Econometrics 1

Lecture 18

Restrictions in Multiple Regression: Restricted Least Square Estimation

(Judge-Hill-Griffith-Lutkopohl-Lee (1988): 236)

OLS procedure to minimise the sum of squared error terms.

$$S = e'e = (Y - X\beta)'(Y - X\beta)$$

$$= Y'Y - 2\beta' X'Y + \beta' X' X\beta$$

Imposing a restriction involves constrained optimisation with a Lagrange multiplier.

$$L = e'e + 2\lambda(r' - \beta' R')$$

$$L = (Y - X\beta)'(Y - X\beta) + 2\lambda(r' - \beta' R')$$

$$L = Y'Y - 2\beta' X'Y + \beta' X' XX\beta + 2\lambda(r' - \beta' R')$$

Partial derivation of this constrained minimisation function (Lagrangian function) wrt β and λ yields

(i) $$\frac{\partial L}{\partial \beta} = -2X'Y + 2X'Xb - 2\lambda R' = 0$$

(ii) $$\frac{\partial L}{\partial \lambda} = 2\left(r - Rb\right) = 0$$

$$X'Xb = X'Y + \lambda R'$$

$$\left(X'X\right)^{-1} X'Xb = \left(X'X\right)^{-1}\left(X'Y + \lambda R'\right)$$

$$b = \left(X'X\right)^{-1} X'Y + \left(X'X\right)^{-1} R'\lambda$$

$$b = \hat{\beta} + \left(X'X\right)^{-1} R'\lambda$$

This is the restricted least square estimator but need still to be solved for λ . For that multiply the above equation both sides by R

$$Rb = R\hat{\beta} + R\left(X'X\right)^{-1}R'\lambda = r$$

$$\lambda = \left[R\left(X'X\right)^{-1}R'\right]^{-1}\left[Rb - R\hat{\beta}\right]$$

$$\lambda = \left[R\left(X'X\right)^{-1}R'\right]^{-1}\left[r - Rb\right]$$

$$b = \hat{\beta} + \left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left[r - Rb\right]$$

Thus the restricted least square estimator is a linear function of the restriction $Rb - r = 0$.

$$E(b) = E(\hat{\beta}) + (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}\left[r - RE(b)\right]$$

$$E(b) = \beta$$

For variance we need to use property of an idempotent matrix AA=A.

Such as $A = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$

Recall in unrestricted case

$$\hat{\beta} = (X'X)^{-1} X'Y = \beta + (X'X)^{-1} X'e$$

$$E(b) - \beta = (X'X)^{-1} X'e + (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}\left[r - RE(b) - R(X'X)^{-1} X'e\right]$$

Since $Rb - r = 0$

$$E(b) - \beta = M(X'X)^{-1} X'e$$

Where M is the idempotent matrix:

$$M = I - (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}R$$

The variance covariance matrix of

$$\text{cov}(b) = [E(b) - \beta][E(b) - \beta]' = E\left[M(X'X)^{-1}X'ee'X(X'X)^{-1}M'\right]$$

$$\text{cov}(b) = \sigma^2 M(X'X)^{-1}M$$

$$\text{cov}(b) = \sigma^2 M(X'X)^{-1}$$

$$M = \sigma^2 (X'X)^{-1}\left[I - (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}\right]R$$

Thus the variance of the restricted least square estimator is smaller than the variance of the unrestricted least square estimator.

$$M = \sigma^2 (X'X)^{-1}\left[I - (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}\right]R$$

Model $y_t = x_{t1}\beta_1 + x_{t2}\beta_2 + x_{t3}\beta_3 + e_t$

One general restriction may be

Null Hypothesis: $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$

A simple restriction $Rb - r = 0$

$$Rb = r \Rightarrow \underset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \underset{r}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

where b are the estimates obtained from the sample estimation.

F-test for Restrictions in Multiple Regression

$$y_t = x_{t,1}\beta_1 + x_{t,2}\beta_2 + x_{t,3}\beta_3 + e_t$$

Estimate β_1 β_2 and β_3, and test the following Test for Restrictions

a. $H_0 : \beta_2 = 0$ against $H_1 : \beta_2 \neq 0$

b. $H_0 : \beta_1 = \beta_2 = 0$ against $H_1 : \beta_1$ or β_2 or both are nonzero

c. $H_0 : \beta_2 = \beta_3 = 0$ against $H_1 : \beta_2$ or β_3 or both are nonzero

d. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ against $H_1 : \beta_1$ or β_2 or β_3 or all are nonzero

$$F = \frac{(Rb - r)'\left[R \operatorname{cov}(b) R'\right]^{-1}(Rb - r)}{J}$$

Estimates of the Model

y_t	x_{t1}	x_{t2}	x_{t3}
1	1	0	-1
-1	-1	1	0
2	1	0	0
0	0	1	0
4	1	2	0
2	0	3	0
2	0	0	1
0	1	-1	1
2	0	0	1

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} \sum_i x_{1,i}^2 & \sum_i x_{1,i}x_{2,i} & \sum_i x_{1,i}x_{3,i} \\ \sum_i x_{1,i}x_{2,i} & \sum_i x_{2,i}^2 & \sum_i x_{2,i}x_{3,i} \\ \sum_i x_{1,i}x_{3,i} & \sum_i x_{2,i}x_{3,i} & \sum_i x_{3,i}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i y_i x_{1,i} \\ \sum_i y_i x_{2,i} \\ \sum_i y_i x_{3,i} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.873 \\ 0.968 \end{bmatrix}$$

Variance of Error and the Parameters

$$e_t = y_t - x_{t1}\beta_1 - x_{t2}\beta_2 - x_{t3}\beta_3$$
$$= y - 1.6x1 - 0.873x2 - 0.968x3$$

$$\sum_i \hat{e}_t^2 = 6.946 \; ; \; \hat{\sigma}^2 = \frac{6.946}{9-3} = 1.158$$

Computed covariance matrix

$$\operatorname{cov}(b_1, b_2) = \hat{\sigma}^2 (X'X)^{-1} = 1.158 \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.063 & 0.0186 \\ 0 & 0.016 & 0.254 \end{bmatrix} = \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}$$

$$\begin{bmatrix} SE\left(\hat{\beta}_1\right) \\ SE\left(\hat{\beta}_2\right) \\ SE\left(\hat{\beta}_3\right) \end{bmatrix} = \begin{bmatrix} \sqrt{0.2315} \\ \sqrt{0.0735} \\ \sqrt{0.294} \end{bmatrix} = \begin{bmatrix} 0.48118 \\ 0.2711 \\ 0.5422 \end{bmatrix}$$

Test of Restrictions -1

ase of (d) $Rb = r \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$F = \frac{\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \right]' \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}{J}$$

$$F = \frac{\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}{J}$$

Test of Restrictions -2

$$F = \frac{\begin{bmatrix} 1.6 & 0.873 & 0.968 \end{bmatrix} \begin{bmatrix} 0.2315 & 0 & 0 \\ 0 & 0.0735 & 0.0184 \\ 0 & 0.0184 & 0.294 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.6 \\ 0.873 \\ 0.968 \end{bmatrix}}{3}$$

= 7.79 .F critical value for d.f.= (3,6) at 5% confidence interval is 4.76.

F calculated is bigger than F critical => Reject null hypothesis, which says $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$.

At least one of these parameters is significant and explains variation in y.

Shazam Program to Test of Restrictions- from HGJ 10.2

Single and Multiple Restrictions	Calculation of F-statistics

OLS y x1 x2 x3/noconstant pcov
test
    test x1 =0
end
OLS y x1 x2 x3/noconstant pcov
test
    test x1 =0
    test x2=0
end
OLS y x1 x2 x3/noconstant pcov
test
    test x1 =0
    test x2=0
end
read pcov / rows=3 cols=3
0.23153 0.0000 0.000
0.0000 0.073503 0.0184
0.0000 0.018376 0.29401
	read bigr/ rows=3 cols=3
	1 0 0
	0 1 0
	0 0 1
	read sr/ rows=3 cols=1
	0
	0
	0
	matrix invcov=inv(pcov)
	print invcov
	matrix F=(beta'*invcov*beta)/3
	matrix inv2 =inv((bigr*pcov*bigr'))
	matrix FF = ((bigr*beta-sr)'*inv2*(bigr*beta-
	    sr))/3
	print F ff

Shazam Program to Test of Restrictions-from HGJ 10.2

Estimates:

VARIABLE NAME	ESTIMATED COEFFICIENT	STANDARD ERROR	T-RATIO 6 DF	P-VALUE	PARTIAL CORR.	STANDARDIZED COEFFICIENT	ELASTICITY AT MEANS
X1	1.6000	0.48118	3.3252	0.0159	0.8051	0.75425	0.40000
X2	0.87302	0.27111	3.2201	0.0181	0.7959	0.71281	0.43651
X3	0.96825	0.54223	1.7857	0.1244	0.5891	0.43034	0.16138

Computed variance-covariance matrix

VARIANCE-COVARIANCE MATRIX OF COEFFICIENTS

	X1	X2	X3
X1	0.23153		
X2	0.0000	0.73503E-01	
X3	0.0000	0.18376E-01	0.29401

```
*F-statistics for the first restriction
|_test
 |_ test x1 =0
 |_end
F STATISTIC =   11.056673     WITH    1 AND    6 D.F.  P-VALUE= 0.01590
WALD CHI-SQUARE STATISTIC =   11.056673     WITH    1 D.F.  P-VALUE= 0.00088
```

=> restriction is not valid because the computed F is greater than the critical F(1,6) = 13.75

```
*F-statistics for the second restriction
F STATISTIC =   10.712866     WITH    2 AND    6 D.F.  P-VALUE= 0.01047
 WALD CHI-SQUARE STATISTIC =   21.425731     WITH    2 D.F.  P-VALUE= 0.00002
```

=> restriction is not valid because the computed F is greater than the critical F(2,6) = 10.92

```
*F-statistics for the third restriction
F STATISTIC =   7.7897623     WITH    3 AND    6 D.F.  P-VALUE= 0.01716
 WALD CHI-SQUARE STATISTIC =   23.369287     WITH    3 D.F.  P-VALUE= 0.00003
```

=> restriction is not valid because the computed F is greater than the critical F(1,6) = 9.15

```
read pcov / rows=3 cols=3
0.23153 0.0000    0.000
0.0000  0.073503  0.0184
0.0000  0.018376  0.29401

read bigr/ rows=3 cols=3
1  0  0
0  1  0
0  0  1

read sr/ rows=3 cols=1
0
0
0

matrix invcov=inv(pcov)
print invcov
matrix F=(beta'*invcov*beta)/3
matrix inv2 =inv((bigr*pcov*bigr'))
matrix FF = ((bigr*beta-sr)'*inv2*(bigr*beta-sr))/3
print F ff

|_read pcov / rows=3 cols=3
       3 ROWS AND           3 COLUMNS, BEGINNING AT ROW       1
 |_read bigr/ rows=3 cols=3
       3 ROWS AND           3 COLUMNS, BEGINNING AT ROW       1
 |_read sr/ rows=3 cols=1
    1 VARIABLES AND        3 OBSERVATIONS STARTING AT OBS       1


 |_matrix invcov=inv(pcov)
 |_print invcov
     INVCOV
     3 BY     3 MATRIX
    4.319095       0.000000       0.000000
    0.000000       13.82113     -0.8649666
    0.000000     -0.8638383       3.455306
 |_matrix F=(beta'*invcov*beta)/3
 |_matrix inv2 =inv((bigr*pcov*bigr'))
 |_matrix FF = ((bigr*beta-sr)'*inv2*(bigr*beta-sr))/3
 |_print F ff
     F
    7.789598
     FF
    7.789598
```