Economic Forecasting

Autocorrelation

Autocorrelation: Causes and Consequences

- Causes:

  inertia

  specification bias

  cobweb phenomena

  manipulation of data

- Consequences

  unbiased and linear estimators

  but

  they there not the best estimators

they are inefficient

Autocorrelation

Assumption behind the OLS

$$\operatorname{cov}(e_i, e_j) = 0 \quad \textit{for all} \quad i \neq j$$

Autocorrelation exists when

$$\operatorname{cov}(e_i, e_j) \neq 0 \quad \textit{for all} \quad i \neq j$$

$$e_i = \rho e_{i-1} + v_i \qquad\qquad v_i \sim N\left(0, \sigma^2\right)$$

ρ **correlation coefficient between –1 and 1**

Variance of the error term

$$\operatorname{var}(e_i) = \operatorname{var}(\rho e_{i-1} + v_i) = \rho^2 \operatorname{var}(e_{i-1}) + \operatorname{var}(v_i)$$
$$+ 2\operatorname{cov}(e_{i-1} v_i)$$

$$\sigma_e^2 = \rho^2 \sigma_e^2 + \sigma_v^2 \quad => \quad \sigma_e^2 = \frac{\sigma_v^2}{1-\rho^2}$$

Specifically:

$$\rho = \frac{\operatorname{cov}(e_t e_{t-1})}{\sqrt{\operatorname{var}(e_t)}\sqrt{\operatorname{var}(e_{t-1})}}; \qquad \operatorname{cov}(e_t e_{t-1}) = \rho \sigma_e^2 \qquad (9.3)$$

Consequence of Autocorrelation

OLS estimate is still unbiased

$$\hat{\beta}_2 = \frac{\sum\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sum\left(x_i - \bar{x}\right)^2} = \frac{\sum\left(x_i - \bar{x}\right)y_i}{\sum\left(x_i - \bar{x}\right)^2} = \sum w_i y_i = \sum w_i\left(\beta_1 + \beta_2 x_i + e_i\right) (9.4$$

$$E\left(\hat{\beta}_2\right) = E\left[\sum w_i y_i\right] = E\left[\sum w_i\left(\beta_1 + \beta_2 x_i + e_i\right)\right] = E\left[\sum w_i \beta_1 + \beta_2 \sum w_i x_i + \sum w_i e_i\right] = \beta_2$$

but the variance of OLS estimator is no longer efficient.

$$\operatorname{var}\left(\hat{\beta}_2\right) = E\left[E\left(\hat{\beta}_2\right) - \beta_2\right] = E\left[\sum w_i e_i\right]^2 = \left[\sum \frac{\left(x_i - \bar{x}\right)}{\sum_i \left(x_i - \bar{x}\right)^2}\right]^2 E\left(e_i\right)^2 + 2\left[\sum \frac{\left(x_i - \bar{x}\right)}{\sum_i \left(x_i - \bar{x}\right)^2}\right]^2 E\left(e_i e_j\right) =$$

$$\left[\sum \frac{\left(x_i - \bar{x}_i\right)}{\sum_i \left(x_i - \bar{x}_i\right)^2}\right]^2 \sigma^2 + 2\left[\sum \frac{\left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_j\right)}{\sum_i \left(x_i - \bar{x}_i\right)^2}\right] \sigma^2 \rho^s \quad (9.6)$$

$$= \hat{\sigma}^2 \frac{1}{\sum_i \left(x_i - \bar{x}\right)^2}\left[1 + \rho \frac{\sum_i x_i x_{i-1}}{\sum_i x_i^2} + 2\rho^2 \frac{\sum_i x_i x_{i-1}}{\sum_i x_i^2} + \ldots. + 2\rho^{n-1} \frac{\sum_i x_i x_{i-1}}{\sum_i x_i^2}\right] (9.6)$$

Difference in the Variance of Parameters in Case of Autocorrelation

The variance of Errors with K number of explanatory variables (including the intercept term)

$$\hat{\sigma}^2 = \frac{\hat{e}\hat{e}'}{T-K} = \frac{\left(Y-\hat{\beta}X\right)\left(Y-\hat{\beta}X\right)'}{T-K}$$

The estimator of β and the variance-covariance of the error matrix can be used to estimate the covariance matrix for $\hat{\beta}$ as following:

$$\hat{\beta} = \left(X'X\right)^{-1}X'Y = \beta + \left(X'X\right)^{-1}X'e$$

$$\operatorname{cov}\left(\hat{\beta}\right) = E\left[\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)'\right]$$

$$= E\left[\left(X'X\right)^{-1}X'e\left(\left(X'X\right)^{-1}X'e\right)'\right] = \hat{\sigma}^2\left(X'X\right)^{-1}$$

But with autocorrelation

$$\operatorname{cov}\left(\hat{\beta}\right) = E\left[\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)'\right]$$

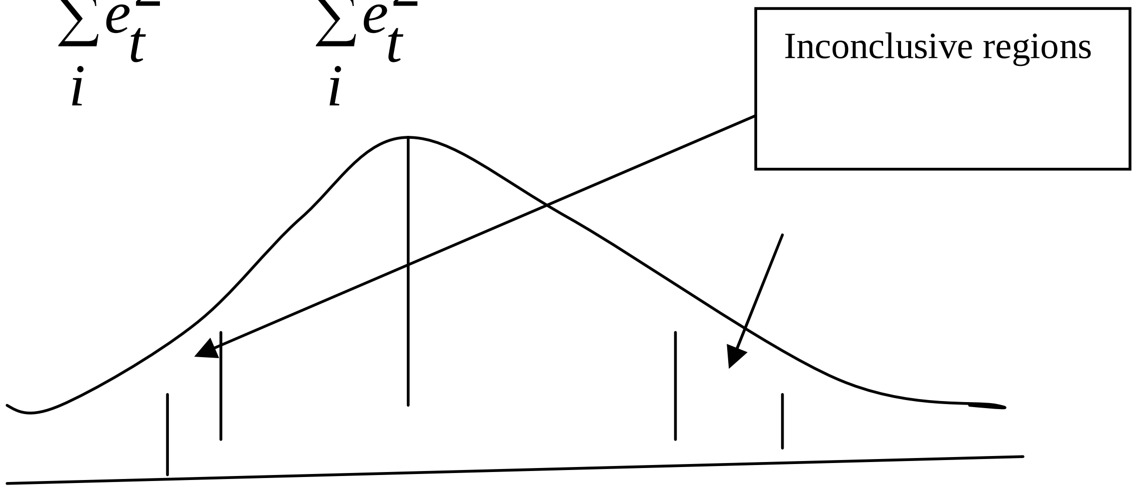
$$= E\left[\left(X'X\right)^{-1}X'e\left(\left(X'X\right)^{-1}X'e\right)'\right] = \hat{\sigma}_{i,j}\left(X'X\right)^{-1}$$

Variance of parameters is large compared to No autocorrelation

Durbin-Watson test

$$\hat{d} = \frac{\sum_{i}^{T}\left(\hat{e}_t - \hat{e}_{t-1}\right)^2}{\sum_i \hat{e}_t^2} = \frac{\sum_i \hat{e}_t^2 - 2\sum_i \hat{e}_t \hat{e}_{t-1} + \sum_i \hat{e}_{t-1}^2}{\sum_i \hat{e}_t^2} \approx 2\left(1 - \hat{\rho}\right) \quad (9.8)$$

$$\hat{d} = \frac{\sum_i \hat{e}_t^2}{\sum_i \hat{e}_t^2} - \frac{2\sum_i \hat{e}_t \hat{e}_{t-1}}{\sum_i \hat{e}_t^2} + \frac{\sum_i \hat{e}_{t-1}^2}{\sum_i \hat{e}_t^2} = \left(2 - 2\rho\right) = 2\left(1 - \hat{\rho}\right)$$



$d = 0; \rho = 1 \qquad d = 2; \rho = 0 \qquad d = 4; \rho = -1$

Remedial Measure when ρ is Known

Take a lag of the original model; and multiply it by ρ; and subtract from the original model to find a transformed model.

$$\rho Y_{t-1} = \rho\beta_1 + \rho\beta_2 x_{t-1} + \rho e_{t-1}$$

$$Y_t - \rho Y_{t-1} = \beta_1 - \rho\beta_1 + \beta_2 x_{t-1} - \rho\beta_2 x_{t-1} + e_{t-1} - \rho e_t$$

Transformed model

$$Y_t^* = \beta_1^* + \beta_2^* x_t^* + e_t^*$$ Where

$$Y_t^* = Y_t - \rho Y_{t-1};\ X_t^* = X_t - \rho X_{t-1};$$

$$e_t^* = e_t - \rho e_{t-1};\ \beta^* = \beta_1 - \rho\beta_1.$$

OLS estimates unknown parameters of this transformed model will be BLUE.

Retrieve β_1 of the original model from estimates of β^*.

Bruce-Godfrey higher order autocorrelation test

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \rho_3 u_{t-3} + \rho_4 u_{t-4} + \rho_5 u_{t-5} + \rho_6 u_{t-6} + \ldots + \rho_7 u_{t-p} + e_t$$

1. regress the original model and get estimates of $\hat{u}_i$
2. form the regression of error in terms of its lagged values and estimate $\hat{\rho}_i$.
3. BG test has a distribution $(n-p).R^2 \sim \chi^2_p$ with p degrees of freedom
4. Compare calculated $(n-p).R^2 \sim \chi^2_p$ with critical χ^2_p distribution from the table and accept $\hat{\rho}_i$ =0 if the calculated value χ^2_p is less than table value.

Cochrane-Orcutt iterative procedure

This method is similar to the above ones, except that it involves multiple iteration for estimating $\hat{\rho}_i$. Steps are as following:

1. Estimate $\hat{\beta}_1$ and $\hat{\beta}_2$ the original model; get error terms and estimate $\hat{\rho}_i$
2. Transform the original model multiplying it by $\hat{\rho}_i$ and by taking the first difference,
3. Estimate $\hat{\hat{\beta}}_1$ and $\hat{\hat{\beta}}_2$ from the transformed model and get errors of this transformed model
4. Then again estimate $\hat{\hat{\rho}}_i$ and use those values to transform the original model

$$Y_t - \hat{\rho}Y_{t-1} = \beta_1 - \hat{\rho}\beta_1 + \beta_2 x_{t-1} - \hat{\rho}\beta_2 x_{t-1} + e_{t-1} - \hat{\rho}e_{t-1}$$

.

5. Continue this iteration process until ρ converges; $\hat{\hat{\rho}}_i$ =0.

Diagnos /ACF options in OLS in Shazam will generate these iterations.

Remedy for Autocorrelation: GLS Estimator

Using a theorem in matrix algebra W can be decomposed into two parts as

$$P'P = W^{-1}$$

Use this partition of W to transform the original model as

$$PY = PX\beta + Pe$$

$$Y^{*} = X^{*}\beta + e^{*}$$

$$\beta_{GLS} = \left(X^{*}{}'X^{*}\right)^{-1} X^{*}{}'Y^{*} = \left(X'P'PX\right)X'P'PY \Rightarrow$$

$$\beta_{GLS} = \left(X^{*}{}'W^{-1}X^{*}\right)^{-1} X^{*}{}'W^{-1}Y^{*}$$

When ρ is not known (assume $\rho = \pm 1$ and transform the model, Variance and Covariance of Errors in Case of Autocorrelation Compared to the Variance of a Normal Error Term

$$E[ee'] = \begin{bmatrix} \text{var}(e_1) & \text{cov}(e_1e_2) & \text{cov}(e_1e_3) & \text{cov}(e_1e_4) & \text{cov}(e_1e_5) \\ \text{cov}(e_2e_1) & \text{var}(e_2) & \text{cov}(e_2e_3) & \text{cov}(e_2e_4) & \text{cov}(e_2e_5) \\ \text{cov}(e_3e_1) & \text{cov}(e_3e_2) & \text{var}(e_3) & \text{cov}(e_3e_4) & \text{cov}(e_3e_5) \\ \text{cov}(e_4e_1) & \text{cov}(e_4e_2) & \text{cov}(e_4e_3) & \text{var}(e_4) & \text{cov}(e_4e_5) \\ \text{cov}(e_5e_1) & \text{cov}(e_5e_2) & \text{cov}(e_5e_3) & \text{cov}(e_5e_4) & \text{var}(e_5) \end{bmatrix}$$

$$E[ee'] = \begin{bmatrix} \sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix} \qquad \longleftarrow$$

$$E[ee'] = \sigma_{i,j} I$$ where I is 5×5 identity matrix.

Variance and Covariance of Errors in Case of Autocorrelation Compared to the Variance of a Normal Error Term

$$E[ee'] = \begin{bmatrix} \sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} & \sigma_{11} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix}$$

$\leftarrow$

$$E[ee'] = \begin{bmatrix} \sigma^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma^2 & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma^2 & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma^2 & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma^2 \end{bmatrix}$$

$$E[ee'] = \sigma_{i,j} I$$ where I is 5×5 identity matrix.

Illustration of Shazam Program Autocorrelation Corrected Estimations

```
read year      y     m4  p     l     r
1970      418032  26643     19.623797    6.56

dim w1 32

ols y p l/list  resid=w1
diagnos/ACF

matrix w=diag(w1)
matrix ww=w*w
auto y p l/order=8 list


*maximum likelihood estimator
auto y p l/ml

*Cochrane-Orcutt iterative procedure
auto y p l/iter=20

*Generalised list square estimator
gls y p l /pmatrix=ww


Stop
```