# Flats in $\mathrm{PG}(9,2)$ external to the Grassmannian $\mathcal{G}_{1,4,2}$.

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**Abstract**

A classification is given of those flats in the projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\mathrm{PG}(4,2)$. In particular it is shown that there exist precisely two $\mathrm{GL}(5,2)$-orbits of external 4-flats, each with stabilizer group $\cong$ 31:5. (No 5-flat is external.) For each $k = 1, 2, 3$, two distinct kinds of external $k$-flats are simply constructed out of certain partial spreads in $\mathrm{PG}(4,2)$ of size $k+2$. A third kind of external plane, with stabilizer $\cong 2^3 : (7 : 3)$, is also shown to exist.

## 1 Introduction

### 1.1 Linearity in the presence of nonlinearity

Let $\mathcal{H}$ be a hypersurface in $\mathrm{PG}(N,q)$ determined by a (homogeneous) polynomial equation $F(x_0, x_1, ..., x_N) = 0$, say. Then one can pose the problem of classifying those flats which are maximal subject to being *internal* to $\mathcal{H}$, and those which are maximal subject to being *external* to $\mathcal{H}$. In the case $\deg F = 2$ the internal problem was solved a long time ago. In particular a non-singular hyperbolic quadric $\mathcal{H}$ in $\mathrm{PG}(N,q)$ is known to have precisely two distinct families of maximal internal flats, each consisting of flats of dimension $\frac{1}{2}(N-1)$.

For $\deg F > 2$, however, the problem becomes in general much harder. Nevertheless, precisely because it is difficult for linearity to secure a place in a highly nonlinear situation, we feel that any such success carries with it the prospect of some interesting mathematics. As an example, consider the problem of classifying the *linear sections* of $\mathrm{GL}(n,q)$. Here we deal with the hypersurface $\mathcal{H}$, whose equation $\det A = 0,\ A \in \operatorname{End} V(n,q)$, has degree $n$, which lies in the projective space $\mathrm{PG}(n^2-1, q) = \mathbb{P}(\operatorname{End} V(n,q))$, and we seek to classify those flats which are *external* to $\mathcal{H}$. It seems to us that this is a genuinely tough problem! For even for the small $n$ and $q$ values $(n,q) = (4,2)$ the complete story was not easy to unravel (see [**?**]). For the description of a particularly interesting maximal linear section of $\mathrm{GL}(4,2)$, which lies inside an $\mathrm{Alt}(7)$ subgroup of $\mathrm{GL}(4,2) \cong \mathrm{Alt}(8)$, see [?].

In the present paper we will be chiefly concerned with those flats in $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V(5,2))$ which are *external* to the Grassmannian $\mathcal{G}_{1,4,2}$ arising from the 155 lines of $\mathrm{PG}(4,2) = \mathbb{P}(V(5,2))$. The general Grassmannian $\mathcal{G}_{r,n,q}$ of the $r$-spaces of $\mathrm{PG}(n,q)$, see [?], is usually thought of as the algebraic variety arising from certain well-known quadratic Grassmann conditions. In particular, in the case $(n,q) = (4,2)$, the Grassmannian $\mathcal{G}_{1,4,2}$ is usually thought of as the variety in $\mathrm{PG}(9,2)$ determined by the simultaneous solutions of the five distinct quadratic conditions $b_{ij}b_{kl} + b_{ik}b_{jl} + b_{il}b_{jk} = 0,\ \{i,j,k,l\} \subset \{1,2,3,4,5\}$. Here the $b_{ij}(= b_{ji})$ are the coordinates of a bivector $b \in V_{10} = V_5 \wedge V_5,\ V_5 = V(5,2)$, with respect to a product basis$\{e_i \wedge e_j\}_{1\le i<j\le 5}$. Since *any* algebraic variety in $\mathrm{PG}(N,q)$ is in fact a hypersurface, see [?, Lemma 3.2], the Grassmannian $\mathcal{G}_{1,4,2}$ is alternatively described as the locus $F(b_{12}, b_{13}, ..., b_{45}) = 0$ for a certain single polynomial $F$ in the 10 coordinates $b_{ij}$. It was proved in [?] that in this case $\deg F = 5$. The next theorem is a consequence of this fact.

**Theorem 1.1** *No 5-flat in* $\mathrm{PG}(4,2)$ *is external to* $\mathcal{G}_{1,4,2}$.

**Proof.** By the particular case $r = 5$ of the theorem in [?], every 5-flat intersects $\mathcal{G}_{1,4,2}$ in an odd number of points. ■

Actually, see theorem **??**, this result can be proved without appeal to the quintic equation of $\mathcal{G}_{1,4,2}$. Indeed all the results in this paper will be obtained without such an appeal — with the sole exception of theorem **??**.

**Note.** The present paper came about as follows. Over a period of several months the first two authors (RS, JGM) engaged in an intense and exciting interchange of ideas, prompted by a need to understand the computer results in [?]. Eventually, stimulated by some further computer results communicated to him by JGM, the first author (RS) constructed the various proofs in this paper (in some cases helped also by computer checks carried out by the third author (NAG)).

## 1.2 Bivectors

Although we will be working over the field GF(2), it is as well to remind ourselves of certain facts concerning the *bivector space* $\wedge^2 V$ which hold for a finite-dimensional vector space $V$ over any field. In this section we use $\lceil\cdot,\cdot\rfloor$ for the natural bilinear pairing of a finite-dimensional vector space $V$ with its dual space $V^*$. Moreover, for $v \in V$ and $f \in V^*$, we write interchangeably $\lceil f, v\rfloor = \lceil v, f\rfloor$. The induced pairing $\lceil\cdot,\cdot\rfloor$ of $\wedge^2 V$ with $\wedge^2 V^*$ is then defined to be that which satisfies $\lceil u\wedge v, f\wedge g\rfloor = \lceil u, f\rfloor\lceil v, g\rfloor - \lceil v, f\rfloor\lceil u, g\rfloor$.

First we take note of two vector spaces, $\mathrm{AltBilin}(V^*)$ and $\mathrm{Alt}(V^*, V)$, which are linearly isomorphic to $\wedge^2 V$. Here $\mathrm{AltBilin}(V^*)$ denotes the space of alternating bilinear forms on $V^*$, and the isomorphism is given by the linear mapping which sends $b \in \wedge^2 V$ to the bilinear form $b(\cdot,\cdot) \in \mathrm{AltBilin}(V^*)$ defined by $b(f,g) = \lceil b, f \wedge g \rfloor$, $f, g \in V^*$. Since $f \wedge f = 0$, the bilinear form $b(\cdot,\cdot)$ is alternating: $b(f,f) = 0$, (and hence skew symmetric: $b(f,g) + b(g,f) = 0$). Also $\mathrm{Alt}(V^*, V)$ here denotes the vector space consisting of all linear maps $B : V^* \to V$ which satisfy the alternating property $\lceil Bf, f \rfloor = 0$ for all $f \in V^*$, and the isomorphism is given by the linear mapping which sends $b \in \wedge^2 V$ to that element $b\llcorner \in \mathrm{Alt}(V^*, V)$ given by

$$\lceil b\llcorner f, g \rfloor = \lceil b, f \wedge g \rfloor = b(f,g) \qquad \text{for all } f, g \in V^*. \tag{1.1}$$

(Here $\llcorner$ is often termed 'right interior multiplication'; in terms of coordinates, using a basis in $V^*$ dual to the basis in $V$, the equation $v = b\llcorner f$ reads $v_i = \sum_j b_{ij} f_j$.) Observe that the alternating property $\lceil b\llcorner f, f \rfloor = 0$ implies the anti-self-dual property $(b\llcorner)^t + b\llcorner = 0$ :

$$\lceil b\llcorner f, g \rfloor + \lceil f, b\llcorner g \rfloor = 0. \tag{1.2}$$

Also note that the matrix $[b_{ij}]$ of a an element of $\mathrm{Alt}(V^*, V)$ is *alternating,* satisfying that is $b_{ij} + b_{ji} = 0$ and $b_{ii} = 0$. An immediate consequence of the anti-self-dual property (**??**) is the result:

$$\ker b\llcorner = (\operatorname{im} b\llcorner)^{\mathrm{O}}, \tag{1.3}$$

where $X^{\mathrm{O}} \subset V^*$ denotes the annihilator of a subspace $X \subset V$.

Secondly, recall the natural action of $\mathcal{G} = \mathrm{GL}(V)$ upon $\wedge^2 V$. Each $A \in \mathcal{G}$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\mathrm{GL}(\wedge^2 V)$ whose effect on the *decomposable bivectors* $u \wedge v \in V \wedge V$ is given by

$$T_A(u \wedge v) = Au \wedge Av, \qquad A \in \mathcal{G} = \mathrm{GL}(V). \tag{1.4}$$

If $X$ is any object in $\wedge^2 V$ its stabilizer $\mathcal{G}_X$ is the subgroup of $\mathcal{G}$ defined by

$$\mathcal{G}_X = \{A \mid A \in \mathcal{G},\ T_A(X) = X\}. \tag{1.5}$$

(More generally a corresponding definition applies, upon replacing $T$ by $\alpha$, for any other object $X$ upon which $\mathcal{G}$ has an action $\alpha$.)

Under this action $T$ of $\mathrm{GL}(V)$ the orbit structure of the bivector space $\wedge^2 V$ is well-known: if $V$ has dimension $2k$ or $2k+1$, then there are $k+1$ distinct orbits $\{\mathrm{Rk}_{2r};\ r = 0, 1, \ldots, k\}$ where $\mathrm{Rk}_{2r} = \mathrm{Rk}_{2r}(V)$ consists of those bivectors having rank $2r$. Here the *rank* of a bivector $b \in \wedge^2 V$ may be defined to be the rank of the associated linear map $b\llcorner$. Thus $\mathrm{rank}(b) = \dim(\operatorname{im} b\llcorner)$. As is well-known, a bivector $b$ has rank $2r > 0$ if and only if there exist linearly independent vectors $e_1, \ldots, e_{2r}$ such that $b = e_1 \wedge e_2 + \ldots + e_{2r-1} \wedge e_{2r}$. (In which case $\operatorname{im} b\llcorner = \prec e_1, \ldots, e_{2r} \succ$ .)

# 2 Preliminaries

From now we work over GF(2), and so *we may identify the nonzero elements of a vector space* $V = V(n+1,2) = V_{n+1}$ *with the points of the associated projective space* $\mathbb{P}V = \mathrm{PG}(n,2)$. Consequently, for $r > 0$, we may view the orbit $\mathrm{Rk}_{2r}(V)$ as a subset, to be denoted $\mathrm{Rk}_{2r}(n,2)$, of the projective space $\mathrm{PG}(N,2) = \mathbb{P}(\wedge^2 V)$ where $N = \frac{1}{2}n(n+1) - 1$.

We use $\prec u, v, \ \dots \succ$ for the *vector* subspace spanned by $u, v, \ \dots$ . The Plücker map $\prec u, v \succ \mapsto \prec u \wedge v \succ$ sends the 2-spaces of $V$ to those 1-spaces of $\wedge^2 V$ which are spanned by decomposable bivectors. Projectively the lines of $\mathrm{PG}(n,2)$ are mapped onto the points of $\mathrm{Rk}_2(V) = \mathcal{G}_{1,n,2}$. *Throughout this paper the images in* $\mathcal{G}_{1,n,2} \subset \mathrm{PG}(N,2)$ *of lines* $\lambda, \mu, \nu, \tau$ *in* $\mathrm{PG}(n,2)$ *will be denoted* $l, m, n, t$. It is easy to see that $\{m_1, m_2, m_3\}$ is a line lying on $\mathcal{G}_{1,n,2}$ if and only if $\{\mu_1, \mu_2, \mu_3\}$ is a *pencil* $(a, \alpha)$ of lines in $\mathrm{PG}(n,2)$, consisting of those three lines $\mu_i$ of the plane $\alpha$ which pass through the point $a \in \alpha$.

The case $n = 3$ of the foregoing, namely that of the Grassmannian $\mathcal{G}_{1,3,2} \subset \mathrm{PG}(5,2)$ arising from the 35 lines of $\mathrm{PG}(3,2)$, is of course extremely well-known, [?]. See appendix **??** for some aspects of relevance to our present concerns.

## 2.1 The Grassmannian $\mathcal{G}_{1,4,2}$

From now onwards our starting space will be $V_5 = V(5,2)$, along with its associated projective space $\mathrm{PG}(4,2)$, and we will be concerned with the corresponding bivector space $V_{10} = V(10,2) = V_5 \wedge V_5$, and with its associated projective space $\mathrm{PG}(9,2) = \mathbb{P}(V_{10})$. From section **??** we know that, under the action $T$ of $\mathcal{G} := \mathrm{GL}(5,2)$, the projective space $\mathrm{PG}(9,2)$ $(= V_{10} \diagdown \{0\})$ is the disjoint union of just two $\mathcal{G}$-orbits:

$$\mathrm{PG}(9,2) = \mathrm{Rk}_2(4,2) \cup \mathrm{Rk}_4(4,2) \qquad (\text{where } \mathrm{Rk}_2(4,2) = \mathcal{G}_{1,4,2}). \tag{2.1}$$

Usually these two orbits will be abbreviated as $\mathrm{Rk}_2$ and $\mathrm{Rk}_4$. Of course $|\,\mathrm{Rk}_2\,| = 155$ (the number of lines in $\mathrm{PG}(4,2)$), and so $|\,\mathrm{Rk}_4\,| = 1023 - 155 = 868$. In the following, the terms *internal, external, tangent* and *bisecant* usually refer to the Grassmannian $\mathrm{Rk}_2 = \mathcal{G}_{1,4,2}$.

### 2.1.1 A useful partition of $\mathrm{Rk}_4$

As well as the $155 + 868$ partition of $\mathrm{PG}(9,2)$ given by (**??**), of great importance (see for example theorem **??** below) will be a certain partition of the 868 points of $\mathrm{Rk}_4$ into 31 subsets of size 28, which arises from the 31 solids of $\mathrm{PG}(4,2)$. For observe that each solid $\sigma = \mathbb{P}(V_4)$ in $\mathrm{PG}(4,2)$ defines a 5-flat $\Pi(\sigma) = \mathbb{P}(\wedge^2 V_4)$ in $\mathrm{PG}(9,2)$, and so, upon defining

$$\mathcal{H}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_2 \qquad \text{and} \qquad \mathcal{W}(\sigma) = \Pi(\sigma) \cap \mathrm{Rk}_4, \tag{2.2}$$

the 63 points of $\Pi(\sigma)$ consist of the 35 points of a hyperbolic quadric $\mathcal{H}(\sigma) = \mathrm{Rk}_2(V_4) = \mathcal{G}_{1,3,2}$ together with the 28 points $\mathcal{W}(\sigma)$ which are external to $\mathcal{H}(\sigma)$. Now an element $b \in \mathrm{Rk}_4$ determines a 4-dimensional subspace $V_4 = \mathrm{im}(b \llcorner) = \{b \llcorner f \,|\, f \in V_5^*\}$ of $V_5$, and also a 1-dimensional subspace $V_1^* = \ker(b \llcorner) = \{f \in V_5^* \,|\, b \llcorner f = 0\,\}$ of $V_5^*$. Projectively we write

$$\sigma = \mathrm{im}\, b \qquad \text{and} \qquad k = \ker b \tag{2.3}$$

for the corresponding solid $\sigma = \mathbb{P}(V_4)$ in $\mathrm{PG}(4,2)$ and point $k = \mathbb{P}(V_1^*)$ in $\mathrm{PG}(4,2)^*$. Observe, see eq. (**??**), that the solid $\sigma$ has equation $\lceil k, x \rfloor = 0$. Since $b \in \mathcal{W}(\sigma)$ if and only if $\mathrm{im}\, b = \sigma$, it follows that the subsets $\mathcal{W}(\sigma)$, $\sigma$ a solid in $\mathrm{PG}(4,2)$, *yield a partition of the 868 points of* $\mathrm{Rk}_4$ *into 31 subsets of size 28:* $868 = 31 \times 28$.

**Definition 2.1** *An external line which lies inside one of the 28-sets* $\mathcal{W}(\sigma)$ *will be called a* special line; *the other external lines, which meet three distinct* $\mathcal{W}(\sigma)$, *will be referred to as* general lines. *(Instead of 'special' and 'general', the terms 'degenerate' and 'non-degenerate' were used in [**?**].) An external plane which contains a special line will be called a* special plane; *the other external planes, which meet seven distinct* $\mathcal{W}(\sigma)$, *will be referred to as* general planes. *Similarly* general solids *are external solids which contain no special lines, and* special solids *contain at least one special line.*

We do not apply this special/general terminology to external 4-flats because, by the next theorem, no external 4-flat contains a special line: every external 4-flat is 'general'.

**Theorem 2.2** *(i) An external 4-flat $E$ meets each* $\mathcal{W}(\sigma)$ *in precisely one point.*

*(ii) A special line possesses no extensions to an external 4-flat $E$.*

*(iii) External 5-flats do not exist; external 4-flats do exist.*

**Proof.** (i) In $\mathrm{PG}(9,2)$ a 4-flat meets a 5-flat in at least one point. So $E$ meets each of the thirty-one $\mathcal{W}(\sigma)$ in at least one point. But $|E| = 31$.

(ii) By (i), $E$ cannot meet $\mathcal{W}(\sigma)$ in a line.

(iii) Within $\mathrm{PG}(9,2)$ any 5-flat meets each 5-flat $\Pi(\sigma)$ in at least a line. So an external 5-flat $F$ would meet each of the thirty-one $\mathcal{W}(\sigma)$ in at least three points. The assumption that $F$ exists thus leads to a contradiction: $31 \times 3 = 93$, yet $|F| = 63$. (See theorem **??** for another proof.) See theorems **??**, **??** below for two constructions of external 4-flats. ■

## 2.2 Classification of the lines in $\mathbb{P}(\wedge^2 V_5)$

First of all we consider the lines in $\mathrm{PG}(9,2)$ through a particular point $b \in \mathrm{Rk}_4$. Recall that $b$ lies in $\mathcal{W}(\sigma)$ for precisely one choice of solid $\sigma =$

$\operatorname{im} b \subset \mathrm{PG}(4,2)$. It should also be noted that the stabilizer $\mathcal{G}_b$ of $b$ within $\mathrm{GL}(5,2)$ has the structure $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$. This comes about because $\mathcal{G}_b$ contains a normal subgroup $\cong (Z_2)^4$, consisting of those transvections which fix $\sigma$ pointwise, and since the stabilizer of $b$ within $\mathrm{GL}(4,2)$ is, see remark **??**, a subgroup $\mathrm{Sp}(4,2) \cong \mathrm{Sym}(6)$. The following lemma is relevant to our immediate concerns, but it will also be of use in the proof of theorem **??** in section **??**.

**Lemma 2.3** *Given* $b \in \mathcal{W}(\sigma)$, *and given* $\sigma' \neq \sigma$, *consider the* 28 *lines* $\langle b, b' \rangle$ *through* $b$ *which arise from the* 28 *choices of* $b' \in \mathcal{W}(\sigma')$. *For precisely four of these choices the line* $\langle b, b' \rangle$ *is a tangent to* $\mathrm{Rk}_2$, *and for the remaining twenty-four choices the line* $\langle b, b' \rangle$ *is a general line.*

**Proof.** Denote by $\alpha$ the plane of intersection of $\sigma$ and $\sigma'$. Recalling (**??**), put $p = \alpha^b$ and $p' = \alpha^{b'}$. Let $\mu$ be some fixed choice of line in $\alpha$ such that $p \notin \mu$ and $p' \notin \mu$. Then $\mu^b$ is a line $\nu \subset \sigma$ which meets $\alpha$ in $p$, and $\mu^{b'}$ is a line $\nu' \subset \sigma'$ which meets $\alpha$ in $p'$. Thus, see after (**??**), we have

$$b = m + n, \quad b' = m + n', \quad \text{and so} \quad b + b' = n + n'. \tag{2.4}$$

Consequently the line $\langle b, b' \rangle$ is a tangent, or a general line, according as $n + n'$ has rank 2, or rank 4 — that is according as the lines $\nu$, $\nu'$ meet, or are skew. Now there are precisely 4 lines $\nu'$ in $\sigma'$ through $p$, but not in $\alpha$. So for 4 of the 28 choices of $b' \in \mathcal{W}(\sigma')$, the point $p'$ coincides with $p$, and for the other 24 choices we have $p' \neq p$. For the former 4 choices the lines $\nu, \nu'$ meet (in the point $p = p'$), whence $\langle b, b' \rangle$ is a tangent. For the remaining 24 choices the lines $\nu, \nu'$ are skew, and so $\langle b, b' \rangle$ is a general line. ■

The 511 lines in $\mathrm{PG}(9,2)$ which pass through the point $b \in \mathcal{W}(\sigma)$ are:
(a) 10 bisecants (the 10 bisecants to $\mathcal{H}(\sigma)$);
(b) 15 tangents to $\mathcal{H}(\sigma)$;
(c) 120 tangents to the 120 points of $\mathrm{Rk}_2 \setminus \mathcal{H}(\sigma)$;
(d) 6 special lines (the other 6 lines through $b$ lying in $\Pi(\sigma)$;
(e) 360 $(= 511 - 10 - 15 - 120 - 6)$ general lines.
Concerning (a), (b), (d), see appendix **??**. Concerning (c), (e), the stated numbers follow from lemma **??**. For firstly there exist 4 points $b'$ in each of the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, such that $\langle b, b' \rangle$ is a tangent: $4 \times 30 = 120$. Secondly there exist 24 points $b'$ in each of the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, such that $\langle b, b' \rangle$ is a general line: $(24 \times 30) \div 2 = 360$. (The fact that the 120 tangents (c) have uniform intersection number 4 with the 30 $\mathcal{W}(\sigma')$, $\sigma' \neq \sigma$, follows also from the fact that $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$ is transitive on the 30 solids $\sigma' \neq \sigma$.)

Next consider the 511 lines through a point $m \in \mathrm{Rk}_2$. Note that $m$ will lie in $\mathcal{H}(\sigma)$ for 7 choices of $\sigma$ (since a line $\mu$ in $\mathrm{PG}(4,2)$ lies in 7 solids). For $m$ in $\mathcal{H}(\sigma)$ consider the 31 lines through $m$ which lie in $\Pi(\sigma)$. Of these, 9 are internal to $\mathcal{H}(\sigma)$, since $\mu$ belongs to 9 pencils in $\sigma$. So there are $35 - 18 - 1 = 16$ points of $\mathcal{H}(\sigma)$ not on these 9 internal lines. Hence $m$ lies

on 16 bisecants in $\Pi(\sigma)$. The remaining $31 - 9 - 16 = 6$ lines are tangents in $\Pi(\sigma)$. It follows that through $m$ pass:
(a) 21 internal lines (since $\mu$ belongs to 21 pencils in $\mathrm{PG}(4,2)$);
(b) 112 bisecants, 16 for each of the 7 $\mathcal{H}(\sigma)$ in the preamble;
(c) 42 tangents which lie in some $\Pi(\sigma)$, 6 for each of the 7 $\mathcal{H}(\sigma)$;
(d) $336\,(= 511 - 21 - 112 - 42)$ other tangents.
As an arithmetical check, note that the bisecants (b) and the tangents (c) account for all 28 points of each of seven $\mathcal{W}(\sigma)$, for $\sigma$ as in the preamble. Considering the remaining 24 $\mathcal{W}(\sigma)$, note that each of the 336 tangents (d) accounts for 2 of their points. Now we have the check $336 \times 2 = 24 \times 28$.

In consequence of the foregoing, the 174,251 lines in $\mathrm{PG}(9,2)$ can be classified as follows:
(i) 1085 internal lines; (ii) 8680 bisecants;
(iii) 6510 tangents which lie in some $\Pi(\sigma)$; (iv) 52080 other tangents;
(v) 1736 special external lines; (vi) 104,160 general external lines.

**Remark 2.4** *These six classes are in fact the six distinct* $\mathrm{GL}(5,2)$*-orbits of lines in* $\mathrm{PG}(9,2)$. *Concerning the lines (*v*), recall from lemma* **??** *that each of the 31* $\mathcal{W}(\sigma)$ *contributes 56 special lines:* $31 \times 56 = 1736$. *Noting, by the same lemma, that the stabilizer within* $\mathrm{GL}(4,2)$ *of a special line* $L \subset \mathcal{W}(\sigma)$ *has structure* $(\mathrm{Alt}(5) \times 3).2$, *the stabilizer* $\mathcal{G}_L$ *within* $\mathrm{GL}(5,2)$ *is seen to have the structure* $\mathcal{G}_L \cong 2^4 : \{(\mathrm{Alt}(5) \times 3).2\}$, *since* $\mathcal{G}_L$ *contains a normal subgroup* $\cong (Z_2)^4$ *consisting of those transvections which fix* $\sigma$ *pointwise. Concerning the general lines (*vi*), see theorem* **??** *for a proof that they form a single* $\mathrm{GL}(5,2)$*-orbit, and for the structure of the stabilizer group.*

## 3 Constructions of general $k$-flats, $0 \le k \le 3$, from regulus-free partial spreads

The internal flats of $\mathcal{G}_{1,4,2}$ are well understood. The maximal ones are of two kinds, *Greek planes* and *Latin solids.* The 7 lines of a plane $\alpha \subset \mathrm{PG}(4,2)$ map onto the 7 points of a 'Greek' plane $P(\alpha) \subset \mathcal{G}_{1,4,2}$, and the 15 lines of $\mathrm{PG}(4,2)$ which pass through the point $a$, forming let us say the *star* $\mathrm{st}(a)$, map onto the 15 points of a 'Latin' solid, say $\mathrm{St}(a) \subset \mathcal{G}_{1,4,2}$. So from now onwards we concentrate upon the external flats.

Because external 5-flats do not exist, our chief concern will be to classify the external 4-flats. To this end, because of theorem **??**, we are particularly interested in general flats (those which are not only external but which also contain no special lines). Our normal notation for a chain of flats in $\mathrm{PG}(9,2)$ of (projective) dimensions $1,2,3,4,5$ will be $L \subset P \subset D \subset E \subset F$. Usually the 4-flat $E$ in such a chain will be external, in which case $L$, $P$, $D$ will be general $k$-flats, $k = 1,2,3$.

### 3.1 Partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ and $r$-caps $\mathcal{C}_r \subset \mathrm{Rk}_2$

Given a partial spread $\mathcal{S}_r = \{\mu_1, \ldots, \mu_r\}$ in $\mathrm{PG}(4,2)$, let the corresponding $r$-set of points of $\mathrm{Rk}_2 \subset \mathrm{PG}(9,2)$ be $\mathcal{C}_r = \{m_1, \ldots., m_r\}$. For $1 \leq i < j < \ \ldots \ \leq r$ we define $m_{ij} = m_i + m_j$, $m_{ijk} = m_i + m_j + m_j$, $\ldots.$ , $m_\Sigma = m_{12\ldots r} = \sum_{i=1}^{r} m_i$. One should be aware that some of these vectors may be zero, and that coincidences may therefore occur amongst the points (= nonzero vectors). For example, if $\mathcal{S}_5 = \{\mu_1, \ldots, \mu_5\}$ is a spread for a solid $\sigma \subset \mathrm{PG}(4,2)$ then, see lemma **??**(iv), $m_\Sigma = 0$, whence $m_{123} = m_{45}$, $m_{1234} = m_5$, etc.

**Lemma 3.1** *(i) $\mathcal{C}_r$ is a $r$-cap: that is, no three points of $\mathcal{C}_r$ are collinear.*
*(ii) All of the points $m_{ij}$ are external points.*
*(iii) All of the points $m_{ijk}$ are external points.*
*(iv) The $\binom{r}{2}$ points $m_{ij}$ are distinct.*

**Proof.** (i) No three points of $\mathcal{C}_r$ can be collinear, since lines internal to $\mathrm{Rk}_2$ arise only from three intersecting lines of a pencil in $\mathrm{PG}(4,2)$.

(ii) Since $\mu_i$ does not meet $\mu_j$, it follows that $m_i + m_j \in \mathrm{Rk}_4$ .

(iii) Whether or not $\{\mu_i, \mu_j, \mu_k\}$ is a regulus we have $m_i = e_1 \wedge e_3$, $m_j = e_2 \wedge e_4$ and $m_k = (e_1 + e_2) \wedge a$, for suitably chosen independent points $e_1, e_2, e_3, e_4$ generating the solid $\sigma_{ij} = \langle \mu_i, \mu_j \rangle$. It follows that $m_{ijk} = e_1 \wedge (e_3 + a) + e_2 \wedge (e_4 + a)$, whence $m_{ijk} \in \mathrm{Rk}_4$, since the lines $\langle e_1, e_3 + a\rangle$ and $\langle e_2, e_4 + a\rangle$ are mutually skew.

(iv) Suppose to the contrary that, for example, $m_{12} = m_{34}$. Then all four lines $\mu_i$, $i = 1,2,3,4$, must lie in the solid $\sigma = \mathrm{im}(m_{12}) = \mathrm{im}(m_{34})$, and so must be members of a spread $\mathcal{S}_5 = \{\mu_1, \ldots \ , \mu_5\}$ for $\sigma$. But then, by lemma **??**(iv), $m_{12} + m_{34} = m_5 \neq 0$, contradicting $m_{12} = m_{34}$. ■

### 3.2 The even hyperplane construction

In the foregoing setting we will be particularly concerned with the projective space $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{\mathrm{even}}$ which is generated by the $\binom{r}{2}$ points $m_{ij}$. In cases where $m_1, \ldots \ , m_r$ are linearly independent, note that $\langle \mathcal{C}_r \rangle_{\mathrm{even}}$ is the *even hyperplane* of the basis $\{m_1, \ldots, m_r\}$ for $\langle \mathcal{C}_r \rangle$; it is the unique hyperplane in $\langle \mathcal{C}_r \rangle$ which contains no element of $\mathcal{C}_r$, and so, amongst the hyperplanes in $\langle \mathcal{C}_r \rangle$, *only $\mathcal{E}(C_r)$ is a candidate for being an external flat.*

The next lemma deals with the case $r = 3$. Take note that in $\mathrm{PG}(4,2)$ there are just two orbits of partial spreads of size 3, one consisting of the *reguli* (any 3 mutually skew lines lying in some solid) and the other *non-reguli* (any 3 mutually skew lines which generate the whole space $\mathrm{PG}(4,2)$).

**Lemma 3.2** *If the 3-cap $\mathcal{C}_3$ arises from a partial spread $\mathcal{S}_3$ then $\mathcal{E}(\mathcal{C}_3)$ is an external line which is special or general according as $\mathcal{S}_3$ is a regulus or a non-regulus .*

**Proof.** The three points $m_{12}, m_{13}, m_{23}$ form an external line $L$. If $\mathcal{S}_3$ is a regulus lying in a solid $\sigma$, each of $\mathrm{im}(m_{ij}) = \sigma$, so each $m_{ij} \in \mathcal{W}(\sigma)$. On the other hand, if $\mathcal{S}_3$ is a non-regulus then the $\mathrm{im}(m_{ij})$ are three distinct solids. ■

Consequently, if we are to use partial spreads in the construction of general flats, our attention is drawn to those partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$, of size $r \geq 2$, which are *regulus-free.* Now we are in the fortunate position of having available a complete classification of all the partial spreads in $\mathrm{PG}(4,2)$: see [**?**]. (The corresponding classification in $\mathrm{PG}(4,q)$ is not known for any other value of $q$.) Of the 63 $\mathrm{GL}(5,2)$-orbits (classes) of partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ of size $r \geq 2$, there are 13 classes of regulus-free partial spreads: for $r = 2,3,4,5,6,7,8$ there are, respectively, $1,1,2,5,2,1,1$ such classes. In [**?**, Table B.2] these 13 classes were labelled IIa.1; IIIa.1; IVa.1, IVb.1; Va.1, Vb.1, Vc.1, Vd.1, Ve.1; VIa.1, VIb.1, VIIa.1, VIIa.1 (the size $r$ of the partial spread being indicated by the corresponding Roman numeral). [*Note: In the following we will also use the same labels for the corresponding* $\mathrm{GL}(5,2)$*-orbits (classes) of r-caps.*]

**Theorem 3.3** *Given a regulus-free partial spread* $\mathcal{S}_r, r \geq 2$, *in* $\mathrm{PG}(4,2)$, *let* $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{even}$ *be defined as previously. Then* $\mathcal{E}(\mathcal{C}_r)$ *is an external flat if and only if one of the following holds:*

*(o)* $r = 2$*;* $\mathcal{E}(\mathcal{C}_2)$ *is an external point;*

*(i)* $r = 3$ *and* $\mathcal{S}_3 \in$ *class IIIa.1;* $\mathcal{E}(\mathcal{C}_3)$ *is a general line;*

*(ii)* $r = 4$ *and* $\mathcal{S}_4 \in$ *class IVb.1;* $\mathcal{E}(\mathcal{C}_4)$ *is a general plane;*

*(iii)* $r = 5$ *and* $\mathcal{S}_5 \in$ *class Ve.1;* $\mathcal{E}(\mathcal{C}_5)$ *is a general solid.*

**Proof.** A straightforward check of the 13 regulus-free classes (for $r \geq 2$) described in [**?**, Table B.2] reveals that $\mathcal{E}(\mathcal{C}_r)$ is an external flat only for the four listed cases (case (i) being that of the non-reguli of lemma **??**). For example, if $\mathcal{S}_4$ is in class IVa.1, then we find that $\mathcal{E}(\mathcal{C}_4)$ is a plane which is not external, since it is tangent to $\mathrm{Rk}_2$ at the point $m_\Sigma = m_{1234}$. ■

**Remark 3.4** *If* $\mathcal{E}(\mathcal{C}_5)$ *is a general solid, then it follows that* $\mathcal{E}(\mathcal{C}_4)$ *is a general plane for each 4-subset* $\mathcal{C}_4$ *of* $\mathcal{C}_5$*. Note that the properties of partial spreads fit in nicely with cases (ii) and (iii) in the theorem: for of the partial spreads* $\mathcal{S}_5$ *of size 5, only those of class Ve.1, see [**?**, lead-in to eq.(3.24)], have the property that* $S_5 \diagdown \{\mu_i\}$ *is of class IVb.1 for each* $\mu_i \in S_5$*.*

A description of the three classes IIIa.1, IVb.1 and Ve.1 of regulus-free partial spreads, and of their stabilizer groups, is given in appendix **??**.

**Remark 3.5** *For* $k = 1,2,3$ *the distinct* $\mathrm{GL}(5,2)$*-orbits of external k-flats in* $\mathrm{PG}(9,2)$ *will be denoted* $\mathrm{orb}(k\alpha)$, $\mathrm{orb}(k\beta)$, ... *. Since each of the classes IIIa.1, IVb.1, Ve.1 of partial spreads is a* $\mathrm{GL}(5,2)$*-orbit, the general flats in parts (i), (ii) and (iii) of the theorem form single* $\mathrm{GL}(5,2)$*-orbits. In each* *case let us agree to label the orbit as the alpha orbit: (i)* $\mathcal{E}(\mathcal{C}_3) \in \mathrm{orb}(1\alpha)$; *(ii)* $\mathcal{E}(\mathcal{C}_4) \in \mathrm{orb}(2\alpha)$; *(iii)* $\mathcal{E}(\mathcal{C}_5) \in \mathrm{orb}(3\alpha)$.

Concerning case (o) in the theorem, every external point arises in this way, since $\mathrm{Rk}_4$ is a single $\mathrm{GL}(5,2)$-orbit. Moreover, recall eq. (**??**), for any external point $b$ we have $\mathcal{E}(\mathcal{C}_2) = \langle b \rangle$ for precisely ten choices of 2-cap $\mathcal{C}_2$. Recall also that the stabilizer of $b$ has the structure $\mathcal{G}_b \cong 2^4 : \mathrm{Sym}(6)$. For case (i), the corresponding facts are as described in the next theorem.

**Theorem 3.6** *(i) All general lines belong to the orbit* $\mathrm{orb}(1\alpha)$.

*(ii) A general line $L$ can be expressed as $L = \mathcal{E}(\mathcal{C}_3)$ for precisely two choices of non-regulus $\mathcal{S}_3$. If $\mathcal{S}_3$ is one choice, then its twin $\mathcal{S}_3^*$, see appendix* **??** *(i), is the only other choice.*

*(iii) The stabilizer $\mathcal{G}_L$ of $L$ is of order 96, and contains, as a subgroup of index 2, the common stabilizer* $\mathcal{G}_{\mathcal{S}_3} = \mathcal{G}_{\mathcal{S}_3^*} \cong \mathrm{Sym}(4) \times Z_2$ *of $\mathcal{S}_3$ and $\mathcal{S}_3^*$.*

**Proof.** (i) In the proof of lemma **??** (in the cases $p' \neq p$), the general line $L$ was seen see in eq. (**??**) to have expression $L = \mathcal{E}(\mathcal{C}_3)$ for the non-regulus $\mathcal{S}_3 = \{\mu, \nu, \nu'\}$.

(ii) In the same proof observe that two choices, say $\mu$ and $\mu^*$, were available for a line $\mu \subset \alpha$ not through $p$ and not through $p'$. Consequently we have $L$ also expressed as $L = \mathcal{E}(\mathcal{C}_3^*)$ for a second non-regulus $\mathcal{S}_3^* = \{\mu^*, \nu^*, \nu^{*\prime}\}$. Note that $\mathcal{S}_3$, $\mathcal{S}_3^*$ are twin non-reguli, their common transversal being $\langle p,p' \rangle$. To complete the proof of (ii), we need to show (upon changing the notation) that the following holds: given $L = \mathcal{E}(\mathcal{C}_3) = \{m_{12}, m_{13}, m_{23}\}$ and $L = \mathcal{E}(\mathcal{C}'_3) = \{m'_{12}, m'_{13}, m'_{23}\}$, where $\mathcal{S}_3 = \{\mu_1, \mu_2, \mu_3\}$ and $\mathcal{S}'_3 = \{\mu'_1, \mu'_2, \mu'_3\}$ are non-reguli, then either $\mathcal{S}'_3 = \mathcal{S}_3$ or $\mathcal{S}'_3 = \mathcal{S}_3^*$. Now we can arrange the labels $1,2,3$ for $\mathcal{C}'_3 = \{m'_1, m'_2, m'_3\}$ so that $m'_{ij} = m_{ij}$. Then the solid $\sigma_{ij} = \langle \mu_i, \mu_j \rangle = \mathrm{im}(m_{ij})$ coincides with the solid $\sigma'_{ij} = \langle \mu'_i, \mu'_j \rangle$. So the non-reguli $\mathcal{S}_3, \mathcal{S}'_3$ share the same transversal $\tau$ (the common intersection of the three solids $\sigma_{ij}$.) Consequently, for $i = 1,2,3$, the plane $\alpha_i := \sigma_{ij} \cap \sigma_{ik} = \langle \mu_i, \tau \rangle$ must coincides with the plane $\alpha'_i = \langle \mu'_i, \tau \rangle$. Let $\mu_i$, $\mu'_i$ meet $\tau$ in the points $a_i$, $a'_i$. Then, to avoid contradiction, we must have $a'_i = a_i$, $i = 1,2,3$. (For example, suppose $a'_1 = a_3$, $a'_2 = a_1$, $a'_3 = a_2$. Then from $m'_{12} = m_{12}$ we have $m'_1 + m_2 = m'_2 + m_1$. Since $\mu'_1 \neq \mu_2$, it follows that $l := m'_2 + m_1$ is $\neq 0$ and so $l$ is the image of some line $\lambda$ passing through the point $a_1$ where $\mu'_2$ meets $\mu_1$. But from $m'_1 + m_2 = l$ it follows that $\mu'_1$ meets $\mu_2$ , whence $\langle \mu'_1, \tau \rangle = \langle \mu_2, \tau \rangle$ — a contradiction, since $\alpha_1 \neq \alpha_2$.) From $a'_i = a_i$, $i = 1,2,3$, and $\alpha'_i = \alpha_i$ we obtain $m'_i = m_i + x_i t$, $x_i \in \{0,1\}$. But then from $m'_{ij} = m_{ij}$ we obtain $x_1 = x_2 = x_3$, whence either $m'_i = m_i$ for each $i$, or else $m'_i = m_i + t = m_i^*$ for each $i$.

(iii) This follows from the bijection $L \longleftrightarrow \{\mathcal{S}_3, \mathcal{S}_3^*\}$ which we have just established upon using lemma **??**. ■

**Theorem 3.7** *(i) A plane* $P \in \operatorname{orb}(2\alpha)$ *can be expressed in the form* $P = \mathcal{E}(\mathcal{C}_4)$ *of theorem* **??***(ii) for a uniquely determined 4-cap* $\mathcal{C}_4$ *of class IVb.1.*

*(ii) The stabilizer* $\mathcal{G}_P$ *of* $P$ *is isomorphic to* $\mathrm{Sym}(3)$*. In its action upon* $P$*, the group* $\mathcal{G}_P$ *fixes a privileged antiflag* $(b, L)$ *in* $P$*; equivalently worded,* $\mathcal{G}_P$ *fixes a privileged conic whose nucleus is* $b$ *and whose external line is* $L$*.*

**Proof.** For $\mathcal{S}_4$ as in (**??**) we have $\mathcal{C}_4 = \{m_1, m_2, m_3, m_4\}$ where, in the shorthand notation of appendix **??**, $m_1 = 1 \wedge 2$, $m_2 = 3 \wedge 4$, $m_3 = 5 \wedge u$, $m_4 = 13 \wedge 15$. Let us consider the seven kernels, see (**??**), $k_{ij} = \ker m_{ij}$ and $k_\Sigma = \ker m_\Sigma$ of the seven points $m_{ij} = m_i + m_j$ and $m_\Sigma = m_{1234}$ of the plane $P = \mathcal{E}(\mathcal{C}_4)$. We quickly see that $P$ contains a distinguished conic $\{m_{14}, m_{24}, m_{34}\}$, picked out by the fact that for this conic alone the corresponding kernels $k_{14}$, $k_{24}$, $k_{34}$ form a line $L^*$ (in the dual geometry $\mathrm{PG}(4,2)^*$). The corresponding fact in $\mathrm{PG}(4,2)$ is that the three solids $\sigma_{14}, \sigma_{24}, \sigma_{34}$ are distinguished as the only three of the solids $\sigma_{ij} = \prec \mu_i, \mu_j \succ$ which share a plane, namely the plane $\alpha = (L^*)^{\mathrm{O}}$. (For $\mathcal{S}_4$ as in (**??**), $\alpha = \prec 1, 3, 5 \succ$.) Consequently $P$ also contains a distinguished point $m_\Sigma = m_{14} + m_{24} + m_{34}$, the nucleus of the conic, and a distinguished line $\{m_{12}, m_{13}, m_{23}\}$, the external line of the conic.

(i) Suppose that $P \in \operatorname{orb}(2\alpha)$ has the two expressions $P = \mathcal{E}(\mathcal{C}_4)$ and $P = \mathcal{E}(\mathcal{C}'_4)$. By the preamble in this proof concerning the distinguished conic of $P$, we must have, after a suitable re-labelling,

$$\text{(a) } m'_{ij} = m_{ij}, \quad \text{(b) } m'_{i4} = m_{i4}, \quad \text{(c) } m'_\Sigma = m_\Sigma \quad (i, j = 1, 2, 3). \qquad (3.1)$$

Since in (a) the distinguished line $\{m_{12}, m_{13}, m_{23}\}$ is general, it follow from theorem **??** that either (1) $m'_i = m_i$, $i = 1, 2, 3$, or else (2) $m'_i = m_i + t$, $i = 1, 2, 3$, where $\tau$ is the transversal of $\{\mu_1, \mu_2, \mu_3\}$ (and of $\{\mu'_1, \mu'_2, \mu'_3\}$). If (2) were to hold, then from (**??**c) we obtain $m'_4 + t = m_4$, which contradicts the fact that $\mu_4$ is skew to $\tau$. So (1) holds, and hence also $m'_4 = m_4$, and we have proved part (i) of the theorem.

(ii) Knowing now that partial spreads $\mathcal{S}$ of class IVb.1 are in bijective correspondence with planes $P \in \operatorname{orb}(2\alpha)$, we must have $\mathcal{G}_P = \mathcal{G}_S$. The rest now follows from our preamble in conjunction with eq. (**??**). ■

# 4 External 4-flats

## 4.1 General solids and external 4-flats

Consider a 5-cap $\mathcal{C} = \{m_1, \ldots, m_5\}$ which arises from a partial spread $\mathcal{S}$ of class Ve.1. Then, see eq. (**??**), $\mathcal{G}_{\mathcal{C}} = \langle B \rangle \cong Z_5$. So, see lemma **??**, $\mathcal{C}$ determines (i) an antiflag $(u, \sigma)$ in $\mathrm{PG}(4,2)$, the unique antiflag which is fixed by $B$, and (ii) a special line $L = \{p, p', p''\} \subset \mathcal{W}(\sigma)$, where $p$, $p'$ and $p''$ are the 3 fixed points of $T_B = \wedge^2 B$ in $\mathrm{PG}(9,2)$. Since $m_\Sigma = \sum_{i=1}^5 m_i$ is clearly fixed by $T_B$, we may as well take $p$ to be $m_\Sigma$.

In the next theorem we consider the general solid $D = \mathcal{E}(\mathcal{C})$ of theorem **??**(iii). Recall, remark **??**, that such solids form the single orbit, orb$(3\alpha)$.

**Theorem 4.1** *(i) The solid $D = \mathcal{E}(\mathcal{C})$ has precisely two extensions $E'$ and $E''$ to an external 4-flat, namely*

$$E' = \langle p', D\rangle \qquad \textit{and} \qquad E'' = \langle p'', D\rangle. \tag{4.1}$$

*(ii) Each $D \in \mathrm{orb}(3\alpha)$ can be expressed as $D = \mathcal{E}(\mathcal{C})$ for a uniquely determined 5-cap $\mathcal{C}$ of class Ve.1.*
*(iii) (a) $\mathcal{G}_D = \mathcal{G}_{\mathcal{C}} = \langle B\rangle \cong Z_5$;  (b) $\mathcal{G}_D \subseteq \mathcal{G}_{E'} \cap \mathcal{G}_{E''}$.*
*(iv) The 5-flat $F = \langle E', E''\rangle = \langle L, D\rangle$ also has stabilizer $\mathcal{G}_F = \langle B\rangle \cong Z_5$.*

**Proof.** (i) We may assume $\mathcal{C}$ is the 5-cap corresponding to the partial spread $\mathcal{S}_5$ in appendix **??**. Thus $D = \langle m_1, \ldots, m_5\rangle_{\text{even}}$ where

$$m_i = e_{i+1} \wedge (e_{i+2} + e_{i+3} + e_{i+4}), \qquad i = 1, 2, 3, 4, 5 \ (\mathrm{mod}\, 5). \tag{4.2}$$

Then $\mathcal{G}_{\mathcal{C}} = \langle B\rangle$ where $B$ effects the cyclic permutation $(e_1e_2e_3e_4e_5)$ of the basis vectors, with $T_B$ effecting the cyclic permutation $(m_1m_2m_3m_4m_5)$ of the points of $\mathcal{C}$. The fixed points $p, p', p''$ of $T_B$ are as in eq. (**??**):

$$p = \textstyle\sum_i e_i \wedge e_{i+1}, \quad p' = \textstyle\sum_i e_i \wedge e_{i+2}, \quad p'' = p + p' = \textstyle\sum_{i<j} e_i \wedge e_j, \tag{4.3}$$

and we see from (**??**) that $p = m_\Sigma$.

Under the action of $T_B$, the solid $D$ consists of three orbits, with representatives $m_{12}$, $m_{13}$ and $m_{1234}$. So to show that the 4-flat $\langle D, p'\rangle$ is external it suffices to show that all three points $p' + m_{12}$, $p' + m_{13}$ and $p' + m_{1234}$ are external. This is easily verified. A corresponding check shows that the 4-flat $\langle D, p''\rangle$ is also external.

We now need to show that $D$ possesses no other extension to an external 4-flat. To this end note that the invariant antiflag of $B$ is $(u, \sigma)$ where $u = \sum_{i=1}^{5} e_i$ and $\sigma$ is the even hyperplane of the basis $\{e_1, e_2, e_3, e_4, e_5\}$. Since $\sigma$ is distinct from the $(10 + 5 =)15$ solids $\mathrm{im}(m_{ij})$ and $\mathrm{im}(m_{ijkl})$, any extension of $D$ to an external 4-flat must be of the form $\langle D, b\rangle$ for some $b \in \mathcal{W}(\sigma)$. Now $\mathcal{W}(\sigma)$ is fixed by $T_B$, decomposing into the three fixed points $p, p', p''$ and five 5-cycles. We have seen that $\langle D, b\rangle$ is an external 4-flat for the two choices $b = p'$ and $b = p''$; however it is not one for the choice $b = p$, since $\langle D, p\rangle$ $(= \langle \mathcal{C}\rangle)$ contains the five internal points $m_i$. If $b_1, \ldots, b_5$ are representatives of the five 5-cycles, then to complete the proof we need to show that each of the five 4-flats $\langle D, b_i\rangle$ meets $\mathrm{Rk}_2$ in at least one point. This is a straightforward check. For, in the shorthand notation of appendix **??**, with the choice $b_1 = 1 \wedge 2 + 3 \wedge 4 + 5 \wedge u$, $b_2 = 1 \wedge 3 + 2 \wedge 4 + 5 \wedge u$, $b_3 = 12 \wedge 14 + 34 \wedge 35$, $b_4 = 12 \wedge 14 + 15 \wedge 34$, $b_1 = 12 \wedge 14 + 25 \wedge 34$ as representatives, we see that each of the points $b_1 + m_{13}$, $b_2 + m_{34}$, $b_3 + m_{23}$, $b_4 + m_{35}$, $b_5 + m_{14}$ lies on $\mathrm{Rk}_2$.

(ii) By part (i), $D = \mathcal{E}(\mathcal{C})$ determines uniquely the 5-flat $F = \langle E', E'' \rangle$. If another 5-cap $\mathcal{C}^*$ exists such that $D = \mathcal{E}(\mathcal{C}^*)$, then both of the 4-flats $\langle \mathcal{C} \rangle$ and $\langle \mathcal{C}^* \rangle$ contain $D$ and lie inside $F$. But there are only three 4-flats $X$ which satisfy $D \subset X \subset F$, and two of these are $E'$ and $E''$. Hence $\langle \mathcal{C} \rangle = \langle \mathcal{C}^* \rangle$, and so $\mathcal{C}(= \langle \mathcal{C} \rangle \cap \mathrm{Rk}_2)$ coincides with $\mathcal{C}^*(= \langle \mathcal{C}^* \rangle \cap \mathrm{Rk}_2)$.

(iii), (iv) Knowing now that partial spreads $\mathcal{S}$ of class Ve.1 are in bijective correspondence with solids $D \in \mathrm{orb}(3\alpha)$, it follows that $\mathcal{G}_D = \mathcal{G}_S = \langle B \rangle$. Similarly $\mathcal{G}_F = \mathcal{G}_S = \langle B \rangle$, since $F$ is uniquely determined by $D$ and hence by $\mathcal{S}$. Concerning (iiib), of course $\mathcal{G}_D$ stabilizes both $E'$ and $E''$ in (**??**), since $T_B$ fixes both $p'$ and $p''$. ■

**Theorem 4.2** *Let $E$ be any external 4-flat and let $F$ be any 5-flat which contains $E$. Put $\mathcal{C}_F = F \cap \mathrm{Rk}_2$, and let $\mathcal{S}_F$ be the corresponding line-set in* $\mathrm{PG}(4,2)$. *Then*

*(i) $\mathcal{C}_F$ is a cap, and $\mathcal{S}_F$ is a regulus-free partial spread;*

*(ii) if $F \neq F'$ then $\mathcal{C}_F \cap \mathcal{C}_{F'} = \emptyset$;*

*(iii) $\{\mathcal{S}_F \,|\, F$ a 5-flat $\supset E\}$ is a partition of the 155 lines of* $\mathrm{PG}(4,2)$*;*

*(iv) for each 5-flat $F \supset E$, $|\mathcal{C}_F| = 5$;*

*(v) $\mathcal{S}_F$ is of class Ve.1 and $\mathcal{G}_{\mathcal{C}_F} \cong Z_5$.*

**Proof.** (i) Any two points $m$, $m' \in \mathcal{C}_F$ lie outside the hyperplane $E$ of $F$; so $m + m'$ lies in $E$, whence $\mathrm{rank}(m + m') = 4$. Hence $\mathcal{C}_F$ is a cap and $\mathcal{S}$ is a partial spread. Suppose that $\mathcal{S}_F$ contains a regulus $\{\mu, \mu', \mu''\}$, and so $\mu$, $\mu'$, $\mu''$ lie in a common solid $\sigma$. This assumption leads to a contradiction as follows. For if $m, m', m''$ lie in $\Pi(\sigma)$, then $m + m', m + m'', m' + m''$ must coincide in the point, see theorem **??**(i), where $E$ meets $\mathcal{W}(\sigma)$, contradicting $\mu$, $\mu'$ and $\mu''$ being distinct. So $\mathcal{S}$ is regulus-free.

(ii) If $\mathcal{C}_F$, $\mathcal{C}_{F'}$ have a point $m$ in common, then $F = \langle \mathcal{C}_F, E \rangle = \langle m, E \rangle$ coincides with $F' = \langle \mathcal{C}_{F'}, E \rangle = \langle m, E \rangle$.

(iii) Every $m \in \mathrm{Rk}_2$ lies in a 5-flat $F$ containing $E$, namely in $F = \langle m, E \rangle$. Hence (iii) follows from (ii).

(iv) For each $F$, $\mathcal{E}(\mathcal{C}_F)$ lies inside $E$, and so is an external flat. From theorem **??** it follows that $|\mathcal{C}_F| \leq 5$. But from (iii) we have $\Sigma_F |\mathcal{C}_F| = |\,\mathrm{Rk}_2\,| = 155$. Hence $|\mathcal{C}_F| = 5$ for each $F$.

(v) This now follows from theorem **??** and eq. (**??**). ■

*We now give an alternative proof that $|C_F| = 5$ for each $F$. It is longer than that just given in (iv), but it has the merit of providing more information, especially concerning the intersections $F \cap W(\sigma)$.* Consider a 5-flat $F \supset E$. Set $L(F, \sigma) = F \cap \Pi(\sigma)$ and $I(F, \sigma) = F \cap \mathcal{W}(\sigma)$, and denote by $b(\sigma)$ the point, theorem **??**, where $E$ meets $\mathcal{W}(\sigma)$. Note that $L(F, \sigma)$, the meet of two 5-flats in $\mathrm{PG}(9,2)$, is at least a line; but it cannot be a plane, since then $L(F, \sigma)$ would meet the hyperplane $E$ of $F$ in a line, contradicting theorem **??**. So, for given $F$, all 31 of the intersections $L(F, \sigma)$ are lines, and the 31 intersections $I(F, \sigma)$ are thus of size 1, 2 or 3. (Not 0, since $b(\sigma)$ is in $I(F,\sigma)$.) Let us consider these three types of intersection in turn. If $|I(F,\sigma)| = 1$ then $L(F,\sigma)$ is a bisecant, of the form $\{m_i, m_j, b(\sigma)\}$. But the $r = |\mathcal{C}_F|$ points $m_i$ of $\mathcal{C}_F$ produce $r(r-1)/2$ bisecants, and hence $r(r-1)/2$ points $m_i + m_j$ of $E$, these points being distinct by lemma **??**(iv). So $|I(F,\sigma)| = 1$ for $r(r-1)/2$ choices of $\sigma$. Next, if $|I(F,\sigma)| = 2$ then $L(F,\sigma)$ is a tangent, of the form $\{m, b(\sigma), b'\}$, for some $b' \in \mathcal{W}(\sigma) \diagdown \{b(\sigma)\}$ and some $m \in \mathcal{C}_F$ such that $\mu \subset \sigma$. Now each line $\mu$ lies in seven solids $\sigma$ of $\mathrm{PG}(4,2)$,and $r-1$ of these seven have been used by the $r-1$ bisecants through $m$ already considered. The $8-r$ remaining solids through $\mu$ yield $8-r$ tangents through $m$ of the considered form. This applies to each $m \in \mathcal{C}_F$, and so $|I(F,\sigma)| = 2$ for $r(8-r)$ choices of $\sigma$. Finally we must have $|I(F,\sigma)| = 3$ for the remaining $31 - r(r-1)/2 - r(8-r)$ choices of $\sigma$. It follows that $r(r-1)/2 + 2r(8-r) + 93 - 3r(r-1)/2 - 3r(8-r) = |F \diagdown \mathcal{C}_F| = 63 - r$ — which gives $r = |\mathcal{C}_F| = 5$ for each $F$. Observe therefore that $|I(F,\sigma)| = 1, 2, 3$ for respectively 10, 15, 6 solids $\sigma \subset \mathrm{PG}(4,2)$.

**Theorem 4.3** *If $E$ is any external 4-flat, its stabilizer $\mathcal{G}_E$ is the normalizer $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of* $\mathrm{GL}(5,2)$.

**Proof.** By theorems **??** and **??** each of the 31 5-flats $F \supset E$ gives rise to a solid $D_F := \langle \mathcal{C}_F \rangle \cap E = \mathcal{E}(\mathcal{C}_F) \in \mathrm{orb}(3\alpha)$. The 31 5-caps $\mathcal{C}_F$ are distinct, and so by theorem **??**(ii) the 31 solids $D_F \subset E$ are also distinct. By theorem **??**(iiia), $\mathcal{G}_{D_F} = \langle B_F \rangle \cong Z_5$; furthermore these 31 stabilizers $\mathcal{G}_{D_F}$ are distinct, since $T_{B_F}$ and $T_{B_{F^*}}$ fix the distinct (if $F \neq F^*$) hyperplanes $D_F$ and $D_{F^*}$ of $E$. Consequently, by theorem **??**(iiib), $\mathcal{G}_E$ contains the 31 distinct $Z_5$-subgroups $\mathcal{G}_{D_F}$.

Now, see [**?**, p.70], there are just three classes, say $\mathfrak{C}_i$, $i = 1, 2, 3$, of maximal subgroups of $\mathrm{GL}(5,2)$ which contain elements of order 5. A subgroup $\mathcal{G} \in \mathfrak{C}_1$ stabilizes a point of $\mathrm{PG}(4,2)$, and a subgroup $\mathcal{G} \in \mathfrak{C}_2$ stabilizes a solid of $\mathrm{PG}(4,2)$. But no common point or solid is stabilized by all the foregoing thirty-one $Z_5$-subgroups of $\mathcal{G}_E$. Hence $\mathcal{G}_E$ lies inside a group $\mathcal{G} \in \mathfrak{C}_3$. But such a $\mathcal{G}$ is the normalizer $\mathcal{N} = N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of some Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5,2)$. But the only subgroup of $\mathcal{N}$ which contains more than one $Z_5$ is $\mathcal{N}$ itself (which indeed contains 31 distinct $Z_5$ subgroups). Hence the theorem. ■

**Theorem 4.4** *Suppose that the $k$-flat $X$, $k = 1, 2, 3$, possesses an extension to an external 4-flat $E$; then $X$ belongs to the orbit* $\mathrm{orb}(k\alpha)$.

**Proof.** Let $E$ be an external 4-flat and let $\mathcal{C} = \{m_1, \ldots, m_5\}$ be one of the 5-caps $\mathcal{C}_F$ of theorem **??**. Then, for $k = 1, 2, 3$, $E$ certainly contains a $k$-flat $X_k \in \mathrm{orb}(k\alpha)$, namely $X_k = \langle m_1, \ldots, m_k \rangle_{\mathrm{even}}$. But, by theorem **??**, $\mathcal{G}_E$ is the normalizer $N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ of a Singer cyclic subgroup $\mathcal{Z} \cong \mathcal{Z}_{31}$ of $\mathrm{GL}(5,2)$, and such a group acts transitively on (i) the 155 lines of $E$, (ii) the 155 planes of $E$, (iii) the 35 solids of $E$. (The $k = 1$ result is otherwise immediate, since the general lines form the single orbit $\mathrm{orb}(1\alpha)$.) ■

## 4.2 The $N(Z_{31})$-construction of external 4-flats.

In this section we find it convenient to adopt the following notation: if $X$ is a projective $r$-flat, then we use $\overline{X}$ for the $(r+1)$-dimensional vector space such that $X = \mathbb{P}\overline{X}$. For example, if the flats $X$, $Y$ are disjoint, then $\wedge^2 V_5 = \overline{X} \oplus \overline{Y}$ is the same thing as $\mathrm{PG}(9,2) = \langle X, Y \rangle$.

Given any subgroup $\mathcal{Z} \cong Z_{31}$ of $\mathrm{GL}(5,2)$, let $\mathcal{N} := N(\mathcal{Z}) \cong Z_{31} \rtimes Z_5$ be its normalizer. We consider the representation $T$ of $\mathcal{N}$ upon $V_{10} = \wedge^2 V_5$ given by $T_C = \wedge^2 C$, $C \in \mathcal{N}$. The first thing to note is that, since $2 \nmid |\mathcal{N}|$, Maschke's theorem applies: the representation $T$ is completely reducible. Now certainly no one-dimensional subspace $\prec b \succ$ of $V_{10}$ is invariant. For note that $A \in \mathcal{Z}$ sends $b \in \mathcal{W}(\sigma)$ to $T_A b \in \mathcal{W}(A(\sigma))$, and that $A(\sigma) \neq \sigma$ for $A \neq I$; and of course no $b \in \mathrm{Rk}_2$ is fixed, since no line of $\mathrm{PG}(4,2)$ is fixed by $\mathcal{Z}$. On the other hand we know from theorem **??** (and from the fact that subgroups $\cong Z_{31} \rtimes Z_5$ of $\mathrm{GL}(5,2)$ form a single class) that there exists a 5-dimensional subspace $\overline{E} = E \cup \{0\}$ of $V_{10}$ which is invariant for $T$, where $E$ is an external 4-flat with $\mathcal{G}_E = \mathcal{N}$. Moreover, only for $n \geq 5$ does $\mathrm{GL}(n,2)$ possess a subgroup $\cong Z_{31}$. Hence, by complete reducibility, it follows that the 10-dimensional representation $T$ of $\mathcal{N}$ decomposes as the direct sum $T_+ \oplus T_-$ of two 5-dimensional irreducible representations $T_+$ and $T_-$, arising from some $\mathcal{N}$-invariant direct sum decomposition $\overline{E_+} \oplus \overline{E_-}$ of $V_{10}$. Observe that the 4-flats $E_+$, $E_-$ are both external: for since $\mathrm{Rk}_2$ is a single $T_{\mathcal{N}}$-orbit, but is not a subspace, each of $E_+$, $E_-$ must lie entirely outside $\mathrm{Rk}_2$.

**Theorem 4.5** *(i) The 5-dimensional irreducible representations $T_+$, $T_-$ of $\mathcal{N}$ just obtained are inequivalent.*

*(ii) For each subgroup $\mathcal{N} \cong Z_{31} \rtimes Z_5$ of $\mathrm{GL}(5,2)$ there exist precisely two external 4-flats $E_+$, $E_-$ which have $\mathcal{N}$ as stabilizer, these lying on different $\mathrm{GL}(5,2)$-orbits, say $\mathrm{orb}(4+)$, $\mathrm{orb}(4-)$.*

**Proof.** (i) Suppose there exists $C \in \mathrm{GL}(5,2)$ such that $T_C$ maps $E_+$ onto $E_-$. It follows that $C\mathcal{G}_{E_+}C^{-1} = \mathcal{G}_{E_-}$, that is $C\mathcal{N}C^{-1} = \mathcal{N}$. But $\mathcal{N}$, being maximal, is its own normalizer, whence $C \in \mathcal{N}$; so $T(C)$ stabilizes $E_+$, contradicting the supposition that $T(C)$ maps $E_+$ onto $E_-$.

(ii) This follows from (i), since the inequivalence of $T_+$ and $T_-$ implies that the direct sum decomposition $\overline{E_+} \oplus \overline{E_-}$ of $V_{10}$ is unique. ■

It is tempting to say that external 4-flat occur as *twins* $\{E_+, E_-\}$; but, because of part (i) of the theorem, it is more accurate to say that external 4-flats which share the same stabilizer occur as *non-identical twins* $\{E_+, E_-\}$. Certainly $E_+, E_-$ *can* be distinguished (unlike twin non-reguli), one way being as follows. In $\mathcal{N}$, the 30 elements $A$ of order 31 fall into 6 conjugacy classes $K_r$ of length 5. Letting $r$ take the values $1, 2, \ldots, 6 \pmod 6$ we may label the classes in such a way that $K_{r+1} = (K_r)^3$. Let $f_r$ be the characteristic polynomial of elements $A \in K_r$. By using the fact that $A$

is diagonalizable over the extension field GF(32) one easily shows that if $\chi_A = f_r$ then $\chi_{T(A)} = f_{r+1}.f_{r+2}$. So, for $A \in K_r$, we may distinguish between $E_+$ and $E_-$ by defining $\overline{E_+} = \ker f_{r+1}(T_A)$ and $\overline{E_-} = \ker f_{r+2}(T_A)$.

We now prove the main result of the present paper.

**Theorem 4.6** *The external 4-flats form just two* GL(5, 2)*-orbits,* orb(4+) *and* orb(4−)*. The stabilizer $\mathcal{G}_E$ of any external 4-flat $E$ has order* 155 *and structure* $\cong Z_{31} \rtimes Z_5$.

**Proof.** We know from theorem **??**(ii) that there exist at least two orbits orb(4+) and orb(4−) of external 4-flats. But, theorem **??**, those solids which extend to an external 4-flat form a single orbit, and so it follows from theorem **??**(i) that there exist at most two orbits. Hence there are precisely two orbits, orb(4+) and orb(4−). The stabilizer $\mathcal{G}_E$ of any external 4-flat $E$ is as described in **??**. ■

# 5 External flats: the remaining orbits

## 5.1 Constructions of special $k$-flats, $k = 1, 2, 3$

In [?] a partial spread in PG(4, 2) was said to be of type I if it contained precisely one regulus, and to be of type O if it was regulus-free. Using partial spreads of type I we now construct special $k$-flats, $k = 1, 2, 3$, in a manner completely analogous to the construction, in theorem **??**, of general $k$-flats out of partial spreads of type O.

**Theorem 5.1** *Given a partial spread $\mathcal{S}_r, r \geq 3$, in* PG(4, 2) *of type I, let $\mathcal{E}(\mathcal{C}_r) = \langle \mathcal{C}_r \rangle_{even}$ be defined as in section* **??**. *Then $\mathcal{E}(\mathcal{C}_r)$ is an external flat if and only if one of the following holds:*

*(i) $r = 3$ and $\mathcal{S}_3 \in$ class IIIb.1; $\mathcal{E}(\mathcal{C}_3)$ is a special line;*

*(ii) $r = 4$ and $\mathcal{S}_4 \in$ class IVc.1; $\mathcal{E}(\mathcal{C}_4)$ is a special plane;*

*(iii) $r = 5$ and $\mathcal{S}_5 \in$ class Vg.1; $\mathcal{E}(\mathcal{C}_5)$ is a special solid.*

**Proof.** From [?] there are 14 classes of partial spreads of type I, namely classes IIIb.1; IVc.1; Vf.1, Vg.1; VIc.1,2, VId.1,2, VIe.1,2, VIf.1; VIIb.1,2,3. A straightforward check of these yields the theorem (case (i) being that of the reguli of lemma **??**). (Actually, on account of theorem **??**, we need to carry out the check only for the four classes IIIb.1; IVc.1; Vf.1, Vg.1.) ■

Since each of the classes IIIb.1, IVc.1, Vg.1 of partial spreads is a GL(5, 2)-orbit, the special flats in parts (i), (ii) and (iii) of the theorem form single GL(5, 2)-orbits. *In each case let us agree to label the orbit as the beta orbit:* (i) $\mathcal{E}(\mathcal{C}_3) \in$ orb($1\beta$); (ii) $\mathcal{E}(\mathcal{C}_4) \in$ orb($2\beta$); (iii) $\mathcal{E}(\mathcal{C}_5) \in$ orb($3\beta$). Recall that the general $k$-flats arising from theorem **??** were assigned to the alpha orbits orb($k\alpha$), $k = 1, 2, 3$. In the next section we show that there exists at least one further orbit, orb($2\gamma$), of general planes.

### 5.2 Construction of another kind of general plane

Let $\mathrm{PG}(4,2) = \mathbb{P}V_5$ be generated by a line $\mu = \{u, v, w\} = \mathbb{P}V_2$ and a plane $\alpha = \mathbb{P}V_3$; so $V_5 = V_2 \oplus V_3$. Choose $A_3 \in \mathrm{GL}(V_3)$ to have order 7 and minimal polynomial $t^3 + t + 1$; then, for any choice of $a_0 \in \alpha$, we have $\alpha = \{a_0, a_1, ..., a_6\}$ where $a_i = (A_3)^i a_0$, the seven lines of $\alpha$ being $\lambda_i = \{a_i, a_{i+1}, a_{i+3}\}$, $i = 0, 1, .., 6 \pmod 7$. Choose $C_3 \in \mathrm{GL}(V_3)$ in the normalizer of $A_3$ to satisfy $C_3 A_3 (C_3)^{-1} = (A_3)^2$ and $C_3 a_0 = a_0$, and note therefore that $C_3 a_i = a_{2i}$. Define also $C_2 \in \mathrm{GL}(V_2)$ to effect the 3-cycle $u \mapsto w \mapsto v \mapsto u$. Setting $A = I_2 \oplus A_3$ and $C = C_2 \oplus C_3$, observe that $A$ and $C$ are elements of $\mathrm{GL}(5,2)$, of orders 7 and 3, which stabilize both $\mu$ and $\alpha$ and which generate a subgroup $F_{21} \cong 7:3$ of $\mathrm{GL}(5,2)$ .

**Theorem 5.2** *Let $P = \{b_0, b_1, ..., b_6\}$ where, in the foregoing notation, the seven points $b_i$ of $\mathrm{PG}(9,2)$ are defined by*

$$b_i = u \wedge a_i + v \wedge a_{i+1}, \quad (i = 0, 1, .., 6 \pmod 7)). \tag{5.1}$$

*Then $P$ is a general plane with stabilizer $\mathcal{G}_P \cong 2^3 : (7 : 3)$.*

**Proof.** *(Outline)* Noting that $b_i + b_{i+1} = b_{i+3}$, we see that $P$ is a plane. Since $\mathrm{im}(b_i) = \langle \mu, \lambda_i \rangle \neq \mathrm{im}(b_j)$ for $i \neq j$, it follows that $P$ is a general plane. Next note that $T_A b_i = b_{i+1}$ and $T_C b_i = b_{2i-1}$, whence $\mathcal{G}_P$ contains $F_{21}$. But for $f \in V_3^*$ each $b_i$ is seen to be fixed by $T_{J(f)}$, where $J(f) \in \mathrm{GL}(5,2)$ is the involution defined by

$$\begin{aligned} &J(f)u = u, \quad J(f)v = v, \\ &J(f)x = x + f(x)u + f(A_3 x)v, \ x \in V_3. \end{aligned} \tag{5.2}$$

Since $AJ(f)A^{-1} = J(\hat{A}f)$ and $CJ(f)C^{-1} = J(\hat{A}\hat{C}f)$, it follows that $\mathcal{G}_P$ contains a subgroup $\mathcal{J} \rtimes F_{21} \cong 2^3 : (7:3)$, where $\mathcal{J} = \{J(f) | \, f \in V_3^*\} \cong (Z_2)^3$. With a little more effort one sees that $\mathcal{G}_P$ is no larger than $\mathcal{J} \rtimes F_{21}$. ∎

Since the stabilizer of a plane $\in \mathrm{orb}(2\alpha)$ is isomorphic to Sym(3), the general plane $P$ in the theorem belongs to a distinct orbit, which we will label $\mathrm{orb}(2\gamma)$. Such a plane $P$ in fact lies inside each solid $D \in \mathrm{orb}(3\beta)$, but if $P$ is arrived at in this way its high symmetry is far from obvious.

### 5.3 Summary of results

We have obtained the following ten orbits of $k$-flats in $\mathrm{PG}(9,2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$:
$k = 0$: $\mathrm{Rk}_4(4,2)$;
$k = 1$: $\mathrm{orb}(1\alpha)$, $\mathrm{orb}(1\beta)$;
$k = 2$: $\mathrm{orb}(2\alpha)$, $\mathrm{orb}(2\beta)$, $\mathrm{orb}(2\gamma)$;
$k = 3$: $\mathrm{orb}(3\alpha)$, $\mathrm{orb}(3\beta)$;
$k = 4$: $\mathrm{orb}(4+)$, $\mathrm{orb}(4-)$.

Moreover, in the cases $k = 0, 1, 4$, but not in the cases $k = 2, 3$, we have proved that this list is complete.

**Claim 5.3** *These ten orbits constitute a complete classification of the $k$-flats in* $\mathrm{PG}(9, 2)$ *which are external to the Grassmannian* $\mathcal{G}_{1,4,2}$.

The justification for this claim will be given in a second paper, where, in particular, we will:
(i) prove the non-existence of further external planes and solids;
(ii) give details of the stabilizer groups of the beta planes and solids,
(iii) provide further constructions of gamma planes.

# 6 Further aspects and comments

## 6.1 The associate $E^{\$}$ of an external 4-flat $E$

**Definition 6.1** *The* associate $X^{\$}$ *of an $r$-flat $X$ of* $\mathrm{PG}(9, 2)$ *consists of those points $y$ of* $\mathrm{PG}(9, 2)$ *for which the following holds: for each hyperplane $H$ of $X$ which does not contain $y$, the join $\langle y, H\rangle$ meets* $\mathrm{Rk}_2$ *in an odd number of points.*

**Remark 6.2** *Let $Q(x) = 0$ be the equation of* $\mathrm{Rk}_2$, *where $Q$ is the quintic polynomial function given in [**?**]. In [**?**] a definition of the* associate $X^{\$}$ *of an $r$-flat $X$ of* $\mathrm{PG}(9, 2)$ *was given in terms of the $r$-th polarization $Q^{(r)}$ of $Q$. However, in the same paper, the definition was shown to be equivalent to the preceding one, which made no appeal to $Q$.*

**Theorem 6.3** *(i) The associate $E^{\$}$ of an external 4-flat $E$ is another external 4-flat; moreover $E^{\$}$ is disjoint from $E$.*

*(ii) If $E$ is an external 4-flat and $F$ is a 5-flat $\supset E$, let $\mathcal{C}_F$ denote the 5-cap $F \cap \mathrm{Rk}_2$ of theorem **??**. Putting $y_F = \sum_{m\in\mathcal{C}_F} m$, the associate of $E$ is*

$$E^{\$} = \{y_F \mid F \text{ a 5-flat containing } E\}. \tag{6.1}$$

*(iii) If $E_\pm$ are as in theorem **??**, then $E_+ = (E_-)^{\$}$ and $E_- = (E_+)^{\$}$.*

*(iv) If $E$ is any external 4-flat then $(E^{\$})^{\$} = E$.*

For the proof of these results see [**?**]. It should be recorded that the proof that $E^{\$}$ is a flat in part (i) makes appeal to the fact that $Q$ is a quintic, and to the consequent fact that $Q^{(4)}$ is quinquelinear in its five arguments.

## 6.2 Further comments

### 6.2.1 Generalization to $\mathrm{PG}(n, q)$?

In this paper we have confined our attention to the field GF(2), and to the projective space $\mathrm{PG}(4, 2)$. Since we have repeatedly taken advantage of many aspects which are peculiar to the space PG$(4,2)$, we do not claim that our methods will generalize easily, or at all, to PG$(n,q)$ for $n>4$ or $q>2$. In particular, in much of our work on the orbit structure of external flats, we have made repeated use of the complete classification of all the partial spreads in PG$(4,2)$ given in [**?**], while the corresponding classification in PG$(4,q)$ is not known for any other value of $q$. As another example (cf. our proof of theorem **??**), we have the property that the normalizer of a Singer cyclic group $Z_{31}$ in PG$(4,2)$ acts regularly on the lines (and also on the planes) of PG$(4,2)$, which is quite exceptional.

Nevertheless some of our results, viz. theorem **??** and theorem **??** parts (i)-(iv), *do* generalize to PG$(4,q)$ for general $q$, as can be seen in [**?**]. The central objective of Cooperstein [**?**] is the construction of certain maximal flats in $\mathbb{P}(\Lambda^2 V(n,q))$ which are external to the algebraic variety $Z_{2r} := Rk_2 \cup Rk_4 \cup ... \cup Rk_{2r}$. In some cases Cooperstein has obtained sharp upper bounds for the dimension of $Z_{2r}$-external flats. However the problem of *classifying the orbits* of these flats was not embarked upon in [**?**].

### 6.2.2 Coding theory

The fact, see theorem **??**, that there are precisely two GL$(5,2)$-orbits of $\mathcal{G}_{1,4,2}$-external 4-flats in PG$(9,2)$ has the following consequence in coding theory: there are exactly two non-equivalent $[32,11,12]$-codes $\mathcal{C}$, $\mathcal{C}'$ which lie between the binary Reed-Muller codes $RM(1,5)$ and $RM(2,5)$ which have the weight set $\{0,12,16,20,32\}$ — the codes $\mathcal{C}$, $\mathcal{C}'$ being *optimal* in the sense that no $[32,k,12]$-code with this weight set exists for $k>11$. For a detailed explanation of the connection with coding theory, see [**?**], where the *weight* of a quadratic form $\phi \in RM(2,5)$, that is the size of the support of $\phi$, is related to the *rank* of the polarization of $\phi$. In the binary case $q=2$ (and indeed for the $q$-ary cases with $q=2^k$) these polarizations are alternating bilinear forms, hence the link with the bivector space $\Lambda^2 V$ of section **??**. More generally, replacing $V(5,2)$ by $V(n,2)$ for any odd $n$, we have a correspondence between certain optimal $[2^n, 2n+1, 2^{n-1}-2^{(n-1)/2}]$-codes $\mathcal{C}$ with $RM(1,n) \subseteq \mathcal{C} \subseteq RM(2,n)$ and $(n-1)$-flats in $\mathbb{P}(\Lambda^2 V(n,2))$ which are external to the variety $Z_{n-3}$. From the coding theory side we know that the number of GL$(n,2)$-orbits of such maximal $Z_{n-3}$-external flats is *at least* $(n-1)/2$. Finally, it should be emphasized that linear codes $\mathcal{C} \subseteq RM(2,n)$ over a field $\mathbb{F}_q$ with char $\mathbb{F}_q \neq 2$ do *not* relate to linear spaces in $\Lambda^2 V(n,q)$, since the polarization of a quadratic form is not an alternating bilinear form in these cases.

# A Appendix

## A.1 The Grassmannian $\mathcal{G}_{1,3,2}$

In this case we write the decomposition $\mathrm{PG}(5,2) = \mathrm{Rk}_2(3,2) \cup \mathrm{Rk}_4(3,2)$ as $\mathrm{PG}(5,2) = \mathcal{H} \cup \mathcal{W}$, where $\mathcal{H} = \mathcal{G}_{1,3,2}$, the Klein quadric, is a hyperbolic quadric in $\mathrm{PG}(5,2)$, and $\mathcal{W}$ consists of the 28 points external to $\mathcal{H}$. It should be noted that *each* $b \in \mathcal{W}$ *can be expressed in the form* $b = m + n$ *for precisely ten pairs* $\{\mu, \nu\}$ *of skew lines* $\mu, \nu$ *of* $\mathrm{PG}(3,2)$. Explicitly, given one expression $b = m + n$, let $\tau_i$, $i = 1, 2, \ldots, 9$, be the nine transversals of the skew pair $\mu, \nu$; then the other nine expressions for $b$ are

$$b = m_i + n_i,\ (i = 1, 2, \ldots, 9) \quad \text{where } m_i = m + t_i, \text{ and } n_i = n + t_i. \tag{A.1}$$

(Here, because $\tau_i$ meets $\mu$, it follows that $m + t_i$ is indeed the image of a line $\mu_i$, namely the third line of the pencil determined by $\mu$ and $\tau_i$; similarly $n + t_i$ is the image of a line $\nu_i$.) Observe therefore that through each point $b \in \mathcal{W}$ there pass ten bisecants of the Klein quadric $\mathcal{H}$.

In the same area, observe that each $b \in \mathcal{W}$ defines a non-degenerate symplectic form $(u, v)_b$ on $V_4$, given by $b \wedge u \wedge v = (u, v)_b \omega$, where $\omega$ is the (unique!) basis vector for the 1-dimensional space $\wedge^4 V_4$. In projective terms, the 28 different symplectic polarities on $\sigma = \mathrm{PG}(3,2)$ thus arise from the 28 choices of $b \in \mathcal{W}$. For a subspace $\alpha \subset \sigma$, we will let $\alpha^b$ denote its polar using the $b$ polarity:

$$\alpha^b = \{u \in \sigma \mid b \wedge u \wedge v = 0 \text{ for all } v \in \alpha\}. \tag{A.2}$$

For a given polarity the ten expressions $b = m + n$ described above correspond to the ten polar pairs $\{\mu, \nu\}$ of lines, with $\nu$ being the polar $\mu^b$ of $\mu$. The remaining $15 (= 35 - 20)$ lines $\lambda$ of $\mathrm{PG}(3,2)$ are self-polar, the join $\langle l, b \rangle$ being tangent to $\mathcal{H}$ at the point $l$. Note that these 15 tangents through $b$ consume 15 further points of $\mathcal{W}$. It follows that, in order to account for the remaining $12 (= 28 - 1 - 15)$ points of $\mathcal{W}$, through $b$ must pass precisely 6 lines external to $\mathcal{H}$. (See also lemma **??**(i) below.)

Of course the subgroup $\mathcal{G}_b$ of $\mathrm{GL}(4,2)$ which stabilizes $b \in \mathcal{W}$ is one of the 28 subgroups of $\mathrm{GL}(4,2)$ which are isomorphic to $\mathrm{Sp}(4,2) \cong \mathrm{Sym}(6)$; see also remark **??** below.

### A.1.1 Use of the isomorphism of $\mathrm{GL}(4,2)$ with $\mathrm{Alt}(8)$

Let *ijklmnrs* be an arbitrary permutation of 12345678. Then we can arrange for the 28 points of $\mathcal{W}$ to be labelled $p_{ij} = p_{ji}$, where the only linear relations amongst the $p_{ij}$ are those of the forms

$$\text{(i) } p_{ij} + p_{ik} + p_{jk} = 0; \qquad \text{(ii) } p_{ij} + p_{kl} + p_{mn} + p_{rs} = 0. \tag{A.3}$$

The 35 points of $\mathcal{H}$ then have double labellings of the kind $q_{ijkl} = q_{mnrs}$, and are given by $q_{ijkl} = p_{ij} + p_{kl}$. (This labelling can be achieved internally, with $p_{ij}$ being defined by $H_i \cap H_j = \{p_{ij}\}$, where $H_i$, $i = 1, 2, \ldots, 8$, are the Conwell heptads, *cf.* [**?**, Lemma 17.5.4]). Any permutation $\pi \in \mathrm{Sym}(8)$ induces, via $p_{ij} \mapsto p_{\pi i \, \pi j}$, a linear transformation of $V_6 = \wedge^2 V_4$ which preserves the Klein quadric; in this way we arrive at the isomorphism of Sym(8) with the full orthogonal group $\mathrm{O}^+(6, 2)$. However elements $A \in \mathrm{GL}(4, 2)$ preserve separately the two systems, Latin and Greek, of internal planes of $\mathcal{H}$, and the elements $T_A$ lie in a subgroup of $\mathrm{O}^+(6, 2)$ of index 2, with $A \mapsto T_A$ giving rise to the well-known isomorphism $\mathrm{GL}(4, 2) \cong \mathrm{Alt}(8)$. The symplectic scalar product $b.c$ on $V_6$ determined by the quadric $\mathcal{H}$ satisfies

$$p_{ij}.p_{ik} = 1, \qquad \text{and} \qquad p_{ij}.p_{kl} = 0 \tag{A.4}$$

whenever $i, j, k, l$ are distinct. (In particular the seven elements $p_{ij}, j \neq i$, of the heptad $H_i$ are pairwise non-conjugate — the defining property of a Conwell heptad.) It should be noted that two lines $\mu$, $\nu$ of $\mathrm{PG}(3, 2)$ meet if and only if the corresponding two points $m, n$ of $\mathcal{H}$ satisfy $m.n = 0$.

The next lemma serves as an indication of the usefulness of the isomorphism $\mathrm{GL}(4, 2) \cong \mathrm{Alt}(8)$.

**Lemma A.1** *(i) There exist in* $\mathrm{PG}(5, 2)$ *precisely fifty-six lines $L$ external to $\mathcal{H}$, these forming a single* $\mathrm{GL}(5, 2)$*-orbit, with six lines passing through each point $b \in \mathcal{W}$.*

*(ii) The 3-flat $L^\perp$ polar to an external line $L$ meets $\mathcal{H}$ in a 5-set $\mathcal{C} = \{m_i\}_{i=1,2,3,4,5}$ which is the image of a spread $\mathcal{S} = \{\mu_i\}_{i=1,2,3,4,5}$ in* $\mathrm{PG}(3, 2)$.

*(iii)* $\mathcal{G}_\mathcal{S} = \mathcal{G}_L \cong (\mathrm{Alt}(5) \times 3).2$, *of order 360.*

*(iv) If $\mathcal{S} = \{\mu_i\}_{i=1,2,3,4,5}$ is a spread in* $\mathrm{PG}(3, 2)$ *then* $\Sigma_{i=1}^5 m_i = 0$.

**Proof.** (i) The 56 lines are the $\binom{8}{3}$ lines $L_{ijk} = \{p_{ij}, p_{ik}, p_{jk}\}$, see (**??**). They form a single $\mathrm{GL}(4, 2)$-orbit, since the triads $ijk$ form a single Alt(8)-orbit. Through the point $p_{78} \in \mathcal{W}$ pass the 6 external lines $L_{i78}$, $i = 1, 2, \ldots, 6$.

(ii) If $m \in \mathcal{H}$ satisfies $m.p_{67} = m.p_{68} = m.p_{78} = 0$ then from (**??**) we see that $m \in \mathcal{C}$ where $\mathcal{C} = \{m_i\}_{i=1,2,3,4,5}$ with $m_i = q_{i678}$. Since, by (**??**), $m_i.m_j = (p_{i6} + p_{78}).(p_{j6} + p_{78}) = 1$, for $i \neq j$, the five lines $\mu_i$ are pairwise skew and so comprise a spread. ($\mathcal{C}$ is in fact an elliptic quadric $\subset L^\perp$.)

(iii) Since there are 56 spreads $\mathcal{S}$ in $\mathrm{PG}(3, 2)$, the construction in part (ii) establishes a bijective correspondence $\mathcal{S} \leftrightarrow L$ with the 56 external lines $L$, and so the stabilizers $\mathcal{G}_\mathcal{S}$, $\mathcal{G}_L$ are equal. The structure of these stabilizers is as indicated, since the stabilizer in Alt(8) of the triad 678 is $\mathcal{G}_0 \cup (12)(78)\mathcal{G}_0$, where $\mathcal{G}_0 = \mathrm{Alt}\{1, 2, 3, 4, 5\} \times \langle (678) \rangle \cong \mathrm{Alt}(5) \times Z_3$.

(iv) From (**??**) we immediately see that $\Sigma_{i=1}^5 q_{i678} = 0$. ■

**Remark A.2** *The isomorphism* $\mathcal{G}_b \cong \mathrm{Sym}(6)$ *at the end of the preceding section can be arrived at in a similar manner to that in part (iii) of the lemma. For if* $b = p_{78}$ *then* $\mathcal{G}_b$ *is isomorphic to the stabilizer in* $\mathrm{Alt}(8)$ *of the duad* 78, *namely to* $\mathcal{A} \cup (12)(78)\mathcal{A}$, *where* $\mathcal{A} = \mathrm{Alt}\{1,2,3,4,5,6\} \cong \mathrm{Alt}(6)$.

## A.2 Some regulus-free partial spreads

We provide information here about three classes (i) IIIa.1, (ii) IVb.1 and (iii) Ve.1, of partial spreads in PG$(4,2)$ which are needed in section **??**. (For the complete classification of partial spreads in PG$(4,2)$, consult [**?**].) At times it will prove convenient to adopt the following shorthand notation. A basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, e_5,\}$ for $V_5$ will be written $\mathcal{B} = \{1,2,3,4,5\}$, and abbreviations such as $235 = e_2+e_3+e_5$, $u = 12345 = e_1+...+e_5$, $5u = 1234$, $2 \wedge 145 = e_2 \wedge (e_1+e_4+e_5)$, ... will be employed. The $3r$-set of points of PG$(4,2)$ which support the $r$ lines of a partial spread $\mathcal{S}_r$ will be denoted $\psi(\mathcal{S}_r)$. A line $\tau$ which meets each of three lines of $\mathcal{S}_r$ will be referred to not only as a transversal of the three lines, *but also as a transversal of* $\mathcal{S}_r$.

**(i) Class IIIa.1.** In PG$(4,2)$ there are only two kinds of partial spreads $\mathcal{S}_3$, reguli (class IIIb.1) and non-reguli (class IIIa.1). With respect to a suitable basis $\mathcal{B}$ any partial spread $\mathcal{S}_3$ which is not a regulus can be taken to be $\mathcal{S}_3 = \{\mu_1, \mu_2, \mu_3\}$ where, in shorthand notation,

$$\mu_1 = \{1,2,12\},\ \mu_2 = \{3,4,34\},\ \mu_3 = \{5,u,5u\}. \tag{A.5}$$

The non-regulus $\mathcal{S}_3$ has a unique transversal $\tau = \{12, 34, 5u\}$, and it also determines the *hyperbasis* $\mathcal{B}_+ = \mathcal{B} \cup \{u\} = \psi(\mathcal{S}_3) \setminus \tau = \{1,2,3,4,5,u\}$. Each permutation $\pi \in \mathrm{Sym}(\mathcal{B}_+)$ gives rise to an element $A_\pi$ of GL$(5,2)$ satisfying $A_\pi : b \mapsto \pi(b),\ b \in \mathcal{B}_+$. Observe that the stabilizer $\mathcal{G}_{\mathcal{S}_3}$ of $\mathcal{S}_3$ must consist of those $A_\pi$, $\pi \in \mathrm{Sym}(\mathcal{B}_+)$, such that $\pi$ preserves the syntheme $12\,34\,5u$. It soon follows that

$$\mathcal{G}_{\mathcal{S}_3} = \mathcal{G}^0_{\mathcal{S}_3} \times \langle J \rangle \cong \mathrm{Sym}(4) \times Z_2, \quad \text{where } J = A_{(12)(34)(5u)}. \tag{A.6}$$

See [**?**], [**?**] for more details of the group $\mathcal{G}^0_{\mathcal{S}_3} \cong \mathrm{Sym}(4)$.

Of considerable importance to our present concerns is the fact that each non-regulus $\mathcal{S}_3$ comes along with a *twin non-regulus* $\mathcal{S}^*_3 = \{\mu^*_1, \mu^*_2, \mu^*_3\}$, where $\mu^*_j$ denotes the third line of the pencil determined by $\mu_j$ and $\tau$ (and so $\mathcal{S}_3$ and $\mathcal{S}^*_3$ share the same transversal $\tau$). For $\mathcal{S}_3$ as in (**??**), $\mathcal{S}^*_3$ is

$$\mu^*_1 = \{1^*, 2^*, 1^*2^*\},\ \mu^*_2 = \{3^*, 4^*, 3^*4^*\},\ \mu^*_3 = \{5^*, u^*, 5^*u^*\}, \tag{A.7}$$

where $1^* = 134$, $2^* = 234$, $3^* = 124$, $4^* = 123$, $5^* = 125$, $u^* = 1^*2^*3^*4^*5^* = 345$. Note that $\tau^* = \{1^*2^*, 3^*4^*, 5^*u^*\}$ indeed coincides with $\tau = \{12, 34, 5u\}$.

**Lemma A.3** *If* $\mathcal{S}_3$ *and* $\mathcal{S}^*_3$ *are twin non-reguli in* PG$(4,2)$ *then*
*(i) The stabilizer* $\mathcal{G}_{\mathcal{S}^*_3}$ *of* $\mathcal{S}^*_3$ *coincides with the stabilizer* $\mathcal{G}_{\mathcal{S}_3}$ *of* $\mathcal{S}_3$.
*(ii) The stabilizer* $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}^*_3\}}$ *of* $\{\mathcal{S}_3, \mathcal{S}^*_3\}$ *is a group* $\mathcal{G}_{\mathcal{S}_3}.2$, *of order 96.*

**Proof.** (i) An element $A \in \mathcal{G}(\mathcal{S}_3)$ permutes the lines $\mu_i$ but fixes $\tau(=\tau^*)$, and hence permutes the lines $\mu_i^*$.

(ii) The element $K \in \mathrm{GL}(5,2)$ which maps $i \in \mathcal{B}$ onto $i^* \in \mathcal{B}^* := \{1^*, 2^*, 3^*, 4^*, 5^*\}$ (and so $u$ onto $u^*$) is an involution ($i \rightleftarrows i^*$) which effects the three interchanges $\mu_j \rightleftarrows \mu_j^*, j = 1, 2, 3$, whence $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}_3^*\}} = \mathcal{G}(\mathcal{S}_3).\langle K\rangle$. ■

(In fact $\mathcal{G}_{\{\mathcal{S}_3, \mathcal{S}_3^*\}}$ is the normalizer in GL(5, 2) of $\mathcal{G}_{\mathcal{S}_3}$.)

**(ii) Class IVb.1.** This class is represented by $\mathcal{S}_4 = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ where

$$\mu_1 = \{1, 2, 12\},\ \mu_2 = \{3, 4, 34\},\ \mu_3 = \{5, u, 5u\},\ \mu_4 = \{13, 15, 35\}. \quad \text{(A.8)}$$

So $\mathcal{S}_4 = \mathcal{S}_3 \cup \{\mu_4\}$, where $\mathcal{S}_3 = \{\mu_1, \mu_2, \mu_3\}$ is as in (**??**). Observe that the line $\mu_4$ is distinguished: each point of $\mu_4$ lies on precisely one transversal of $\mathcal{S}_4$, but in contrast the points $1, 3, 5$ lie on precisely two transversals, and the points $2, 4, u$ lie on no transversals. It follows that $\mathcal{G}_{\mathcal{S}_4}$ is a subgroup of $\mathcal{G}_{\mathcal{S}_3}$, and that any $T \in \mathcal{G}_{\mathcal{S}_4}$ must permute amongst themselves the three points $1, 3, 5$, and also three points $2, 4, u$. It quickly follows that

$$\text{class IVb.1:} \qquad \mathcal{G}(\mathcal{S}_4) = \langle A, C\rangle \cong \mathrm{Sym}(3), \qquad \text{(A.9)}$$

where $A$ and $C$ in GL(5, 2) effect the permutations $(135)(24u)$ and $(13)(24)$ of the hyperbasis $\mathcal{B}_+$. Incidentally, observe that $\mathcal{G}(\mathcal{S}_4)$ fixes the point $135 = 24u$ (and no other).

**(iii) Class Ve.1.** This class is represented by $\mathcal{S}_5 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ where the lines $\mu_1, \ldots, \mu_5$ are, in shorthand notation,

$$\{1u, 2, 345\},\ \{2u, 3, 451\},\ \{3u, 4, 512\},\ \{4u, 5, 123\},\ \{5u, 1, 234\}. \quad \text{(A.10)}$$

Note that $\mathcal{S}_5$ is cyclic, since if $B \in \mathrm{GL}(5,2)$ is defined by $B : 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 5 \mapsto 1$, then $B$ effects the permutation $(\mu_1\mu_2\mu_3\mu_4\mu_5)$. In fact, since any $T \in \mathcal{G}(\mathcal{S}_5)$ must permute amongst themselves those points which lie on only one transversal, namely the points $1, 2, 3, 4, 5$, it then quickly follows from (**??**) that $\mathcal{G}(\mathcal{S}_5)$ is precisely $\langle B\rangle$:

$$\text{class Ve.1:} \qquad \mathcal{G}(\mathcal{S}_5) = \langle B\rangle \cong Z_5. \qquad \text{(A.11)}$$

## A.3 Elements of order 5

Let $B \in \mathrm{GL}(n+1, 2)$ be of order 5, generating a subgroup $\langle B\rangle \cong Z_5$. Note that we must have $n + 1 > 3$, since $5 \nmid |\,\mathrm{GL}(3,2)|$. Moreover, since we work in characteristic 2, and $2 \nmid |Z_5|$, Maschke's theorem applies: the vector space $V_{n+1} = V(n+1, 2)$ decomposes into a direct sum of subspaces upon which $B$ acts irreducibly. If $3 \leq n \leq 6$ it follows that we have the direct sum decomposition

$$B = B_4 \oplus I_{n-3}, \qquad \text{acting upon} \qquad V_{n+1} = V_4 \oplus V_{n-3}, \qquad \text{(A.12)}$$

with $B_4$ an element of $\mathrm{GL}(V_4)$ of order 5 and $I_{n-3}$ the identity in $\mathrm{GL}(V_{n-3})$. Note that the minimal polynomial of $B$ is $\mu_B = t^5 + 1 = (t+1)f$ where $f$ is the irreducible polynomial $f := t^4 + t^3 + t^2 + t + 1$, and so we have $V_4 = \ker(B^4 + B^3 + B^2 + B + I)$ and $V_{n-3} = \ker(B + I)$. The next lemma lists some consequent properties of elements of $\mathrm{GL}(n+1, 2)$ of order 5.

**Lemma A.4** *Let* $B \in \mathrm{GL}(n+1, 2)$ *be of order 5 and act on* $\mathrm{PG}(n, 2)$, $3 \leq n \leq 6$. *Then:*
*(i)* $n = 3$*:* $\mathrm{PG}(3, 2)$ *is a union of three 5-cycles of B, each of which forms a hyperbasis (i.e. base + unit point) for* $V_4$.
*(ii)* $n = 4$*:* $\mathrm{PG}(4, 2)$ *is a union of six 5-cycles of B, together with one fixed point; three of the 5-cycles are bases for* $V_5$, *and the other three 5-cycles lie in the common even hyperplane of the first three 5-cycles.*
*(iii)* $n = 5$*:* $\mathrm{PG}(5, 2)$ *is a union of twelve 5-cycles, together with a line of fixed points;*
*(iv)* $n = 6$*:* $\mathrm{PG}(6, 2)$ *is a union of twenty-four 5-cycles, together with a plane of fixed points.*
*In each of the cases* $n = 3, 4, 5, 6$ *there is a single class of elements of order 5 in* $\mathrm{GL}(n+1, 2)$.

**Proof.** Concerning (i), applying $B_4$ to any nonzero $v \in V_4$ generates a 5-cycle whose five members $(B_4)^i v$, $i = 0, 1, 2, 3, 4$, sum to zero, since $f(B_4) = 0$, and so form a hyperbasis (no other relations holding between the five because $B_4$ acts irreducibly). It follows that $\mathrm{GL}(4, 2)$ contains just one class of elements of order 5. The rest now follows upon using the direct sum decompositions (**??**). ∎

Consider next $T_{B_4} = \wedge^2 B_4$, acting upon $V_6 = \wedge^2 V_4$. Then from (**??**) $V_6$ has a $T_{B_4}$-invariant direct sum decomposition $V_6 = V_4 \oplus V_2$, upon which $T_{B_4}$ acts as $B_4 \oplus I_2$.. Consider finally $T_B = \wedge^2 B$, where $B \in \mathrm{GL}(5, 2)$ has order 5, acting upon $V_{10} = \wedge^2 V_5$. Now, from (**??**), $B = B_4 \oplus I_1$, whence, see [**?**, eq. (4.12)], $T_B = T_{B_4} \oplus B_4$. But, as just noted, $T_{B_4} \cong B_4 \oplus I_2$. So $V_{10}$ has $T_B$-invariant decompositions of the kind $V_{10} = V_4 \oplus V_4 \oplus V_2$. (Here $V_2$ is unique and fixed pointwise, but there are many choices for the two $V_4$.)

In the case of an element $B \in \mathrm{GL}(5, 2)$ of order 5 we set down in the next lemma aspects of the foregoing in projective terms.

**Lemma A.5** *Each element* $B \in \mathrm{GL}(5, 2)$ *of order 5 defines a unique antiflag* $(u, \sigma)$ *in* $\mathrm{PG}(4, 2)$ *which is fixed by* $B : Bu = u$ *and* $B(\sigma) = \sigma$. *In its induced action* $T_B$ *upon* $\mathrm{PG}(9, 2) = \mathbb{P}(\wedge^2 V_5)$, *there is a unique* $T_B$*-invariant Latin solid, namely* $\mathrm{St}(u)$. *Moreover there are precisely three fixed points, these forming a special line L which lies in* $\mathcal{W}(\sigma)$.

**Proof.** Recall that $B = I_1 \oplus B_4$ acting upon $V_5 = V_1 \oplus V_4$. This last $B$-invariant decomposition $V_5 = V_1 \oplus V_4$ translates into the $B$-invariant antiflag

$(u,\sigma)$, where $\prec u \succ = V_1$ and $\sigma = \mathbb{P}(V_4)$. In the decomposition $\wedge^2(\prec u \succ \oplus V_4) = (\prec u \succ \otimes V_4) \oplus \wedge^2 V_4$ , the $T_B$-invariant subspace $\prec u \succ \otimes V_4$ is projectively the $T_B$-invariant Latin solid $\mathrm{St}(u) = \{u \wedge x \,|\, x \in \sigma\}$. In the further decomposition $\wedge^2 V_4 = V_4 \oplus V_2$, the $T_B$-invariant $V_2$ is projectively a line $L$ of fixed points which lies in $\mathbb{P}(\wedge^2 V_4) = \Pi(\sigma)$. Now, by lemma **??**(i), no line of $\sigma = \mathbb{P}(V_4)$ is fixed by $B_4$, so all three points of $L$ are external: $L \subset \mathcal{W}(\sigma)$. ■

**Remark A.6** *If* $B \in \mathrm{GL}(5,2)$ *is defined to permute the basis elements* $\{e_1,e_2,e_3,e_4,e_5\}$ *in the manner* $(e_1e_2e_3e_4e_5)$, *then* $B$ *has order 5. The invariant antiflag in this case is clearly* $(u,\sigma_{even})$ *where* $u = \sum_i e_i$ *and* $\sigma_{even}$ *is the even hyperplane of the basis* $\{e_1,e_2,e_3,e_4,e_5\}$ *with* $\sigma_{even}$ *comprising the ten points* $e_i + e_j$ *and the five points* $u + e_i$. *Observe that* $\wedge^2 B$ *fixes the three points*

$$p = \textstyle\sum_i e_i \wedge e_{i+1}, \quad p' = \sum_i e_i \wedge e_{i+2}, \quad p'' = p + p' = \sum_{1\le i<j\le 5} e_i \wedge e_j \qquad (A.13)$$

*of* $\mathrm{PG}(9,2)$, *where in the first two summations* $i$ *runs from* 1 *to* 5 (mod 5). *So* $L = \{p,p',p''\}$ *is the special line of fixed points of lemma* **??** *for the given* $B$. *Of course, by lemma* **??**, *any* $B \in \mathrm{GL}(5,2)$ *of order 5 can be expressed as in this remark relative to a suitable choice of basis.*

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