Principal Agent Model: Moral Hazard and Adverse Selection
Advanced Microeconomics

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Moral hazard

Owners of a firm are principals (P) and workers are agents (A). They play a production game in which an agent exerts effort \((a)\) in return of income \((y)\).

Principal maximises net profit.

Agent can put high or low efforts and P cannot observe it. Utility of agent at work is given by

\[
V = u(y) - a \geq V^0
\]  

(1)

This must be greater than a reservation utility \(V^0\) that is available from alternative work. The income level that an individual worker requires is then given by

\[
y = V^{-1} (V^0 + a)
\]  

(2)
Moral hazard

Less informed P can make sure that A exerts good effort by making wage contract as \((y_1 < y^*)\)

\[
V = \pi v(y^*) + (1 - \pi) v(y_1) < V^0
\]  
(3)

Principal’s objective when \(a\) is observable is then to maximize profit by producing \(x\) subject to the participation constraint

\[
\max z_i = \pi_i (x_1 - y_1) + (1 - \pi_i) (x_2 - y_2) \quad i=h,l
\]  
(4)

subject to

\[
\pi_i v(y_1) + (1 - \pi_i) v(y_2) - a_i \geq V^0
\]  
(5)

There is **uncertainty** in production resulting in \(x_1\) and \(x_2\) levels of production, \(x_1 < x_2\). Because of this uncertainty \(x_1\) may happen despite A putting high level effort, which P cannot observe.
Full information scenario

\[ \mathcal{L} = \pi_i (x_1 - y_1) + (1 - \pi_i) (x_2 - y_2) + \lambda [\pi_i \nu(y_1) + (1 - \pi_i) \nu(y_2) - a_i - V^0] \]  \hspace{1cm} (6)

First order conditions (for high effort case)

\[ \frac{\partial z_h}{\partial y_1} = -\pi_h + \lambda \pi_h \nu'(y_1) = 0 \]  \hspace{1cm} (7)

\[ \frac{\partial z_h}{\partial y_2} = -(1 - \pi_h) + \lambda (1 - \pi_h) \nu'(y_2) = 0 \]  \hspace{1cm} (8)

\[ \frac{\partial z_h}{\partial \lambda} = \pi_i \nu(y_1) + (1 - \pi_i) \nu(y_2) - a_i - V^0 = 0 \]  \hspace{1cm} (9)
Optimal efforts

From the first two first order conditions in the full information case

\[ v'(y_1) = v'(y_2) = \frac{1}{\lambda} \implies y_1 = y_2 \]  

(10)

Thus the owners of the company force managers to put the same level of efforts. Risk-neutral owners bear all risk.

P can design contracts similarly if they like A to put low efforts.

\[ L = \pi_l (x_1 - y_1) + (1 - \pi_l) (x_2 - y_2) + \lambda \left[ \pi_l v(y_1) + (1 - \pi_l) v(y_2) - a_l - V^0 \right] \]  

(11)
Incomplete information scenario

$P$ cannot observe $a$ of $A$; therefore they must design a contract which makes $A$ put $a_h$

This requires adding an incentive compatibility constraint.

$$
\pi_h v(y_1) + (1 - \pi_h) v(y_2) - a_h \geq \pi_l v(y_1) + (1 - \pi_l) v(y_2) - a_l \quad (12)
$$

Then the problem is modified as

$$
\max z_i = \pi_i (x_1 - y_1) + (1 - \pi_i) (x_2 - y_2) \quad i=h,l \quad (13)
$$

subject to

$$
\pi_i v(y_1) + (1 - \pi_i) v(y_2) - a_i \geq V^0 \quad (14)
$$

$$
\pi_h v(y_1) + (1 - \pi_h) v(y_2) - a_h \geq \pi_l v(y_1) + (1 - \pi_l) v(y_2) - a_l \quad (15)
$$
Incomplete information scenario

\[
\mathcal{L} = \pi_l (x_1 - y_1) + (1 - \pi_l) (x_2 - y_2) + \lambda \left[ \pi_l \nu(y_1) + (1 - \pi_l) \nu(y_2) - a_l - V^0 \right.
\]

\[
\mu \left[ \pi_h \nu(y_1) + (1 - \pi_h) \nu(y_2) - a_h - \pi_l \nu(y_1) - (1 - \pi_l) \nu(y_2) + a_l \right].
\]

The optimising conditions in this case are given by

\[
\frac{\partial z_h}{\partial y_1} = -\pi_h + \lambda \nu'(y_1) + \mu (\pi_h - \pi_l) \nu'(y_1) = 0 \quad (17)
\]

\[
\frac{\partial z_h}{\partial y_2} = - (1 - \pi_h) + \lambda (1 - \pi_h) \nu'(y_2) + \mu (\pi_h - \pi_l) \nu'(y_2) = 0 \quad (18)
\]

\[
\frac{\partial z_h}{\partial \lambda} = \pi_l \nu(y_1) + (1 - \pi_l) \nu(y_2) - a_l - V^0 = 0 \quad (19)
\]

\[
\frac{\partial z_h}{\partial \lambda} = \pi_h \nu(y_1) + (1 - \pi_h) \nu(y_2) - a_h - \pi_l \nu(y_1) - (1 - \pi_l) \nu(y_2) + a_l = 0
\]
Incomplete information scenario

From these conditions

\[ \lambda + \mu \left( \frac{\pi_h - \pi_l}{\pi_h} \right) = \frac{1}{v'(y_1)} \]  \hspace{1cm} (21)

\[ \lambda - \mu \left( \frac{\pi_h - \pi_l}{\pi_h} \right) = \frac{1}{v'(y_2)} \]  \hspace{1cm} (22)

\[ \frac{1}{v'(y_1)} < \lambda < \frac{1}{v'(y_2)} \implies y_1 > y^* > y_2 \]  \hspace{1cm} (23)

Payment to A now varies according to the contribution in the gross profit. It gives A an incentive to choose \( a_h \) than \( a_l \)

\[ (\pi_h - \pi_l) \left[ v'(y_1) - v'(y_2) \right] = a_h - a_l \]  \hspace{1cm} (24)
Adverse selection

First best

\[
\max x_i - y_i \tag{25}
\]

subject to

\[
v(y) - a \geq V^0 \quad \text{and} \quad x_i = x(a_i, \theta_i) \tag{26}
\]

Second best

\[
\mathcal{L} = \pi (x_1 - y_1) + (1 - \pi) (x_2 - y_2) + \lambda \left[ v(y_1) - \psi_1 v(x_2, \theta_1) - V^0 \right] + \mu \left[ v(y_2) - \psi_1 v(x_2, \theta_1) - v(y_1) + \psi_1 v(x_2, \theta_1) \right] \tag{27}
\]
First order conditions
\[ \mathcal{L}_{x_1} = \]
\[ \mathcal{L}_{y_1} = \]
\[ \mathcal{L}_{x_2} = \]
\[ \mathcal{L}_{y_2} = \]
\[ \mathcal{L}_{\lambda} = \]
\[ \mathcal{L}_{\mu} = \]
Example (Tirole): Consider a model of shareholders \((P)\) and managers \((A)\) with continuous efforts \(e\). Managers’ utility function is positively related with wages and negatively with the efforts as

\[
u \left( w - \frac{Re^2}{2} \right)
\]

Here \(R\) is a disutility from work parameter; \(u' (w) > 0\) and \(u'' (w) < 0\). and reservation wage is \(w_0\). Thus the participant constraints for \(A\) is

\[
Eu \left( w - \frac{Re^2}{2} \right) \geq w_0
\]
Full knowledge equilibrium

Profit is a stochastic variable with a random variable

\[ \Pi = e + \varepsilon \]  (30)

Here \( \varepsilon \) is a random variable with \( E\varepsilon = 0 \)

If the shareholder could observe efforts, the optimal contract would be \( w = \overline{w} \), is a fixed wage. Here this from the participation constraint is

\[ \overline{w} = w_0 + \frac{Re^2}{2} \]  (31)

Maximisation of shareholder’s expected profit is:

\[ E (\Pi) = E \left( e + \varepsilon - w_0 - \frac{Re^2}{2} \right) = e - w_0 - \frac{Re^2}{2} \]  (32)
Shareholders and Manager

\[ \frac{\partial E(\Pi)}{\partial e} = 1 - Re = 0 \iff e = \frac{1}{R} \text{ if } w_0 \leq \frac{1}{2R} \]  

(33)

This is the optimal solution when shareholders could observe the effort of managers.

Now suppose the efforts are not observable. Consider a **linear incentive scheme**:

\[ w(\Pi) = a + b\Pi \]  

(34)

What is the expected utility of A with linear scheme:

\[ Eu \left( a + be + b\epsilon - \frac{Re^2}{2} \right) \iff b - Re \iff e = \frac{b}{R} \]  

(35)
Effort grows with the slope of the incentive scheme. If $b = 1$; then $e = e^*$. The expected utility at this level of efforts is

$$Eu \left( a + \frac{b^2}{R} + b\varepsilon - \frac{b^2}{2R} \right) \iff Eu \left( a + \frac{b^2}{2R} + b\varepsilon \right)$$  \hspace{1cm} (36)

Shareholder’s expected profit

$$\Pi^e = E [e + \varepsilon - a - be - b\varepsilon] = \frac{b}{R} - a - \frac{b^2}{R} = \frac{b}{R} (1 - b) - a$$  \hspace{1cm} (37)

Linear optimal scheme:

$$\max \Pi^e = \frac{b}{R} (1 - b) - a$$  \hspace{1cm} (38)
Shareholders and Manager

Linear optimal scheme:

$$\max \Pi^e = \frac{b}{R} (1 - b) - a$$

subject to

$$Eu \left( a + \frac{b^2}{2R} + b \epsilon \right) \geq w_0$$
Substitute $a$ from the participation constraint

$$Eu \left( -\Pi^e + \frac{b}{R} - \frac{b^2}{2R} + b\epsilon \right) = w_0 \quad (41)$$

differentiating wrt $b$

$$Eu' \left( \frac{1 - b}{R} \right) + Eu'\epsilon = 0 \quad (42)$$

This gives $b = 1$.
This value of the linear scheme optimises profit for the shareholder as the agent puts maximum efforts at work.
## Popular Principal Agent Games

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Principal Agent Model in Job Market: Incomplete Information and Adverse Selection

- Principal wants to produce output employing workers with a scheme of wage contract that matches efforts put by a worker to produce.
- Worker knows his type but the principal does not.
- Principal knows the distribution of quality of workers $F(s)$, where $s$ denotes either good or bad state such as probability of observing good is 0.5 and of bad 0.5.
- Principal offers the agent a wage contract $W(q)$.
- Worker accepts or rejects this contract based on self-selection and participation constraints.
Objective of Principal and Agents

- Basically worker evaluates the utility from the wage and disutility from work and decides the amount of work to put in.
- Output from good workers is \( q(e, \text{good}) = 3e \) and from bad state is \( q(e, \text{bad}) = e \)
- If agent rejects the contract there is no work both worker and principal get zero payoff.
- If worker accepts the contract
  - Agent’s utility:
    \[
    U_A(e, w, s) = w - e^2 \tag{43}
    \]
  - Principal’s profit:
    \[
    V_p(q, w) = q - w \tag{44}
    \]
Optimal level of efforts by good and bad workers

- Good worker maximises

\[
\max_{e_G} U_G = w_G - e_G^2 = 3e_G - e_G^2
\]  \hspace{1cm} (45)

- The first part is wage income and the second part of disutility of work.
- The optimal level of efforts by good agent is:

\[
3 - 2e_G = 0 \implies e_G = 1.5
\]  \hspace{1cm} (46)

Bad worker’s Objective and Optimal Efforts

\[
\max_{e_B} U_B = w_B - e_B^2 = e_B - e_B^2
\]  \hspace{1cm} (47)

\[
1 - 2e_G = 0 \implies e_G = 0.5
\]  \hspace{1cm} (48)

- The principal does not know what levels of efforts are appropriate for good and bad workers.
Principal’s Objective

Principal maximises expected profit

\[ \max_{q_G, q_B, w_G, w_B} \quad U_P = [0.5 (q_G - w_G) + 0.5 (q_B - w_B)] \]  \quad (49) 

by designing separate contracts for good \((q_G, w_G)\) and bad workers \((q_B, w_B)\) and .

Wage for good worker: \(w_G = q(e, \text{good}) = 3e\) or \(e = \frac{q_G}{3}\)

Wage for bad worker: \(w_B = q(e, \text{bad}) = e\) or \(e = q_B\)
Incentive Compatibility Constraints for Agents

Self selection constraint for good worker

\[ U_G = w_G - e_G^2 = w_G - \left( \frac{qG}{3} \right)^2 \geq U_G = w_B - e_B^2 = w_B - \left( \frac{qB}{3} \right)^2 \] (50)

Self selection constraint for bad worker

\[ U_B = w_B - e_B^2 = w_B - (q_B)^2 \geq U_B = w_G - e_G^2 \] (51)

Participation constraints for good worker

\[ U_G = w_G - \left( \frac{qG}{3} \right)^2 \geq 0 \] (52)

Participation constraint for bad worker

\[ U_B = w_B - (q_B)^2 \geq 0 \] (53)
Binding Constraints

Participation constraint of bad worker

\[ w_B = q_B^2 \]  \hfill (54)

Self selection constraint for good worker

\[ w_G = \left( \frac{q_G}{3} \right)^2 + w_B - \left( \frac{q_B}{3} \right)^2 \implies w_G = \left( \frac{q_G}{3} \right)^2 + q_B^2 - \left( \frac{q_B}{3} \right)^2 \]  \hfill (55)
Principal’s Optimal Solution

- Principal includes agents’ optimal choices into his utility function:
  \[ \max_{q_G, q_B, w_G, w_B} U_P = [0.5(q_G - w_G) + 0.5(q_B - w_B)] \]

- Including binding constraints of agents:
  \[ \max_{q_G, q_B, w_G, w_B} U_P = \left[ 0.5 \left( q_G - \left( \frac{q_G^2}{3} + q_B^2 - \left( \frac{q_B}{3} \right)^2 \right) \right) + 0.5(q_B - q_B^2) \right] \]

Now principal decides how much to produce from each type of worker

- First order conditions with respect to \( q_G \) and \( q_B \)
  \[
  \frac{\partial U_P}{\partial q_G} = 0.5 \left(1 - \frac{2q_G}{9}\right) = 0 \implies q_G = 4.5 \quad (56)
  \]
  \[
  \frac{\partial U_P}{\partial q_B} = 0.5 \left(-2q_B - \frac{2q_B}{9}\right) + 0.5 \left(1 - 2q_B\right) = 0 \implies 34q_B = 9 \implies q_B = 0.265 \quad (57)
  \]
Incentive Compatible First Best Choices of Good and Bad Worker

- Now wages can be found from the constraints

\[ w_B = q_B^2 = (0.265)^2 = 0.07 \quad (58) \]

\[ w_G = \left( \frac{q_G}{3} \right)^2 + q_B^2 - \left( \frac{q_B}{3} \right)^2 = \left( \frac{4.5}{3} \right)^2 + (0.265)^2 - \left( \frac{0.265}{3} \right)^2 = 2.32 \quad (59) \]

- Thus in the presence of information asymmetry, the efforts by the good worker is at the first best level as the bad effort by him is not as attractive as the good effort.
- It is not profitable for good worker to pretend as a bad worker. Good worker is not attracted by the contract for bad worker.
- It is very costly for the bad worker to accept the contract of good worker. Bad worker’s first best to put low effort.
Incentive compatible game on renting a piece of agricultural land

If a worker puts $x$ amount of effort, the land produces $y = f(x)$
Then the land owner pays worker $s(y)$.
The land owner wants to maximise profit
$$\pi = f(x) - s(y) = f(x) - s(f(x))$$
Worker has cost of putting effort $c(x)$ and has a reservation utility, $\bar{u}$
The participation constraint is given by
$$s(f(x)) - c(x) \geq \bar{u}$$
Including this constraint maximisation problem becomes
$$\max \quad \pi = f(x) - s(f(x))$$
subject to
$$sf(x) - c(x) \geq \bar{u}$$
Solution: marginal productivity equals marginal efforts $f'(x)) = c'(x)$
Incentive compatible game on rendering a piece of agricultural land

(a) renting the land where the workers pays a fixed rent $R$ to the owner and takes the residual amount of output, at equilibrium

$$f(x^*) - c(x^*) - R = \bar{u}$$

(60)

(b) Take it or leave it contract where the owner gives some amount such as

$$B - c(x^*) = \bar{u}$$

(61)

(c) hourly contract

$$s(f(x)) = wx + K$$

(62)

(d) sharecropping, in which both worker and owner divide the output in a certain way.

In (a)-(c) burden of risks due to fluctuations in the output falls on the worker but it is shared by both owner and worker in (d).

Which of these incentives work best depends on the situation.
Holt Charles (2007) Markets, Games and Strategic Behaviour, Pearson,
Rasmusen E (2007) Games and Information, Blackwell,