

# Macroeconometric Models

Simultaneous Equations Model  
Estimation, Simulations and Forecasts

## A simple Example of the Simultaneous Equation System

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$y_{1i} = \frac{(a_{22}b_{11} - a_{12}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(a_{22}b_{12} - a_{12}b_{22})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{1i}$$

$$y_{2i} = \frac{(-a_{21}b_{11} + a_{11}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(-a_{21}b_{12} + a_{11}b_{22})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{2i}$$

# General Form for the Simultaneous Equation System

$$a_{11}y_{1i} + a_{12}y_{2i} + \dots + a_{1m}y_{mi} + b_{11}x_{1i} + b_{12}x_{2i} + \dots + b_{1k}x_{ki} = e_{1i}$$

$$a_{21}y_{1i} + a_{22}y_{2i} + \dots + a_{2m}y_{mi} + b_{21}x_{1i} + b_{22}x_{2i} + \dots + b_{2k}x_{ki} = e_{2i}$$

.....

$$a_{m1}y_{1i} + a_{m2}y_{2i} + \dots + a_{mm}y_{mi} + b_{m1}x_{1i} + b_{m2}x_{2i} + \dots + b_{mk}x_{ki} = e_{mi}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \cdot \\ \cdot \\ y_{mi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mk} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \\ \cdot \\ \cdot \\ x_{mi} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \\ \cdot \\ \cdot \\ e_{mi} \end{bmatrix}$$

# Impact and Shock Analysis in a Simultaneous Equation System

$$AY_i + BX_i = U_i$$

$$Y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \cdot \\ \cdot \\ y_{mi} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & & & \\ & & & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \quad X_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \cdot \\ \cdot \\ x_{mi} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ & & & \\ & & & \\ b_{m1} & b_{m2} & \dots & b_{mk} \end{bmatrix} \quad U_i = \begin{bmatrix} e_{1i} \\ e_{2i} \\ \cdot \\ \cdot \\ e_{mi} \end{bmatrix}$$

$$Y_i = -A^{-1}BX_i + A^{-1}U_i \quad \hat{Y}_i = -A^{-1}BX_i$$

$$\text{Var}(Y_i - \hat{Y}_i) = A^{-2}\sigma_u^2$$

# Macroeconometric (IS-LM) Model

Consumption function:  $C_t = \beta_0 + \beta_1(Y_t - T_t) + \varepsilon_{1t}$  (1)

Taxation function:  $T_t = c_0 + c_1 Y_t + \varepsilon_{2t}$  (2)

Import function:  $M_t = m_0 + m_1 Y_t + \varepsilon_{3t}$  (3)

Investment function:  $I_t = \mu_0 + \mu_1 R_t + \phi \Delta Y_{t-1} + \varepsilon_{4t}$  (4)

National income identity:  $Y_t = C_t + I_t + G_t + X_t - M_t$  (5)

Money market (LM curve):  $\left(\frac{MM_t}{P_t}\right) = b_0 + b_1 Y_t - b_2 R_t + \varepsilon_{6t}$  (6)

Money market (LM curve):  $R_t = \frac{b_0}{b_2} - \frac{1}{b_2} \left(\frac{MM_t}{P_t}\right) + \frac{b_1}{b_2} Y_t + \varepsilon_{6t}$  (6)

$Y_t, C_t, M_t, I_t, R_t$  and  $T_t$  are endogenous variables and  $G_t, X_t, MM_t$  and  $\Delta Y_{t-1}$  are predetermined or exogenous variables representing government expenditure, exports, money stock and change in  $Y_t$  in the previous period respectively. Set the price level  $P_t$  equal to unity for simplicity.

# Reduced form of the Macro Model

$$C_t = \Pi_{10} + \Pi_{11}G_t + \Pi_{12}X_t + \Pi_{13}MM_t + \Pi_{14}\Delta Y_{t-1}$$

$$M_t = \Pi_{20} + \Pi_{21}G_t + \Pi_{22}X_t + \Pi_{23}MM_t + \Pi_{24}\Delta Y_{t-1}$$

$$I_t = \Pi_{30} + \Pi_{31}G_t + \Pi_{32}X_t + \Pi_{33}MM_t + \Pi_{34}\Delta Y_{t-1}$$

$$R_t = \Pi_{40} + \Pi_{41}G_t + \Pi_{42}X_t + \Pi_{43}MM_t + \Pi_{44}\Delta Y_{t-1}$$

$$T_t = \Pi_{50} + \Pi_{51}G_t + \Pi_{52}X_t + \Pi_{53}MM_t + \Pi_{54}\Delta Y_{t-1}$$

$$Y_t = \Pi_{60} + \Pi_{61}G_t + \Pi_{62}X_t + \Pi_{63}MM_t + \Pi_{64}\Delta Y_{t-1}$$

# Rank and Order Conditions of Identification

Order condition:  $K - k \geq m - 1$

Rank condition:  $\rho(A) \geq (M - 1)(M - 1) \Rightarrow$  order of the matrix

$M$  = number of endogenous variables in the model

$K$  = number of exogenous variables in the model

including the intercept

$m$  = number of endogenous variable in an equation

$k$  = number of exogenous variables in a given equation

Rank condition is defined by the rank of the matrix, which should have a dimension  $(M-1)$ , where  $M$  is the number of endogenous variables in the model.

## How to Determine the Rank of the Matrix?

Rank of matrix is the order of non-singular matrix

Rank matrix is formed from the coefficients of the variables (both endogenous and exogenous) excluded from that particular equation but included in the other equations in the model.

The rank condition tells us whether the equation under consideration is identified or not.

The order condition tells us if it is exactly identified or overidentified.

## Order and rank conditions of identification

1. If  $K - k > m - 1$  and the rank of the  $\rho(A)$  is  $M - 1$  then the concerned equation is overidentified.
2. If  $K - k = m - 1$  and the rank of the  $\rho(A)$  is  $M - 1$  then the equation is exactly identified.
3. If  $K - k \geq m - 1$  and the rank of the  $\rho(A)$  is less than  $M - 1$  then the equation is underidentified.

If  $K - k < m - 1$  the structural equation is unidentified. The rank of the  $\rho(A)$  is less than  $M - 1$  in this case.

# Steps for rank condition

1. Write down the system in the tabular form
2. Strike out the coefficients corresponding to the equation to be identified
3. Strike out the columns corresponding to those coefficients in 2 which are nonzero.
4. The entries left in the table will give only the coefficients of the variables included in the system but not in the equation under consideration. From these coefficients form all possible  $A$  matrices of order  $M-1$  and obtain a corresponding determinant. If at least one of these determinants is non-zero then that equation is identified

Rank condition:  $\rho(A) \geq (M - 1)(M - 1) \Rightarrow$  order of the matrix.

Table of Coefficients in a Macro Econometric Model

	Constant	$Y_t$	$C_t$	$M_t$	$I_t$	$R_t$	$T_t$	$G_t$	$X_t$	$MM_t$	$\Delta Y_{t-1}$
$C_t$	$-\beta_0$	$-\beta_1$	1	0	0	0	$\beta_1$	0	0	0	0
$T_t$	$-t_0$	$-t_1$	0	$-t_2$	0	0	1	0	0	0	0
$M_t$	$-m_0$	$-m_1$	1	1	0	0	$-t_2$	0	0	0	0
$I_t$	$-\mu_0$	0	0	0	1	$-\mu_1$	0	0	0	0	0
$R_t$	$-\frac{b_0}{b_2}$	$-\frac{b_1}{b_2}$	0	0	1	0	0	0	0	$\frac{1}{b_2}$	0
$Y_t$	0	1	-1	1	-1	0	0	-1	-1	0	0

## Checking Rank Identification Condition for the above System

Consumption function:

$$A_1 = \begin{bmatrix} t_2 & 0 & 0 & 0 \\ 1 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad |A_1| = -\frac{1}{b_2} \phi m_2 t_2 \quad \rho(A_1) = 4$$

It is obvious that there exists at least one non-singular matrix of order  $M-1$ .

Tax function:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad \rightarrow \quad |A_1| = -\frac{1}{b_2} \phi m_2 \quad \rho(A_1) = 4$$

## Checking Rank Identification Condition for the above System

Import

$$A_1 = \begin{bmatrix} 1 & \beta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad |A_1| = -\frac{1}{b_2} \phi \quad \rho(A_1) = 4$$

Interest Rate

$$A_1 = \begin{bmatrix} \beta_1 & 0 & -\beta_1 & 0 \\ t_1 & t_2 & 1 & 0 \\ m_1 & 1 & m_3 & 0 \\ \frac{b_1}{b_2} & 0 & 0 & \frac{1}{b_2} \\ b_2 & & & \end{bmatrix} \quad |A_1| = \beta_1 t_2 m_3 \frac{1}{b_2} + \beta_1 t_2 m_1 \frac{1}{b_2} \quad \rho(A_1) = 4$$

Investment

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 1 & m_2 & 0 \\ 0 & 0 & -\mu_1 & \phi \end{bmatrix} \quad |A_1| = t_2 m_2 \phi \quad \rho(A_1) = 4$$

# Some Techniques for Estimation of the Simultaneous Equations System

Single Equations Methods

Ordinary Least Squares

Indirect Least Squares

Two Stage Least Squares Method

System Method

Three Stage LS: Generalised Least Square

Seemingly Unrelated Regression Equations

Full Information Maximum Likelihood

## Estimation by the Generalised Least Square Method

$$\text{cov}[vv'] = \begin{bmatrix} \text{var}(v_1) & \text{cov}(v_1v_2) & \text{cov}(v_1v_3) & \cdot & \text{cov}(v_1v_5) \\ \text{cov}(v_2v_1) & \text{var}(v_2) & \text{cov}(v_2v_3) & \cdot & \text{cov}(v_2v_5) \\ \text{cov}(v_3v_1) & \text{cov}(v_3v_2) & \text{var}(v_3) & \cdot & \text{cov}(v_3v_5) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \text{cov}(v_5v_1) & \text{cov}(v_5v_2) & \text{cov}(v_5v_3) & \cdot & \text{var}(v_v) \end{bmatrix} = \Omega$$

$$P'P = \Omega^{-1}$$

$$PY = PX\beta + Pe \qquad Y^* = X^*\beta + e^*$$

$$\beta_{GLS} = (X^{*'}X^*)^{-1}X^{*'}Y^* = (X'P'PX)X'P'PY$$

$$\beta_{GLS} = (X^{*'}\Omega^{-1}X^*)^{-1}X^{*'}\Omega^{-1}Y^*$$

## Estimation of Simultaneous System using PcGive

$$C = + 0.2128 * M4 + 1.407 * G + 0.1767 * X + 8.059e+004$$

(SE) (0.0266) (0.212) (0.154) (1.63e+004)

$$Y = + 0.2617 * M4 + 2.86 * G + 1.348 * X + 8.862e+004$$

(SE) (0.0454) (0.361) (0.262) (2.78e+004)

$$T = + 0.3204 * M4 + 0.9521 * G - 0.08909 * X - 7.533e+004$$

(SE) (0.02) (0.159) (0.116) (1.23e+004)

$$M = + 0.06508 * M4 - 0.4738 * G + 1.003 * X + 4.34e+004$$

(SE) (0.0198) (0.157) (0.114) (1.21e+004)

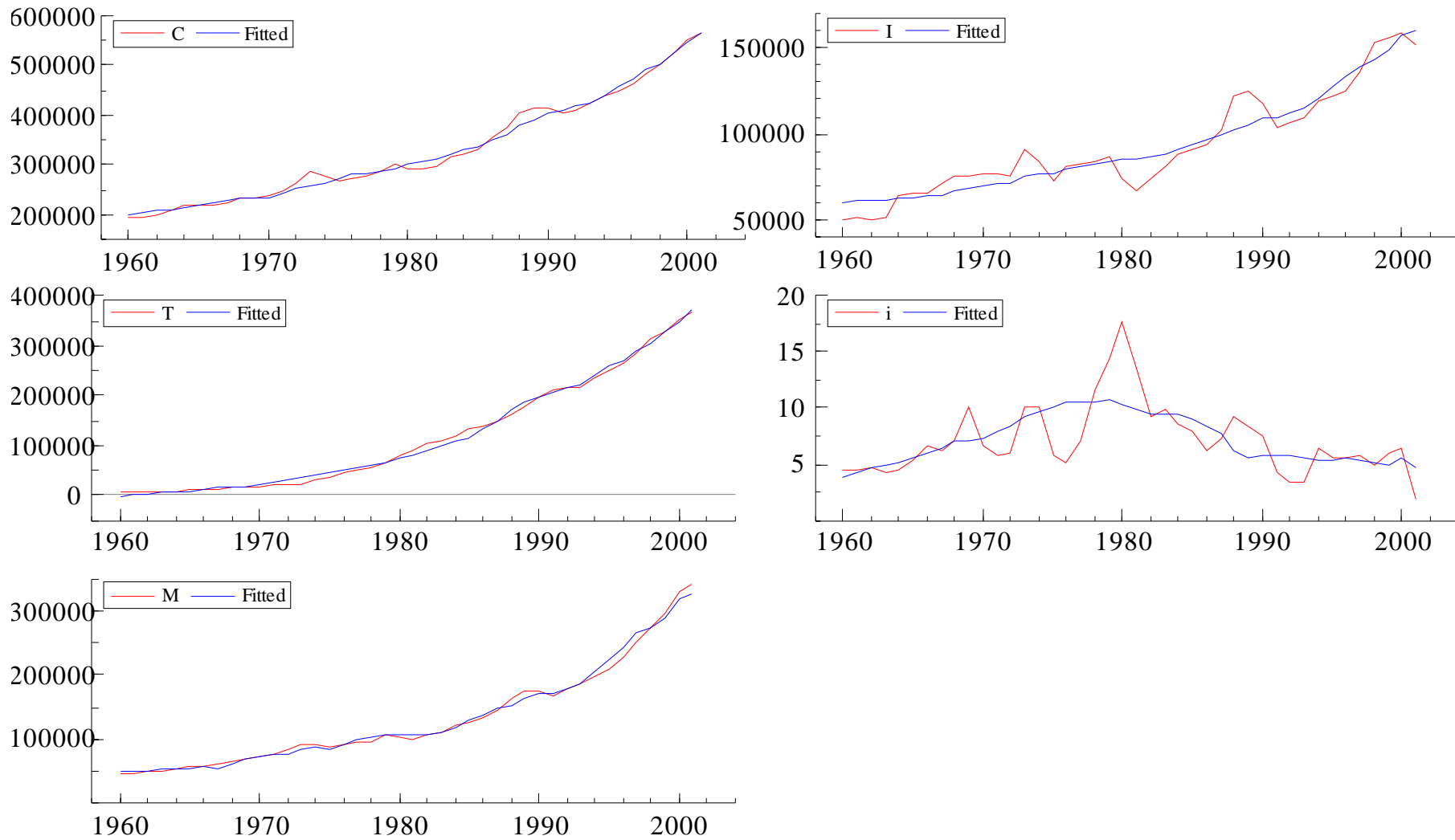
$$r = - 2.384e-005 * M4 + 0.0001148 * G + 6.273e-005 * X - 7.408$$

(SE) (6.2e-006) (4.93e-005) (3.58e-005) (3.79)

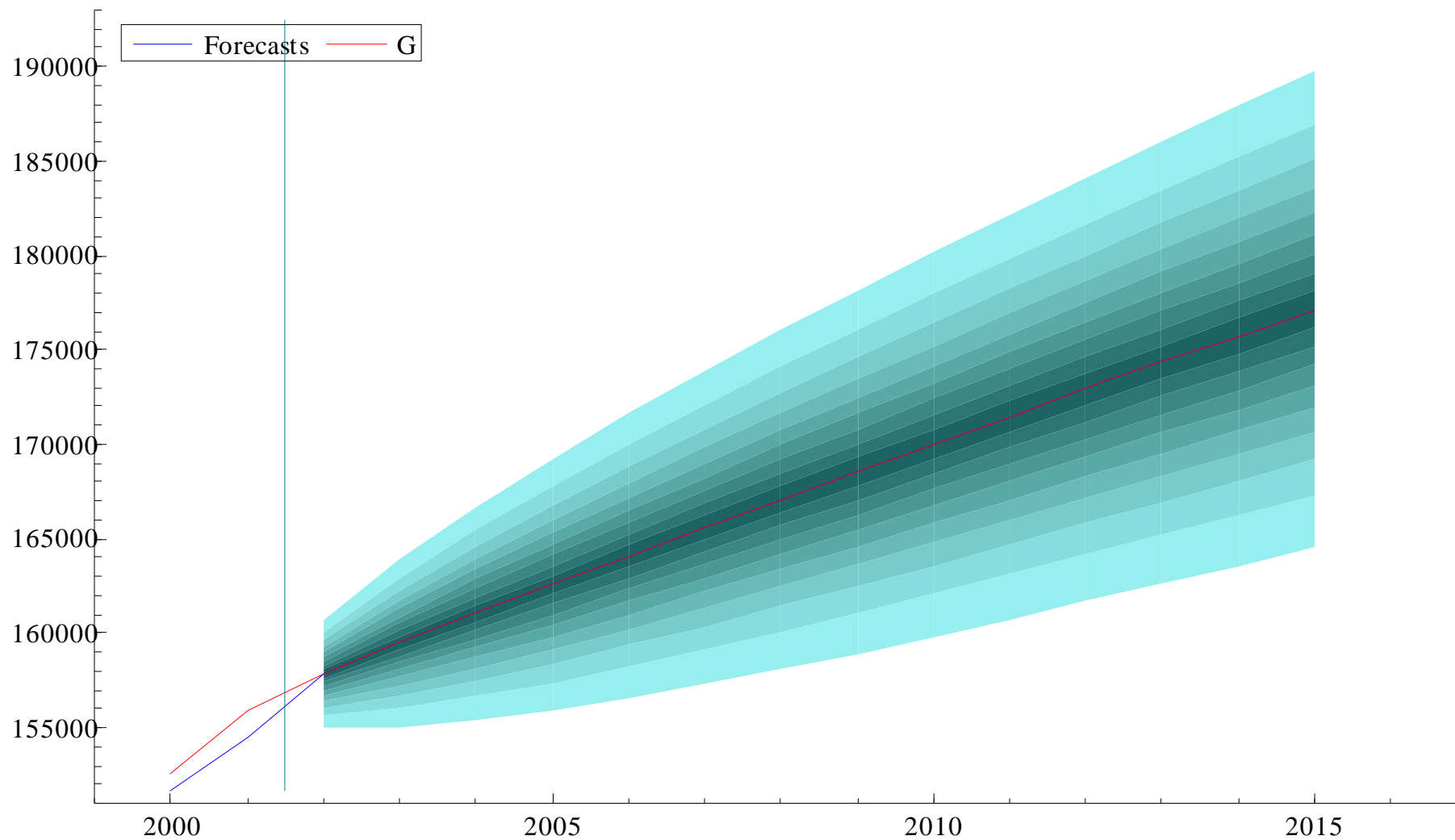
$$I = + 0.02907 * M4 + 0.0684 * G + 0.2681 * X + 4.292e+004$$

(SE) (0.0229) (0.182) (0.132) (1.4e+004)

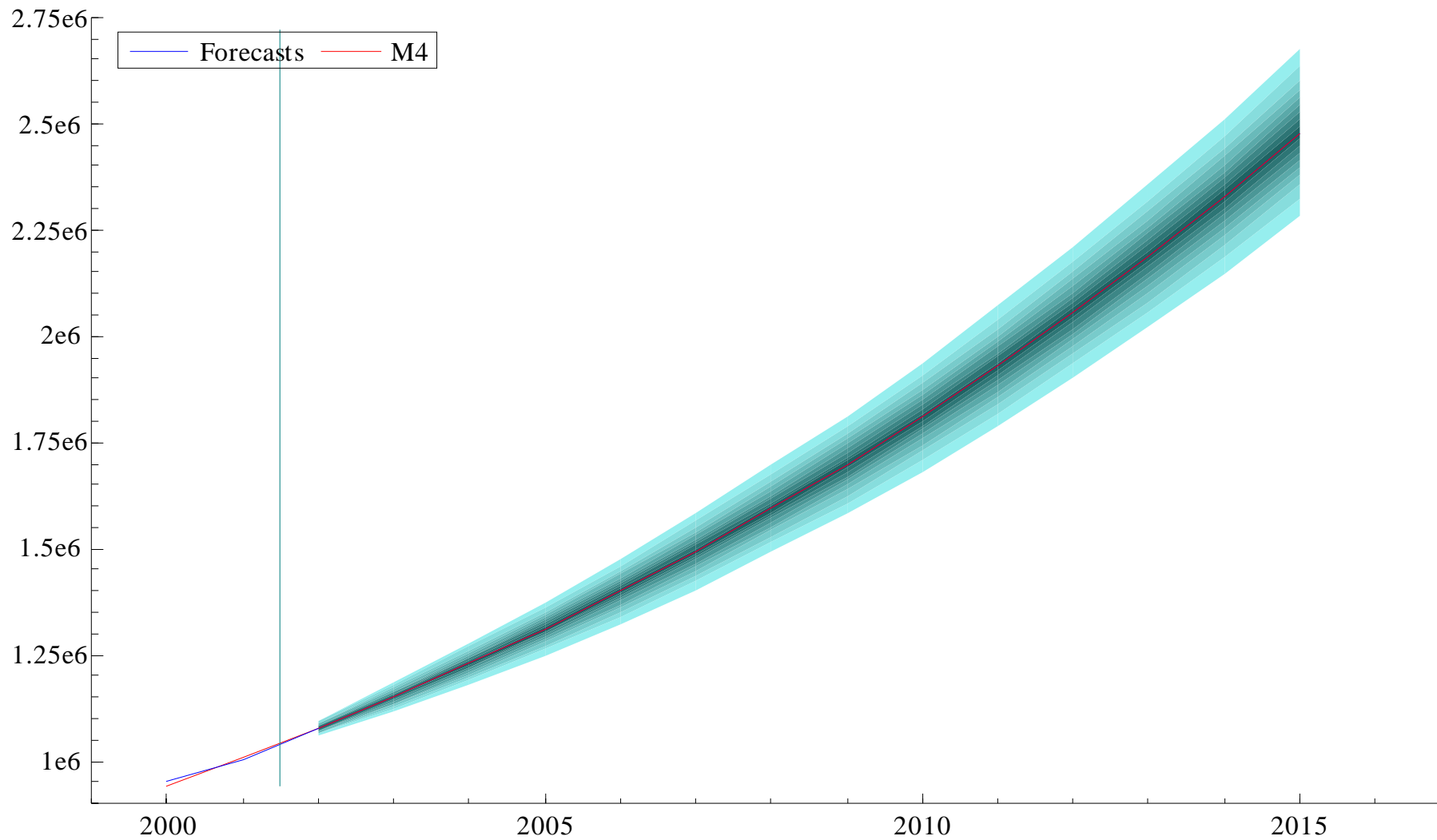
# Actual and Predicted Values in above Macroeconomic Model Historical Simulations



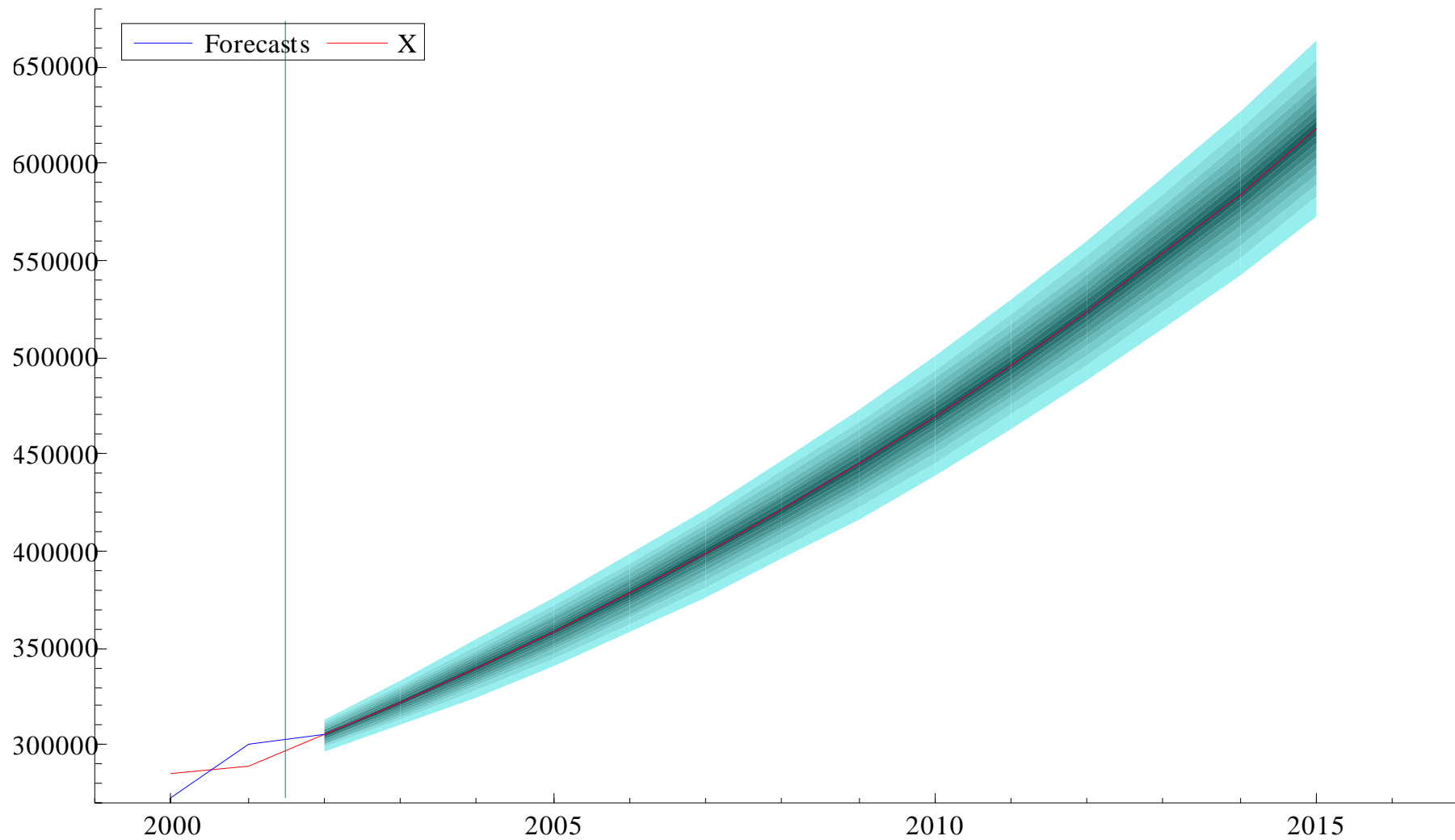
# Forecast of Exogenous Variable AR(2) : Government Spending



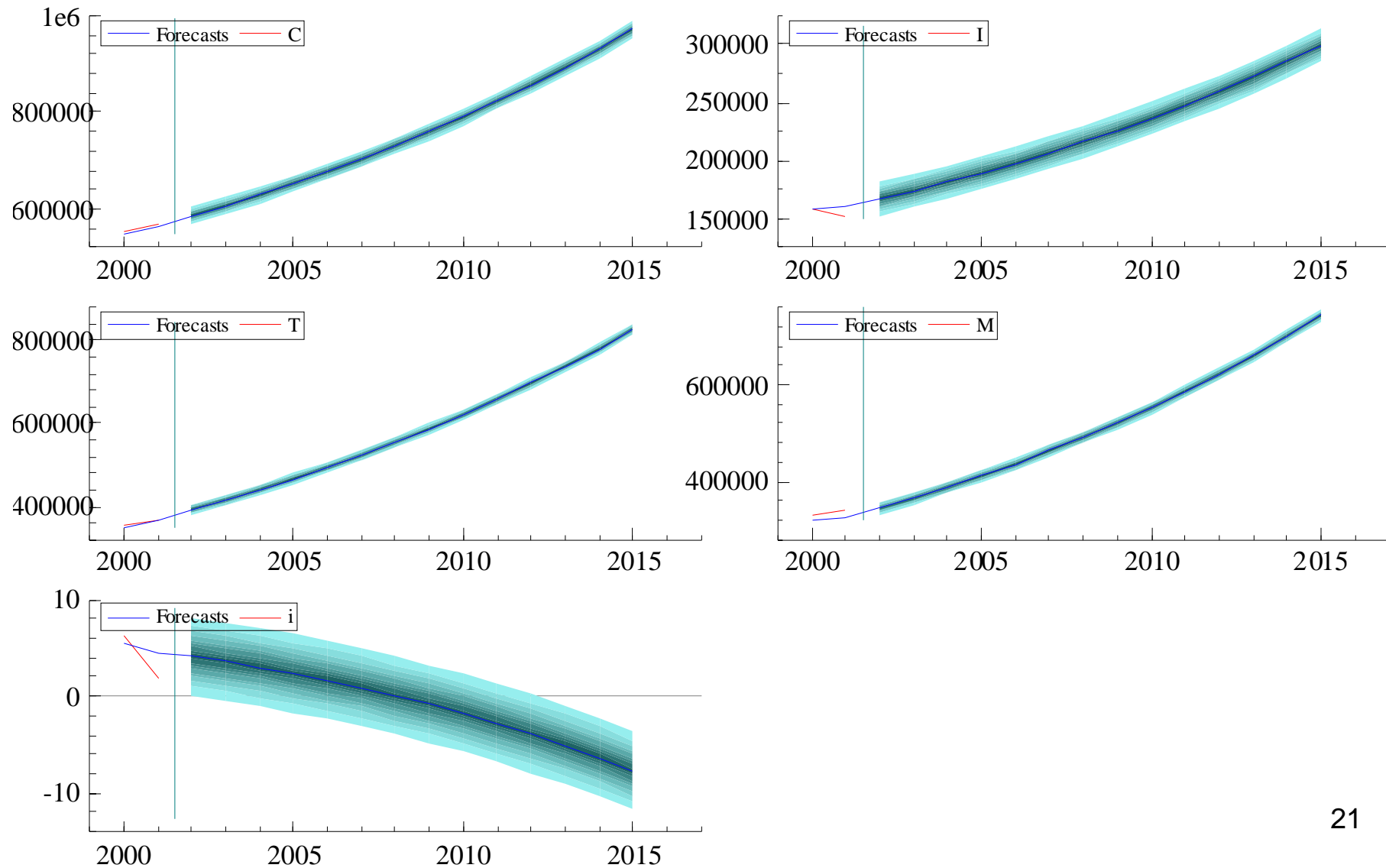
# Forecast of Exogenous Variable AR(2) : Money Supply



# Forecast of Exogenous Variable AR(2) : Exports



# Ex-ante Forecast Endogenous Variables from the Simultaneous Equation Model Consumption, Tax, Investment and the Interest Rate



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