Macroeconometric Models

Simultaneous Equations Model

Estimation, Simulations and Forecasts

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A simple Example of the Simultaneous Equation System

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{21} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$y_{1i} = \frac{(a_{22}b_{11} - a_{12}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(a_{22}b_{12} - a_{12}b_{12})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{1i}$$

$$y_{2i} = \frac{(-a_{21}b_{11} + a_{11}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(-a_{21}b_{12} + a_{11}b_{22})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{2i}$$

General Form for the Simultaneous Equation System

$$a_{11}y_{1i} + a_{12}y_{2i} + .. + a_{1m}y_{mi} + b_{11}x_{1i} + b_{12}x_{2i} + .. + b_{1k}x_{ki} = e_{1i}$$

$$a_{21}y_{1i} + a_{22}y_{2i} + .. + a_{2m}y_{mi} + b_{21}x_{1i} + b_{22}x_{2i} + .. + b_{2k}x_{ki} = e_{2i}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}y_{1i} + a_{m2}y_{2i} + .. + a_{mm}y_{mi} + b_{m1}x_{1i} + b_{m2}x_{2i} + .. + b_{mk}x_{ki} = e_{mi}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ . \\ . \\ y_{mi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1m} \\ a_{21} & a_{22} & .. & a_{2m} \\ & & & \\ & & & \\ a_{m1} & a_{m2} & .. & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} & .. & b_{1k} \\ b_{21} & b_{22} & .. & b_{2k} \\ & & & \\ & & & \\ b_{m1} & b_{m2} & .. & b_{mk} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \\ . \\ . \\ x_{mi} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & .. & a_{1m} \\ a_{21} & a_{22} & .. & a_{2m} \\ & & & \\ & & & \\ a_{m1} & a_{m2} & .. & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \\ \\ \\ e_{mi} \end{bmatrix}$$

Impact and Shock Analysis in a Simultaneous Equation System

$$AY_i + BX_i = U_i$$

$$Y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ . \\ . \\ y_{mi} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1m} \\ a_{21} & a_{22} & .. & a_{2m} \\ & & & \\ a_{m1} & a_{m2} & .. & a_{mm} \end{bmatrix} \quad X_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ . \\ . \\ x_{mi} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & .. & b_{1k} \\ b_{21} & b_{22} & .. & b_{2k} \\ & & & \\ b_{m1} & b_{m2} & .. & b_{mk} \end{bmatrix} \quad U_i = \begin{bmatrix} e_{1i} \\ e_{2i} \\ \\ e_{mi} \end{bmatrix}$$

$$Y_i = -A^{-1}BX_i + A^{-1}U_i \qquad \hat{Y}_i = -A^{-1}BX_i$$

$$Var\left(Y_i - \hat{Y}_i\right) = A^{-2}\sigma_u^2$$

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Macroeconometric (IS-LM) Model

Consumption function: $$C_t = \beta_0 + \beta_1 \left(Y_t - T_t \right) + \varepsilon_{1t} \qquad (1)$$

Taxation function: $$T_t = c_0 + c_1 Y_t + \varepsilon_{2t} \qquad (2)$$

Import function: $$M_t = m_0 + m_1 Y_t + \varepsilon_{3t} \qquad (3)$$

Investment function $$I_t = \mu_0 + \mu_1 R_t + \phi \Delta Y_{t-1} + \varepsilon_{4t} \qquad (4)$$

National income identity $$Y_t = C_t + I_t + G_t + X_t - M_t \qquad (5)$$

Money market (LM curve): $$\left(\frac{MM_t}{P_t} \right) = b_0 + b_1 Y_t - b_2 R_t + \varepsilon_{6t} \qquad (6)$$

Money market (LM curve): $$R_t = \frac{b_0}{b_2} - \frac{1}{b_2} \left(\frac{MM_t}{P_t} \right) + \frac{b_1}{b_2} Y_t + \varepsilon_{6t} \qquad (6)$$

Y_t, C_t, M_t, I_t, R_t and T_t are endogenous variables and G_t, X_t, MM_t and ΔY_{t-1} are predetermined or exogenous variables representing government expenditure, exports, money stock and change in Y_t in the previous period respectively. Set the price level P_t equal to unity for simplicity.

Reduced form of the Macro Model

$$C_t = \Pi_{10} + \Pi_{11} G_t + \Pi_{12} X_t + \Pi_{13} MM_t + \Pi_{14} \Delta Y_{t-1} + V_1$$

$$M_t = \Pi_{20} + \Pi_{21} G_t + \Pi_{22} X_t + \Pi_{23} MM_t + \Pi_{24} \Delta Y_{t-1} + V_2$$

$$I_t = \Pi_{30} + \Pi_{31} G_t + \Pi_{32} X_t + \Pi_{33} MM_t + \Pi_{34} \Delta Y_{t-1} + V_3$$

$$R_t = \Pi_{40} + \Pi_{41} G_t + \Pi_{42} X_t + \Pi_{43} MM_t + \Pi_{44} \Delta Y_{t-1} + V_4$$

$$T_t = \Pi_{50} + \Pi_{51} G_t + \Pi_{52} X_t + \Pi_{53} MM_t + \Pi_{54} \Delta Y_{t-1} + V_5$$

$$Y_t = \Pi_{60} + \Pi_{61} G_t + \Pi_{62} X_t + \Pi_{63} MM_t + \Pi_{64} \Delta Y_{t-1} + V_6$$

Rank and Order Conditions of Identification

Order condition: $K - k \geq m - 1$

Rank condition: $\rho(A) \geq (M-1)(M-1)$ => order of the matrix

M = number of endogenous variables in the model

K = number of exogenous variables in the model including the intercept

m = number of endogenous variable in an equation

k = number of exogenous variables in a given equation

Rank condition is defined by the rank of the matrix, which should have a dimension (M-1), where M is the number of endogenous variables in the model.

How to Determine the Rank of the Matrix?

Rank of matrix is the order of non-singular matrix

Rank matrix is formed from the coefficients of the variables (both endogenous and exogenous) excluded from that particular equation but included in the other equations in the model.

The rank condition tells us whether the equation under consideration is identified or not.

The order condition tells us if it is exactly identified or overidentified.

Order and rank conditions of identification

1. If $K-k>m-1$ and the rank of the $\rho(A)$ is M-1 then the concerned equation is overidentified.

2. If $K-k=m-1$ and the rank of the $\rho(A)$ is M-1 then the equation is exactly identified.

3. If $K-k\geq m-1$ and the rank of the $\rho(A)$ is less than M-1 then the equation is underidentified.

If $K-k<m-1$ the structural equation is unidentified. The rank of the $\rho(A)$ is less than M-1 in this case.

Steps for rank condition

1. Write down the system in the tabular form
2. Strike out the coefficients corresponding to the equatio to be identified
3. Strike out the columns corresponding to those coefficients in 2 which are nonzero.
4. The entries left in the table will give only the coefficier of the variables included in the system but not in the equation under consideration. From these coefficients form all possible A matrices of order M-1 and obtain a corresponding determinant. If at least one of these determinants is non-zero then that equation is identifiec

Rank condition: $\rho(A) \geq (M-1)(M-1)$ => order of the matrix.

Table of Coefficients in a Macro Econometric Model

	Constant	Y_t	C_t	M_t	I_t	R_t	T_t	G_t	X_t	MM_t	ΔY_{t-1}
C_t	$-\beta_0$	$-\beta_1$	1	0	0	0	β_1	0	0	0	0
T_t	$-t_0$	$-t_1$	0	$-t_2$	0	0	1	0	0	0	0
M_t	$-m_0$	$-m_1$	1	1	0	0	$-t_2$	0	0	0	0
I_t	$-\mu_0$	0	0	0	1	$-\mu_1$	0	0	0	0	0
R_t	$-\frac{b_0}{b_2}$	$-\frac{b_1}{b_2}$	0	0	1	0	0	0	0	$\frac{1}{b_2}$	0
Y_t	0	1	-1	1	-1	0	0	-1	-1	0	0

Checking Rank Identification Condition for the above System

Consumption function:

$$A_1 = \begin{bmatrix} t_2 & 0 & 0 & 0 \\ 1 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \qquad |A_1| = -\frac{1}{b_2}\phi m_2 t_2 \qquad \rho(A_1) = 4$$

It is obvious that there exists at least on non-singular matrix of order *M*-1.

Tax function:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad \Rightarrow \quad |A_1| = -\frac{1}{b_2}\phi m_2 \qquad \rho(A_1) = 4$$

Checking Rank Identification Condition for the above System

Import

$$A_1 = \begin{bmatrix} 1 & \beta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad |A_1| = -\frac{1}{b_2}\phi \quad \rho(A_1) = 4$$

Interest Rate

$$A_1 = \begin{bmatrix} \beta_1 & 0 & -\beta_1 & 0 \\ t_1 & t_2 & 1 & 0 \\ m_1 & 1 & m_3 & 0 \\ \frac{b_1}{b_2} & 0 & 0 & \frac{1}{b_2} \end{bmatrix} \quad |A_1| = \beta_1 t_2 m_3 \frac{1}{b_2} + \beta_1 t_2 m_1 \frac{1}{b_2} \quad \rho(A_1) = 4$$

Investment

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 1 & m_2 & 0 \\ 0 & 0 & -\mu_1 & \phi \end{bmatrix} \quad |A_1| = t_2 m_2 \phi \quad \rho(A_1) = 4$$

Some Techniques for Estimation of the Simultaneous Equations System

Single Equations Methods
Ordinary Least Squares
Indirect Least Squares
Two Stage Least Squares Method

System Method
Three Stage LS:Generalised Least Square
Seemingly Unrelated Regression Equations
Full Information Maximum Likelyhood

Estimation by the Generalised Least Square Method

$$\operatorname{cov}[vv'] = \begin{bmatrix} \operatorname{var}(v_1) & \operatorname{cov}(v_1 v_2) & \operatorname{cov}(v_1 v_3) & . & \operatorname{cov}(v_1 v_5) \\ \operatorname{cov}(v_2 v_1) & \operatorname{var}(v_2) & \operatorname{cov}(v_2 v_3) & . & \operatorname{cov}(v_2 v_5) \\ \operatorname{cov}(v_3 v_1) & \operatorname{cov}(v_3 v_2) & \operatorname{var}(v_3) & . & \operatorname{cov}(v_3 v_5) \\ . & . & . & . & . \\ \operatorname{cov}(v_5 v_1) & \operatorname{cov}(v_5 v_2) & \operatorname{cov}(v_5 v_3) & . & \operatorname{var}(v_v) \end{bmatrix} = \Omega$$

$$P'P = \Omega^{-1}$$

$$PY = PX\beta + Pe \qquad Y^* = X^*\beta + e^*$$

$$\beta_{GLS} = \left(X^{*}{}'X^{*}\right)^{-1} X^{*}{}'Y^{*} = \left(X'P'PX\right)X'P'PY$$

$$\beta_{GLS} = \left(X^{*}{}'\Omega^{-1}X^{*}\right)^{-1} X^{*}{}'\Omega^{-1}Y^{*}$$

Estimation of Simultaneous System using PcGive

C = + 0.2128*M4 + 1.407*G + 0.1767*X + 8.059e+004
(SE) (0.0266) (0.212) (0.154) (1.63e+004)

Y = + 0.2617*M4 + 2.86*G + 1.348*X + 8.862e+004
(SE) (0.0454) (0.361) (0.262) (2.78e+004)

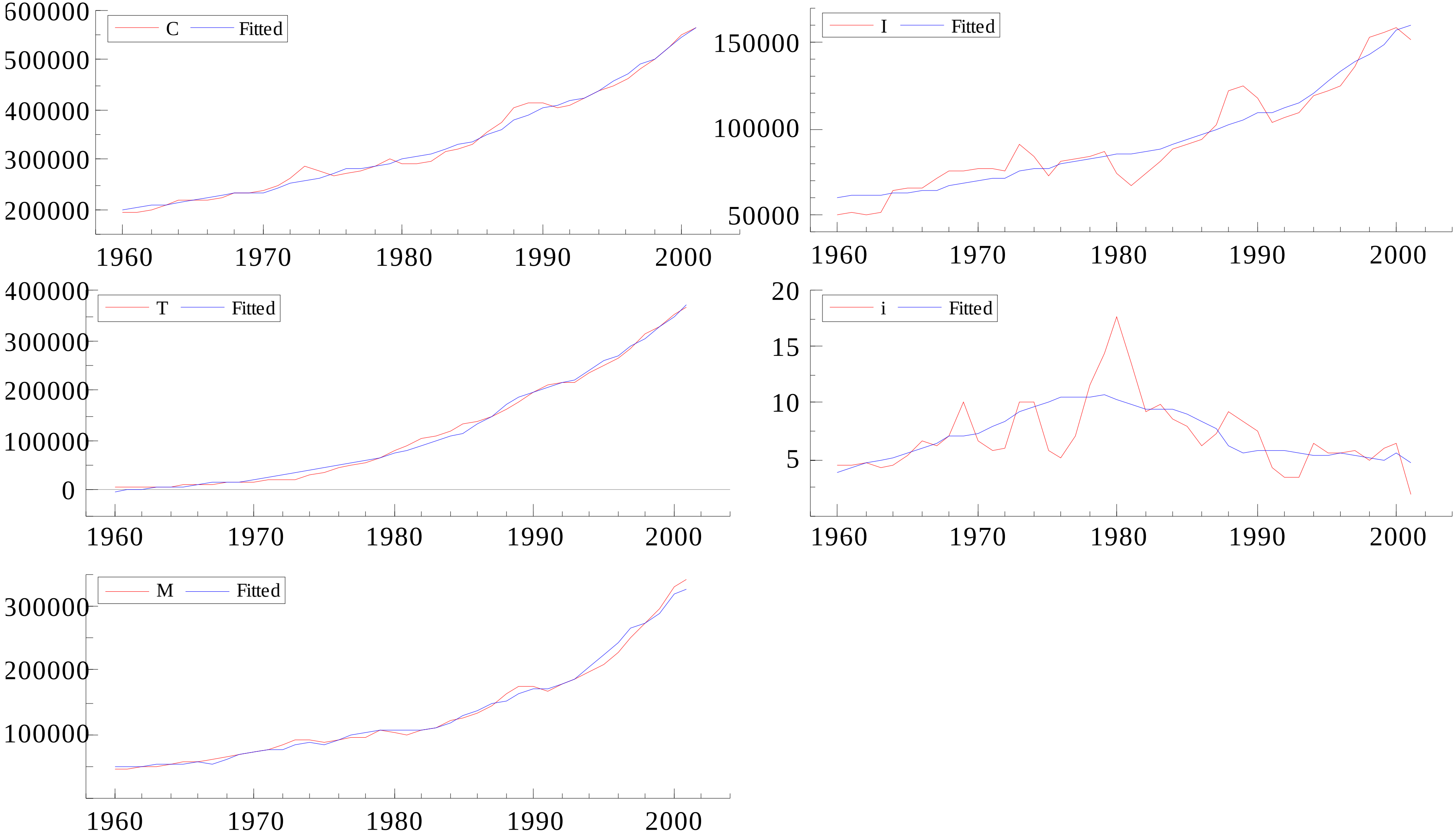
T = + 0.3204*M4 + 0.9521*G - 0.08909*X - 7.533e+004
(SE) (0.02) (0.159) (0.116) (1.23e+004)

M = + 0.06508*M4 - 0.4738*G + 1.003*X + 4.34e+004
(SE) (0.0198) (0.157) (0.114) (1.21e+004)

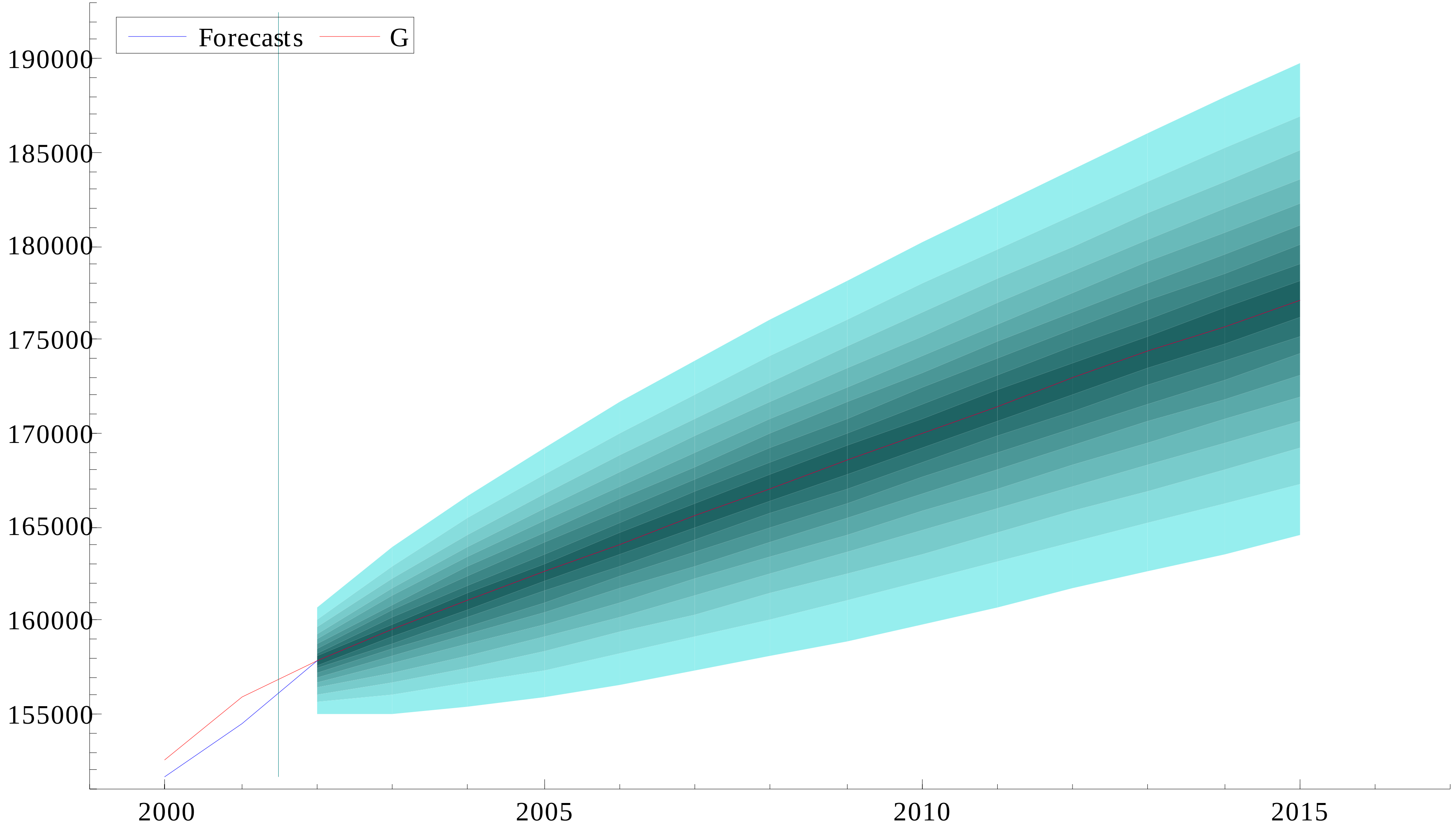
r = - 2.384e-005*M4 + 0.0001148*G + 6.273e-005*X - 7.408
(SE) (6.2e-006) (4.93e-005) (3.58e-005) (3.79)

I = + 0.02907*M4 + 0.0684*G + 0.2681*X + 4.292e+004
(SE) (0.0229) (0.182) (0.132) (1.4e+004)

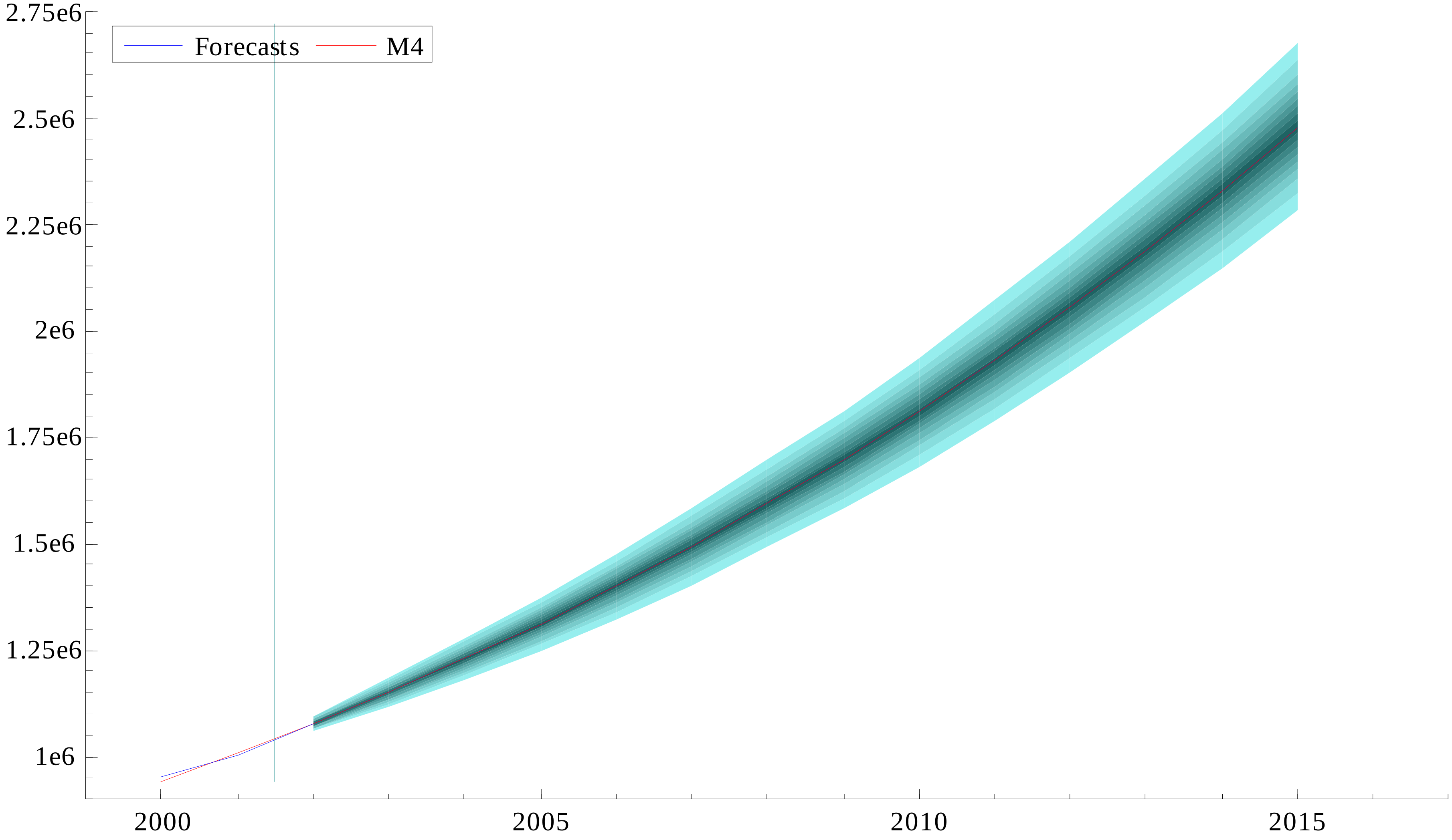
Actual and Predicted Values in above Macroeconomic Model
Historical Simulations



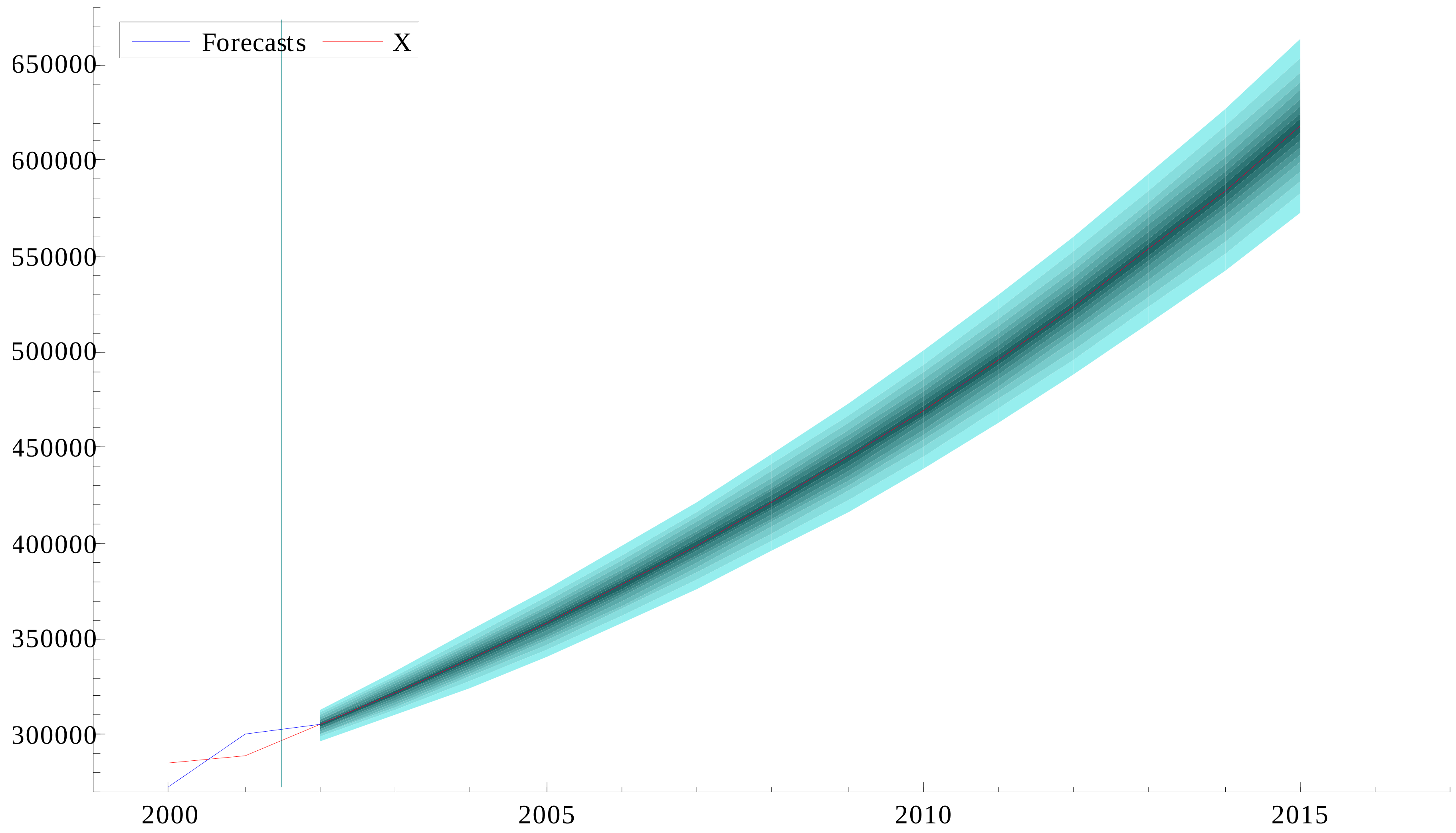
Forecast of Exogenous Variable AR(2) : Government Spending



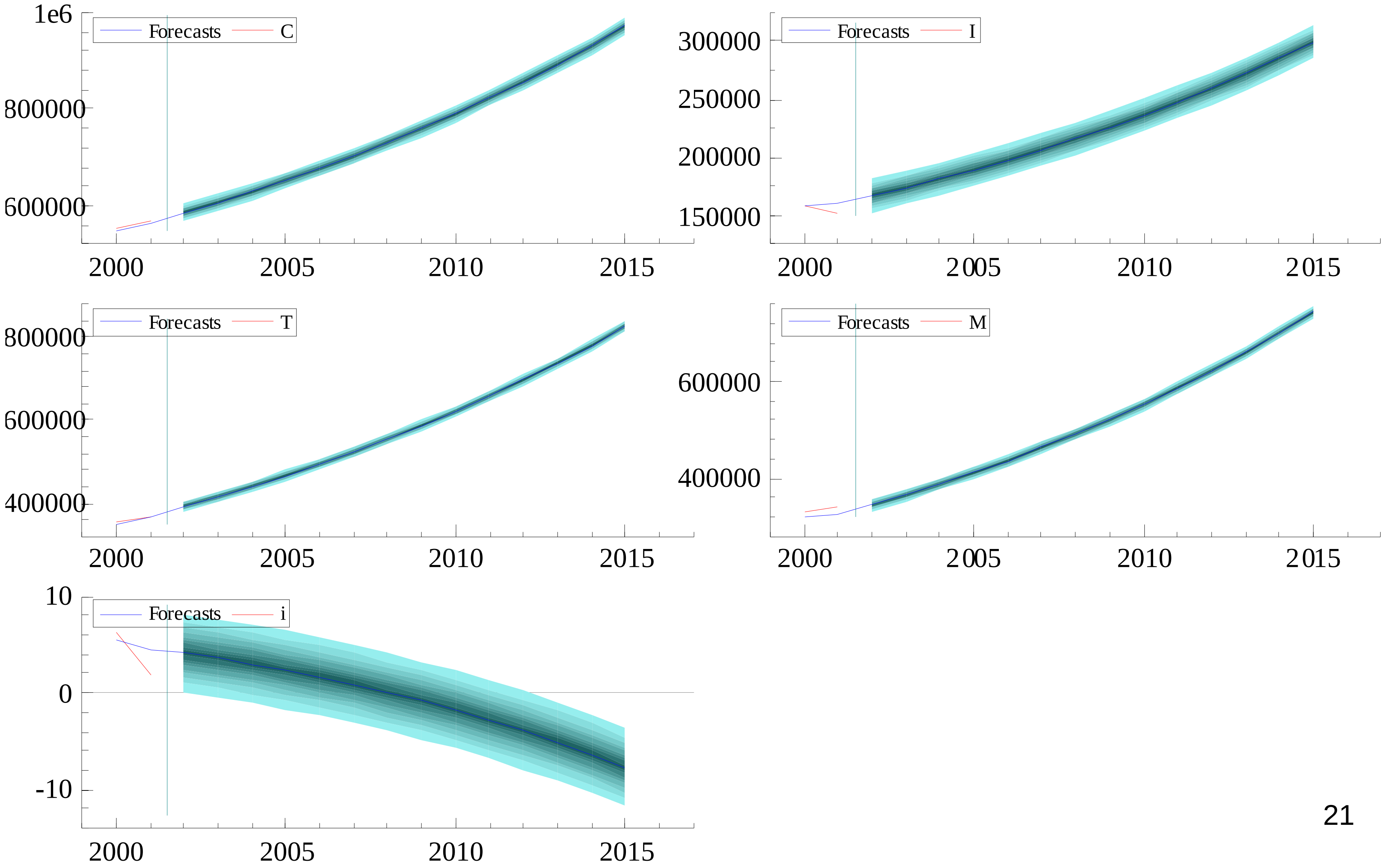
Forecast of Exogenous Variable AR(2) : Money Supply



Forecast of Exogenous Variable AR(2) : Exports



Ex-ante Forecast Endogenous Variables from the Simultaneous Equation Model Consumption, Tax, Investment and the Interest Rate



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