Restrictions in Multiple Regression Analysis

Restricted Least Square

F-Test on validity of restriction

Consider a Multiple Regression with Three Variables and No Constant (data from the text)

y_t	x_{t1}	x_{t2}	x_{t3}
1	1	0	-1
-1	-1	1	0
2	1	0	0
0	0	1	0
4	1	2	0
2	0	3	0
2	0	0	1
0	1	-1	1
2	0	0	1

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} \sum_i x_{1,i}^2 & \sum_i x_{1,i}x_{2,i} & \sum_i x_{1,i}x_{3,i} \\ \sum_i x_{1,i}x_{2,i} & \sum_i x_{2,i}^2 & \sum_i x_{2,i}x_{3,i} \\ \sum_i x_{1,i}x_{3,i} & \sum_i x_{2,i}x_{3,i} & \sum_i x_{3,i}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i y_i x_{1,i} \\ \sum_i y_i x_{2,i} \\ \sum_i y_i x_{3,i} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.873 \\ 0.968 \end{bmatrix}$$

Variance of Errors and Parameters

$$e_t = y_t - x_{t1}\beta_1 - x_{t2}\beta_2 - x_{t3}\beta_3$$
$$= y - 1.6x1 - 0.873x2 - 0.968x3$$

$$\sum_i \hat{e}_t^2 = 6.946\ ; \quad \hat{\sigma}^2 = \frac{6.946}{9-3} = 1.158$$

Computed covariance matrix

$$\operatorname{cov}(b_1, b_2) = \hat{\sigma}^2 (X'X)^{-1} = 1.158 \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.063 & 0.0186 \\ 0 & 0.016 & 0.254 \end{bmatrix} = \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}$$

$$\begin{bmatrix} SE(\hat{\beta}_1) \\ SE(\hat{\beta}_2) \\ SE(\hat{\beta}_3) \end{bmatrix} = \begin{bmatrix} \sqrt{0.2315} \\ \sqrt{0.0735} \\ \sqrt{0.294} \end{bmatrix} = \begin{bmatrix} 0.48118 \\ 0.2711 \\ 0.5422 \end{bmatrix}$$

Test of Restrictions -1

ase of (d) $Rb = r \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$F = \frac{\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \right]' \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}{J}$$

$$F = \frac{\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}{J}$$

Test of Restrictions -2

$$F = \frac{\begin{bmatrix} 1.6 & 0.873 & 0.968 \end{bmatrix} \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.6 \\ 0.873 \\ 0.968 \end{bmatrix}}{3}$$

= 7.79 .F critical value for d.f.= (3,6) at 5% confidence interval is 4.76.

F calculated is bigger than F critical => Reject null hypothesis, which says $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$.

At least one of these parameters is significant and explains variation in y.

Restricted Least Square Method

OLS procedure to minimise the sum of squared error terms.

$$S = e'e = (Y - X\beta)'(Y - X\beta)$$

$$= Y'Y - 2\beta' X'Y + \beta' X' X\beta$$

Imposing a restriction involves constrained optimisation with a Lagrange multiplier.

$$L = e'e + 2\lambda(r' - \beta' R')$$

$$L = (Y - X\beta)'(Y - X\beta) + 2\lambda(r' - \beta' R')$$

$$L = Y'Y - 2\beta' X'Y + \beta' X' XX\beta + 2\lambda(r' - \beta' R')$$

Restricted Least Square Method

Partial derivation of this constrained minimisation function (Lagrangian function) wrt β and λ yields

(i) $$\frac{\partial L}{\partial \beta} = -2X'Y + 2X'Xb - 2\lambda R' = 0$$

(ii) $$\frac{\partial L}{\partial \lambda} = 2(r - Rb) = 0$$

$$X'Xb = X'Y + \lambda R'$$

$$(X'X)^{-1}X'Xb = (X'X)^{-1}(X'Y + \lambda R')$$

$$b = (X'X)^{-1}X'Y + (X'X)^{-1}R'\lambda$$

$$b = \hat{\beta} + (X'X)^{-1}R'\lambda$$

This is the restricted least square estimator but need still to be solved for λ . For that multiply the above equation both sides by R

Restricted Least Square Method

$$Rb = R\hat{\beta} + R\left(X'X\right)^{-1} R'\lambda = r$$

$$\lambda = \left[R\left(X'X\right)^{-1} R'\right]^{-1} \left[Rb - R\hat{\beta}\right]$$

$$\lambda = \left[R\left(X'X\right)^{-1} R'\right]^{-1} \left[r - Rb\right]$$

$$b = \hat{\beta} + \left(X'X\right)^{-1} R'\left[R\left(X'X\right)^{-1} R'\right]^{-1} \left[r - Rb\right]$$

Thus the restricted least square estimator is a linear function of the restriction $Rb - r = 0$.

Restricted Least Square Method

$$E(b) = E(\hat{\beta}) + (X'X)^{-1} R' \left[R(X'X)^{-1} R' \right]^{-1} \left[r - RE(b) \right]$$

$$E(b) = \beta$$

For variance we need to use property of an idempotent matrix AA=A.

Such as $A = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$

Recall in unrestricted case

$$\hat{\beta} = (X'X)^{-1} X'Y = \beta + (X'X)^{-1} X'e$$

$$E(b) - \beta = (X'X)^{-1} X'e + (X'X)^{-1} R' \left[R(X'X)^{-1} R' \right]^{-1} \left[r - RE(b) - R(X'X)^{-1} X'e \right]$$

Since $Rb - r = 0$

$$E(b) - \beta = M (X'X)^{-1} X'e$$

Restricted Least Square Method

Where M is the idempotent matrix:

$$M = I - (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1} R$$

The variance covariance matrix of

$$\operatorname{cov}(b) = [E(b) - \beta][E(b) - \beta]' = E\left[M(X'X)^{-1} X'ee'X(X'X)^{-1} M'\right]$$

$$\operatorname{cov}(b) = \sigma^2 M(X'X)^{-1} M$$

$$\operatorname{cov}(b) = \sigma^2 M(X'X)^{-1}$$

$$M = \sigma^2 (X'X)^{-1}\left[I - (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}\right]R$$

Thus the variance of the restricted least square estimator is smaller than the variance of the unrestricted least square estimator.

$$M = \sigma^2 (X'X)^{-1}\left[I - (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}\right]R$$