL(3,4) may also coalesce to form a single class of GL(3,4) — see the 4C and 12A entries of table 7.)

As an example, consider the elements of $\Gamma\,\mathrm{L}(2,8)$ of order 7. Since $3 \nmid 7$, such elements lie in GL(2,8). In fact GL(2,8) has 27 classes of elements of order 7, and these fuse in threes to form nine classes of $\Gamma\,\mathrm{L}(2,8)$. First of all there are the six singleton classes $\{U^i\}$, $i = 1, \ldots, 6$, of GL(2,8) which fuse to yield two classes of $\Gamma\,\mathrm{L}(2,8)$, namely $\{U, U^2, U^4\}$ and $\{U^{-1}, U^{-2}, U^{-4}\}$. Since $U = S_3 \oplus S_3$, the elements U^i, $i = 1,2,4$, belong to class 7A of GL(6,2) and the elements U^i, $i = 6,5,3$, belong to class 7B of GL(6,2). The remaining 21 classes have representatives of the form $S_3^i \oplus S_3^j (\sim_{\mathrm{GL}(2,8)} S_3^j \oplus S_3^i)$ with $0 \le i < j \le 6$. Since in each case the centralizer is $\cong Z_7 \times Z_7$, each of these 21 classes of GL(2,8) has length 72. Belonging to class 7A of GL(6,2) are the three classes with representatives $S_3 \oplus S_3^2$, $S_3^2 \oplus S_3^4$ and $S_3^4 \oplus S_3$. (Using $I_3 \oplus K_3^{-1}$, note that $S_3 \oplus S_3^2$ is similar to $S_3 \oplus S_3$; however $I_3 \oplus K_3^{-1} \notin \Gamma\,\mathrm{L}(2,8)$.) The inverses of these last three GL(2,8) classes accordingly belong to class 7B of GL(6,2). The six elements $I_3 \oplus S_3^i$, $i = 1, \ldots, 6$, represent six further classes of GL(2,8) which fuse in threes, $i = 1,2,4$ and $i = 6,5,3$, to form two classes of $\Gamma\,\mathrm{L}(2,8)$, with the first three belonging to class 7C, and the second three belonging to class 7D, of GL(6,2). Next there are the three classes of GL(2,8) with representatives

$S_3^i \oplus S_3^{-i}$, $i = 1, 2, 3$; these lie in the subgroup $\mathrm{SL}(2,8)$, they belong to class 7E of $\mathrm{GL}(6,2)$ and they fuse to form a single class of $\Gamma\mathrm{L}(2,8)$. Finally there are six further classes of $\mathrm{GL}(2,8)$ which also belong to class 7E of $\mathrm{GL}(6,2)$; three of these have representatives $S_3 \oplus S_3^3$, $S_3^2 \oplus S_3^6$, $S_3^4 \oplus S_3^5$, which fuse to form a single class of $\Gamma\mathrm{L}(2,8)$, and the other three are obtained upon taking inverses.

Table 7. $\mathrm{SL}(6/d, 2^d)$, $\mathrm{GL}(6/d, 2^d)$ and $\Gamma\mathrm{L}(6/d, 2^d)$ classes inside $\mathrm{GL}(6,2)$, $d = 2, 3$

GL(6,2) class	Representative	Cycle type	SL(3,4)	GL(3,4)	ΓL(3,4)	SL(2,8)	GL(2,8)	ΓL(2,8)
1A	I_6	1^{63}	1	1+0	1+0	1	1+0	1+0
2B	$I_2 \oplus J_2 \oplus J_2$	$2^{24}\,1^{15}$	1	1+0	1+0	—	—	—
2C	$J_2 \oplus J_2 \oplus J_2$	$2^{28}\,1^{7}$	—	—	0+1	1	1+0	1+0
3A	$S_2 \oplus S_2 \oplus S_2$	3^{21}	2	2+2	2+0	1	1+0	1+0
3B	$I_4 \oplus S_2$	$3^{16}\,1^{15}$	—	0+2	1+0	—	—	—
3C	$I_2 \oplus S_2 \oplus S_2$	$3^{20}\,1^{3}$	1	1+2	2+0	—	—	0+2
4C	$J_3 \oplus J_3$	$4^{12}\,2^{6}\,1^{3}$	3	1+0	1+0	—	—	—
4E	$J_2 \oplus J_4$	$4^{12}\,2^{6}\,1^{3}$	—	—	0+1	—	—	—
5A	$I_2 \oplus (S_4)^3$	$5^{12}\,1^{3}$	2	2+0	1+0	—	—	—
6A	$S_2 \oplus (J_2 \otimes S_2)$	$6^{8}\,3^{5}$	2	2+2	2+0	—	—	—
6D	$J_2 \oplus J_2 \oplus S_2$	$6^{6}\,3^{4}\,2^{6}\,1^{3}$	—	0+2	1+0	—	—	—
6E	$I_2 \oplus (J_2 \otimes S_2)$	$6^{8}\,3^{4}\,1^{3}$	—	0+2	1+0	—	—	—
6F	$J_2 \oplus (J_2 \otimes S_2)$	$6^{9}\,3^{2}\,2^{1}\,1^{1}$	—	—	0+1	—	—	0+2
7A,B	$S_3 \oplus S_3$, $S_3^{-1} \oplus S_3^{-1}$	7^{9}	1,1	$(1+0)^2$	$(1+0)^2$	—	$(0+6)^2$	$(2+0)^2$
7C,D	$I_3 \oplus S_3$, $I_3 \oplus S_3^{-1}$	$7^{8}\,1^{7}$	—	—	—	—	$(0+3)^2$	$(1+0)^2$
7E	$S_3 \oplus S_3^{-1}$	7^{9}	—	—	—	3	3+6	3+0
8A	$I_1 \oplus J_5$	$8^{4}\,4^{6}\,2^{2}\,1^{3}$	—	—	0+1	—	—	—
9A	$(S_6)^7$	9^{7}	—	0+2	1+0	3	3+0	1+2
12A	$J_3 \otimes S_2$	$12^{4}\,6^{2}\,3^{1}$	6	2+0	1+0	—	—	—
14A,B	$J_2 \otimes S_3$, $J_2 \otimes S_3^{-1}$	$14^{4}\,7^{1}$	—	—	$(0+1)^2$		$(0+3)^2$	$(1+0)^2$
15A,B	$I_2 \oplus S_4$, $I_2 \oplus S_4^{-1}$	$15^{4}\,1^{3}$	—	$(0+2)^2$	$(1+0)^2$	—	—	—
15C	$S_2 \oplus (S_4)^3$	$15^{3}\,5^{3}\,3^{1}$	—	0+4	2+0	—	—	—
15D,E	$S_2 \oplus S_4$, $S_2 \oplus S_4^{-1}$	$15^{4}\,3^{1}$	2, 2	$(2+2)^2$	$(2+0)^2$	—	—	—
21A,B	$(S_6)^3, (S_6)^{-3}$	21^{3}	2, 2	$(2+0)^2$	$(1+0)^2$	—	$(0+3)^2$	$(1+0)^2$
63A...F	S_6, ... S_6^{-11}	63^{1}	—	$(0+2)^6$	$(1+0)^6$	—	$(0+3)^6$	$(1+0)^6$
Total number of classes			28	60	39	9	63	29

Observe from the table that the following 24 classes of $\mathrm{GL}(6,2)$ cannot be represented by an element of $\Gamma\mathrm{L}(3,4)$ nor by an element of $\Gamma\mathrm{L}(2,8)$:

2A, 4A, 4B, 4D, 6B, 6C, 8B, 10A, 12B, 12C, 14C, 14D, 21C,D, 28A,B, 30A,B, 31A...F

If A belongs to one of these classes and $\mathcal{S}_{3,4}$ is a set of generators for the subgroup $\mathrm{SL}(3,4)$ of $\mathrm{GL}(6,2)$ it follows, since the normalizer $\Gamma\mathrm{L}(3,4)$ of $\mathrm{SL}(3,4)$ is maximal in $\mathrm{GL}(6,2)$, that $\langle \mathcal{S}_{3,4}, A\rangle = \mathrm{GL}(6,2)$. Similarly, if $\mathcal{S}_{2,8}$ is a set of generators for $\mathrm{SL}(2,8) < \mathrm{GL}(6,2)$, then $\langle \mathcal{S}_{2,8}, A\rangle = \mathrm{GL}(6,2)$. In particular these last statements hold for $A \in$ class 2A, that is if A is any transvection in $\mathrm{GL}(6,2)$. (Incidentally if $A \in \mathrm{GL}(3,4) < \mathrm{GL}(6,2)$ is a transvection *qua its action upon* $V(3,4)$ then A belongs to class 2B of $\mathrm{GL}(6,2)$, and if $A \in \mathrm{GL}(2,8) < \mathrm{GL}(6,2)$ is a transvection *qua its action upon* $V(2,8)$ then A belongs to class 2C of $\mathrm{GL}(6,2)$.)

REMARK 7.1. *It is worth noting that the subgroup* $G_\otimes = \{A_2 \otimes A_3 \mid A_i \in \mathrm{GL}(V_i)\} \cong \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$ *of* $\mathrm{GL}(V_6)$, *associated with a tensor product structure* $V_6 = V_2 \otimes V_3$, *is (unlike the situation for* $q > 2$) not *a maximal subgroup of* $\mathrm{GL}(V_6)$; *indeed* $G_\otimes$ *lies inside a* $\Gamma\mathrm{L}(3,4)$ *subgroup. To see this note that* $W = S_2 \otimes I_3$ *is in class 3A, and so the* $\mathrm{GL}(3,4)$ *and* $\Gamma\mathrm{L}(3,4)$ *subgroups of* $\mathrm{GL}(6,2)$ *considered previously may* *be taken to be* $C(W)$ *and* $C(W) \cup JC(W)$, *with* $J = J_2 \otimes I_3$ *satisfying* $JWJ^{-1} = W^2$. *Since* $\mathrm{GL}(V_2) = \langle S_3 \rangle \cup J_2 \langle S_3 \rangle$, *it follows that* $G_\otimes$ *is subgroup of* $C(W) \cup JC(W) \cong \Gamma\,\mathrm{L}(3,4)$. *(In this connection contrast the isomorphism* $\Gamma\,\mathrm{L}(1,4) \cong \mathrm{GL}(2,2)$ *with the strict inequality* $|\Gamma\,\mathrm{L}(1,q^2)| < |\,\mathrm{GL}(2,q)|$ *which holds for* $q > 2$.*) Recall from theorem 5.3 that the following* 16 *classes of* $\mathrm{GL}(6,2)$ *may be represented by elements belonging to a* $G_\otimes$ *subgroup of* $\mathrm{GL}(6,2)$*:*

1A, 2B, 2C, 3A, 3C, 4C, 4E, 6A, 6F, 7A,B, 12A, 14A,B, 21A,B.

From table 7 we see that these 16 *classes are indeed amongst the* 33 *classes represented by elements belonging to a* $\Gamma\,\mathrm{L}(3,4)$ *subgroup of* $\mathrm{GL}(6,2)$.

Appendix A. Tables of GL$(n, 2)$ results, $n = 4, 5, 6$.

For the $n = 2$ and $n = 3$ results see tables 1 and 2 in section 5. In the Notes column of the following tables we have abbreviated lemma by L, remark by Rmk and theorem by T.

Table 3. Classes of GL$(4, 2)$

Class	(Class)^p	Reptve.	χ	μ	F.p.'s	Cycle type	$C(A)$	$\lvert C(A)\rvert$	Length	Notes
1A		I_4	$(f_1)^4$	f_1	PG$(3,2)$	1^{15}	GL$(4,2)$	20,160	1	
2A		$J_2 \oplus I_2$	$(f_1)^4$	$(f_1)^2$	plane	$2^4\,1^7$		192	105	L 3.8(iii), L 4.2
2B		$J_2 \oplus J_2$	$(f_1)^4$	$(f_1)^2$	line	$2^6\,1^3$	eq. (5.3)	96	210	§5.3.1(i)
3A		$S_2 \oplus S_2$	$(f_2)^2$	f_2	f.p.f.	3^5	GL$(2,4)$	180	112	(Singer)^5, L 3.1
3B		$S_2 \oplus I_2$	$(f_1)^2 f_2$	$f_1 f_2$	line	$3^4 1^3$	$Z_3 \times$ GL$(2,2)$	18	1120	L 2.8
4A	A	$J_3 \oplus I_1$	$(f_1)^4$	$(f_1)^3$	line	$4^2\,2^2\,1^3$		16	1260	eq. (3.19)
4B	B	J_4	$(f_1)^4$	$(f_1)^4$	point	$4^3\,2^1\,1^1$	eq. (2.10)	8	2520	L 3.7, L 3.8(i)
5A		$(S_4)^3$	g_4	g_4	f.p.f.	5^3	$\langle S_4\rangle \cong Z_{15}$	15	1344	(Singer)^3, T 3.6(iii)
6A	AB	$J_2 \otimes S_2$	$(f_2)^2$	$(f_2)^2$	f.p.f.	$6^2\,3^1$	$Z_3 \times (Z_2)^2$	12	1680	§5.3.1(ii), Rmk 5.1
6B	BA	$S_2 \oplus J_2$	$(f_1)^2 f_2$	$(f_1)^2 f_2$	point	$6^1\,3^2\,2^1\,1^1$	$Z_3 \times Z_2 = Z_6$	6	3360	L 2.7(ii), Rmk 5.1
7A		$S_3 \oplus I_1$	$f_1 f_3$	$f_1 f_3$	point	$7^2\,1^1$	Z_7	7	2880	L 2.8
7B		$S_3^{-1} \oplus I_1$	$f_1 \hat{f}_3$	$f_1 \hat{f}_3$	point	$7^2\,1^1$	Z_7	7	2880	L 2.8
15A	AA	S_4	f_4	f_4	f.p.f.	$(15)^1$	Z_{15}	15	1344	Singer,
15B	AA	S_4^{-1}	$\hat{f}_4$	$\hat{f}_4$	f.p.f.	$(15)^1$	Z_{15}	15	1344	§ 3.2

Table 4 Classes of GL(5, 2)

Class	(Class)^p	Representative	χ	μ	F.p.'s	Cycle type	$C(A)$	$\lvert C(A)\rvert$	Length	Notes
1A		I_5	$(f_1)^5$	f_1	PG(4, 2)	1^{31}	GL(5, 2)	9,999,360	1	
2A		$J_2 \oplus I_3$	$(f_1)^5$	$(f_1)^2$	PG(3, 2)	$2^8\,1^{15}$		21,504	465	L 3.8(iii), L 4.2
2B		$J_2 \oplus J_2 \oplus I_1$	$(f_1)^5$	$(f_1)^2$	plane	$2^{12}\,1^7$	eq. (5.6)	1,536	6,510	§5.4
3A		$S_2 \oplus I_3$	$(f_1)^3 f_2$	$f_1 f_2$	plane	$3^8\,1^7$	$Z_3 \times$ GL(3, 2)	504	19,840	L 2.8
3B		$S_2 \oplus S_2 \oplus I_1$	$f_1(f_2)^2$	$f_1 f_2$	point	$3^{10}\,1^1$	GL(2, 4)	180	55,552	L 2.8
4A	A	$J_3 \oplus I_2$	$(f_1)^5$	$(f_1)^3$	plane	$4^4\,2^4\,1^7$		384	26,040	L 3.8(iii)
4B	A	$J_3 \oplus J_2$	$(f_1)^5$	$(f_1)^3$	line	$4^4\,2^6\,1^3$		128	78,120	L 2.11
4C	B	$J_4 \oplus I_1$	$(f_1)^5$	$(f_1)^4$	line	$4^6\,2^2\,1^3$		32	312,480	eq. (3.19)
5A		$(S_4)^3 \oplus I_1$	$f_1 g_4$	$f_1 g_4$	point	$5^6\,1^1$	Z_{15}	15	666,624	L 2.8
6A	AA	$S_2 \oplus J_2 \oplus I_1$	$(f_1)^3 f_2$	$(f_1)^2 f_2$	line	$6^2\,3^4\,2^2\,1^3$	$Z_3 \times D_8$	24	416,640	L 2.7(ii)
6B	BB	$(J_2 \otimes S_2) \oplus I_1$	$f_1(f_2)^2$	$f_1(f_2)^2$	point	$6^4\,3^2\,1^1$	$Z_3 \times (Z_2)^2$	12	833,280	L 2.8
7A,B		$S_3^{\pm 1} \oplus I_2$	$(f_1)^2 f_3$ & $(\hat{\ })$	$f_1 f_3$ & $(\hat{\ })$	line	$7^4\,1^3$	$Z_7 \times$ GL(2, 2)	42	238,080	L 2.8
8A	B	J_5	$(f_1)^5$	$(f_1)^5$	point	$8^2\,4^3\,2^1\,1^1$	see eq. (2.10)	16	624,960	L 3.7, L 3.8(i)
12A	AA	$S_2 \oplus J_3$	$(f_1)^3 f_2$	$(f_1)^3 f_2$	point	$12^1\,6^1\,4^1\,3^2\,2^1\,1^1$	$Z_3 \times Z_4 = Z_{12}$	12	833,280	L 2.7(ii)
14A,B	AA, AA	$S_3^{\pm 1} \oplus J_2$	$(f_1)^2 f_3$ & $(\hat{\ })$	$(f_1)^2 f_3$ & $(\hat{\ })$	point	$14^1\,7^2\,2^1\,1^1$	$Z_7 \times Z_2 = Z_{14}$	14	714,240	L 2.7(ii)
15A,B	AB, AB	$S_4^{\pm 1} \oplus I_1$	$f_1 f_4$ & $(\hat{\ })$	$f_1 f_4$ & $(\hat{\ })$	point	$15^2\,1^1$	Z_{15}	15	666,624	L 2.8
21A,B	BA, AA	$S_3^{\pm 1} \oplus S_2$	$f_2 f_3$ & $(\hat{\ })$	$f_2 f_3$ & $(\hat{\ })$	f.p.f.	$21^1\,7^1\,3^1$	$Z_7 \times Z_3 = Z_{21}$	21	476,160	L 2.7
31A,B		$S_5,\ S_5^{-1}$	$f_5,\ \hat{f}_5$	$f_5,\ \hat{f}_5$	f.p.f.	31^1	Z_{31}	31	322,560	Singer,
31C,D		$(S_5)^5,\ (S_5)^{-5}$	$\hat{g}_5,\ g_5$	$\hat{g}_5,\ g_5$	f.p.f.	31^1	Z_{31}	31	322,560	§3.2,
31E,F		$(S_5)^{-6},\ (S_5)^6$	$\hat{h}_5,\ h_5$	$\hat{h}_5,\ h_5$	f.p.f.	31^1	Z_{31}	31	322,560	L 4.4

Table 5a. Classes of GL(6, 2): elements of order < 8

Class	(Class)^p	Representative	χ	μ	F.p.'s	Cycle type	$\lvert C(A)\rvert$	Length	Notes
1A		I_6	$(f_1)^6$	f_1	PG(5, 2)	1^{63}	$\lvert \mathrm{GL}(6,2)\rvert$	1	
2A		$I_4 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	PG(4, 2)	$2^{16}\,1^{31}$	10,321,920	1,953	L 3.8(iii), L 4.2
2B		$I_2 \oplus J_2 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	PG(3, 2)	$2^{24}\,1^{15}$	147,456	136,710	§5.5.1(ii)
2C		$J_2 \oplus J_2 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	plane	$2^{28}\,1^{7}$	86,016	234,360	§5.5.1(iii)
3A		$S_2 \oplus S_2 \oplus S_2$	$(f_2)^3$	f_2	f.p.f.	3^{21}	181,440	111,104	(Singer)^{21}, §5.5.1(iv)
3B		$I_4 \oplus S_2$	$(f_1)^4 f_2$	$f_1 f_2$	PG(3, 2)	$3^{16}\,1^{15}$	60,480	333,312	L 2.8
3C		$I_2 \oplus S_2 \oplus S_2$	$(f_1)^2 (f_2)^2$	$f_1 f_2$	line	$3^{20}\,1^{3}$	1,080	18,665,472	L 2.8
4A	A	$I_3 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	PG(3, 2)	$4^{8}\,2^{8}\,1^{15}$	43,008	468,720	L 3.8(iii)
4B	A	$I_1 \oplus J_2 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	plane	$4^{8}\,2^{12}\,1^{7}$	2,048	9,843,120	§5.5.1(v)
4C	B	$J_3 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	line	$4^{12}\,2^{6}\,1^{3}$	1,536	13,124,160	§5.5.1(vi)
4D	B	$I_2 \oplus J_4$	$(f_1)^6$	$(f_1)^4$	plane	$4^{12}\,2^{4}\,1^{7}$	768	26,248,320	L 3.8(iii)
4E	B	$J_2 \oplus J_4$	$(f_1)^6$	$(f_1)^4$	line	$4^{12}\,2^{6}\,1^{3}$	256	78,744,960	L 2.11, §5.5.2
5A		$I_2 \oplus (S_4)^3$	$(f_1)^2 g_4$	$f_1 g_4$	line	$5^{12}\,1^{3}$	90	223,985 ,664	L 2.8
6A	AB	$S_2 \oplus (J_2 \otimes S_2)$	$(f_2)^3$	$(f_2)^2$	f.p.f.	$6^{8}\,3^{5}$	576	34,997,760	§5.5.1(vii)
6B	BA	$I_2 \oplus J_2 \oplus S_2$	$(f_1)^4 f_2$	$(f_1)^2 f_2$	plane	$6^{4}\,3^{8}\,2^{4}\,1^{7}$	576	34,997,760	L 2.7(ii)
6C	CA	$J_2 \oplus S_2 \oplus S_2$	$(f_1)^2 (f_2)^2$	$(f_1)^2 f_2$	point	$6^{5}\,3^{10}\,2^{1}\,1^{1}$	360	55,996,416	L 2.7(ii)
6D	BB	$J_2 \oplus J_2 \oplus S_2$	$(f_1)^4 f_2$	$(f_1)^2 f_2$	line	$6^{6}\,3^{4}\,2^{6}\,1^{3}$	288	69,995,520	L 2.7(ii)
6E	CB	$I_2 \oplus (J_2 \otimes S_2)$	$(f_1)^2 (f_2)^2$	$f_1 (f_2)^2$	line	$6^{8}\,3^{4}\,1^{3}$	72	279,982,080	L 2.8
6F	CC	$J_2 \oplus (J_2 \otimes S_2)$	$(f_1)^2 (f_2)^2$	$(f_1)^2 (f_2)^2$	point	$6^{9}\,3^{2}\,2^{1}\,1^{1}$	24	839,946,240	L 2.7(ii)
7A,B		$S_3 \oplus S_3$, $S_3^{-1} \oplus S_3^{-1}$	$(f_3)^2$, $(\hat{f}_3)^2$	f_3 , $\hat{f}_3$	f.p.f.	7^{9}	3,528	5,713,920	$(S_6)^{-9}, (S_6)^{9}$; §5.5.1(viii), L 4.5
7C,D		$I_3 \oplus S_3$, $I_3 \oplus S_3^{-1}$	$(f_1)^3 f_3$, $(f_1)^3 \hat{f}_3$	$f_1 f_3$, $f_1 \hat{f}_3$	plane	$7^{8}\,1^{7}$	1,176	17,141,760	L 2.8
7E		$S_3 \oplus S_3^{-1}$	$f_3 \hat{f}_3$	$f_3 \hat{f}_3$	f.p.f.	7^{9}	49	411,402,240	L 2.7(ii)

Table 5b. Classes of GL(6, 2): elements of order ≥ 8

Class	$(\text{Class})^p$	Representative	χ	μ	F.p.'s	Cycle type	$\|C\|$	Length	Notes
8A	B	$I_1 \oplus J_5$	$(f_1)^6$	$(f_1)^5$	line	$8^4\, 4^6\, 2^2\, 1^3$	64	314,979,840	eq. (3.19)
8B	C	J_6	$(f_1)^6$	$(f_1)^6$	point	$8^6\, 4^3\, 2^1\, 1^1$	32	629,959,680	L 3.7, L 3.8(i)
9A	A	$(S_6)^7$	l_6	l_6	f.p.f.	9^7	63	319,979,520	$(\text{Singer})^7$, T 3.6(iii), L 4.5
10A	AA	$J_2 \oplus (S_4)^3$	$(f_1)^2 g_4$	$(f_1)^2 g_4$	point	$10^3\, 5^6\, 2^1\, 1^1$	30	671,956,992	L 2.7(ii)
12A	AC	$J_3 \otimes S_2$	$(f_2)^3$	$(f_2)^3$	f.p.f.	$12^4\, 6^2\, 3^1$	48	419,976,960	§5.5.2
12B	BA	$I_1 \oplus S_2 \oplus J_3$	$(f_1)^4 f_2$	$(f_1)^3 f_2$	line	$12^2\, 6^2\, 4^2\, 3^4\, 2^2\, 1^3$	48	419,976,960	L 2.7(ii)
12C	DD	$S_2 \oplus J_4$	$(f_1)^4 f_2$	$(f_1)^4 f_2$	point	$12^3\, 6^1\, 4^3\, 3^2\, 2^1\, 1^1$	24	839,946,240	L 2.7(ii)
14A,B	AC, BC	$J_2 \otimes S_3,\ J_2 \otimes S_3^{-1}$	$(f_3)^2,\ (\hat{f}_3)^2$	$(f_3)^2,\ (\hat{f}_3)^2$	f.p.f.	$14^4\, 7^1$	56	359,976,960	§5.5.2
14C,D	CA, DA	$I_1 \oplus J_2 \oplus S_3^{\pm 1}$	$(f_1)^3 f_3$ & (ˆ)	$(f_1)^2 f_3$ & (ˆ)	line	$14^2\, 7^4\, 2^2\, 1^3$	56	359,976,960	L 2.7(ii)
15A,B	AC, AC	$I_2 \oplus S_4,\ I_2 \oplus S_4^{-1}$	$(f_1)^2 f_4$ & (ˆ)	$f_1 f_4$ & (ˆ)	line	$15^4\, 1^3$	90	223,985,664	L 2.8
15C	AB	$S_2 \oplus (S_4)^3$	$f_2 g_4$	$f_2 g_4$	f.p.f.	$15^3\, 5^3\, 3^1$	45	447,971,328	L 2.7(ii)
15D,E	AA, AA	$S_2 \oplus S_4,\ S_2 \oplus S_4^{-1}$	$f_2 f_4$ & (ˆ)	$f_2 f_4$ & (ˆ)	f.p.f.	$15^4\, 3^1$	45	447,971,328	L 2.7(ii)
21A,B	BA, AA	$(S_6)^{-3}, (S_6)^3$	$k_6,\ \hat{k}_6$	$k_6,\ \hat{k}_6$	f.p.f.	21^3	63	319,979,520	$(\text{Singer})^3$, T 3.6(iii), L 4.5
21C,D	DB, CB	$I_1 \oplus S_2 \oplus S_3^{\pm 1}$	$f_1 f_2 f_3$ & (ˆ)	$f_1 f_2 f_3$ & (ˆ)	point	$21^2\, 7^2\, 3^2\, 1^1$	21	959,938,560	L 2.7(ii)
28A,B	CA, DA	$J_3 \oplus S_3,\ J_3 \oplus S_3^{-1}$	$(f_1)^3 f_3$ & (ˆ)	$(f_1)^3 f_3$ & (ˆ)	point	$28^1\, 14^1\, 7^2\, 4^1\, 2^1\, 1^1$	28	719,953,920	L 2.7(ii)
30A,B	AAC, BAC	$J_2 \oplus S_4,\ J_2 \oplus S_4^{-1}$	$(f_1)^2 f_4$ & (ˆ)	$(f_1)^2 f_4$ & (ˆ)	point	$30^1\, 15^2\, 2^1\, 1^1$	30	671,956,992	L 2.7(ii)
31A...F		$I_1 \oplus S_5, \ldots, I_1 \oplus (S_5)^6$	$f_1 x_5,\ f_1 \hat{x}_5$	$f_1 x_5,\ f_1 \hat{x}_5$	point	$31^2\, 1^1$	31	650,280,960	L 2.8, table 4, $x_5 = f_5, \hat{g}_5, \hat{h}_5$
63A,B	BA, AA	$S_6,\ S_6^{-1}$	$f_6,\ \hat{f}_6$	$f_6,\ \hat{f}_6$	f.p.f.	63^1	63	319,979,520	Singer,
63C,D	BA,AA	$(S_6)^{-5},\ (S_6)^5$	$g_6,\ \hat{g}_6$	$g_6,\ \hat{g}_6$	f.p.f.	63^1	63	319,979,520	§3.2,
63E,F	BA,AA	$(S_6)^{11},\ (S_6)^{-11}$	$h_6,\ \hat{h}_6$	$h_6,\ \hat{h}_6$	f.p.f.	63^1	63	319,979,520	L 4.5

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