# Rational Expectation Models and Estimation Techniques

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# Rational Expectation

- Rational people take account of all available information while making their decision.
- information set is given by $\Omega_t$
- Unconditional expextation $\left( E_{t-1} X_{t+1} / \Omega_t \right) = \tilde{X}_t$
- Conditional expextation $\left( E_t^* X_{t+1} / \Omega_t \right) = \tilde{X}_t$
- Three methods of forming a rational expectation

1. Survey of opinion – This relies on asking people, for instance to the CEOs or consumers, about their confidence about the economy.
2. Using current value of variable as the best predictor of future.
3. Extrapolative model based forecasts (Lucas (1976), Wallis (1977), Lee et.a. (2000)).

# Naive Expectation

If investment $y_t$ of a firm depends on expected profits next periods $x^*_{t+1}$ it is difficult to estimated a and b parameters because $x^*_{t+1}$ is not yet observed. This just a subjective estimate.

$$y_t = a + bx^*_{t+1} + u_t$$

Naive approach to form expectation is to use available information to estimate $x^*_{t+1}$.
Linearly as $x^*_{t+1} = x_t$ or $x^*_{t+1} - x_t = x_t - x_{t-1}$

$$x^*_{t+1} = 2x_t - x_{t-1}$$

In percentage terms as $\frac{x^*_{t+1}}{x_t} = \frac{x_t}{x_{t-1}}$ or $x^*_{t+1} = \frac{x_t^2}{x_{t-1}}$; In case of quarterly or monthly series this can be written as $\frac{x^*_{t+1}}{x_{t-3}} = \frac{x_t}{x_{t-4}}$; $\frac{x^*_{t+1}}{x_{t-11}} = \frac{x_t}{x_{t-12}}$

Accuracy of such models is can be judged looking at the average absolute error

$$AEA = \frac{1}{n}\sum\left|actual - predicted\right|$$

## Adaptive Expectation: Definition

$$x_{t+1}^* = \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \ldots\ldots + \beta_k x_{t-k}$$

$$x_{t+1}^* = \sum_{t=0}^{\infty} \beta_i x_{t-i} \quad \text{where } \beta_i = \beta_0 \lambda^i \text{ and } 0 \le \lambda \le 1$$

$$\sum \beta_0 \lambda^i = 1 \quad = \frac{\beta_0}{1-\lambda}$$

$$x_{t+1}^* = \sum_{t=0}^{\infty} \beta_i x_{t-i} = \sum_{t=0}^{\infty} (1-\lambda)\lambda^i x_{t-i}$$

Taking one period lag and by multiplying $\lambda$

$$\lambda x_t^* = \lambda \sum_{t=0}^{\infty} (1-\lambda)\lambda^i x_{t-i} = \sum_{t=0}^{\infty} (1-\lambda)\lambda^{i+1} x_{t-i-1}$$

By subtraction

$$x_{t+1}^* - \lambda x_t^* = (1-\lambda) x_t$$

$$x_{t+1}^* - x_t^* = (1-\lambda)(x_t - x_t^*)$$

Thus revision in expectation $x_{t+1}^* - x_t^*$ is function of the past error $(x_t - x_t^*)$.

# Adaptive Expectation: Estimation

$$y_t = a + bx^*_{t+1} + u_t$$

Taking one period lag and by multiplying $\lambda$

$$\lambda y_{t-1} = \lambda a + \lambda b x^*_t + \lambda u_{t-1}$$

$$y_t - \lambda y_{t-1} = (1-\lambda)a + b\left(x^*_{t+1} - \lambda x^*_t\right) + u_t - \lambda u_{t-1}$$

replacing $x^*_{t+1} - \lambda x^*_t = (1-\lambda)x_t$ term

$$y_t - \lambda y_{t-1} = (1-\lambda)a + b(1-\lambda)x_t + u_t - \lambda u_{t-1}$$

$$y_t = a' + \lambda y_{t-1} + b' x_t + v_t$$

where $a' = (1-\lambda)a$ ; $b' = b(1-\lambda)$; $v_t = u_t - \lambda u_{t-1}$

Thus estimation of unobserved expectation variables can be done by observed variables.

There is one problem. The $y_{t-1}$ term in the transformed model is not free from $v_t$ because of $v_t = u_t - \lambda u_{t-1}$. This needs looking for an instruments that are orthogonal to this $\sum_{t=0}^{\infty} v_t x_t = 0$ and $\sum_{t=0}^{\infty} v_t x_{t-1} = 0$.

## Rational Expectation: Definition

If the economic system changes the way how expectations are from need to be changes (Muth (1961)). The adaptive expectation model does not allow such changes to occur $x^*_{t+1} - x^*_t = (1-\lambda)(x_t - x^*_t)$.

Rational expectation should fulfil certain conditions:

First all available information $I_{t-1}$ should have been used and only unanticipated errors should reflect the noise or the prediction error as

$$\varepsilon_t = y_t - y^*_t$$

where $y^*_t$ is the rational expectation of $y_t$ with assumption that $E(\varepsilon_t) = 0$.

$$y_t = y^*_t + \varepsilon_t$$

$$\text{var}(y_t) = \text{var}(y^*_t) + \text{var}(\varepsilon_t)$$

$$y^*_t = E(y_t / I_{t-1})$$

$y^*_t$ is the subjective expectation and $E(y_t / I_{t-1})$ is the objective expectation. The rational expectation rule establishes connection between these two.

## Estimation of Rational Expectation Model

There are mainly two ways to estimate a rational expectation model: instrumental variables method and substation method.

Under the instrumental variable method it is necessary to find out instruments for $y_t^* = y_t - \varepsilon_t$. For instance consider a money demand model

$$m_t - p_t = a + b\left(p_{t+1}^* - p_t\right) + u_t$$

$$p_{t+1}^* = p_{t+1} - v_{t+1}$$

$$m_t - p_t = a + b\left(p_{t+1} - p_t\right) + u_t - bv_{t+1}$$

It is assumed here that $p_{t+1}$ and $v_{t+1}$ are uncorrelated. It is standard to use lags $m_{t-1}$ and $p_{t-1}$ to instrument $p_{t+1}$ or other longer lags. Use two stage least square method to estimate the above equation.

# Strong and weak tests for rationality

One may have survey based information on expectation $y_t^*$ but can test whether rationality is corrected using strong and weak tests of rationality.

Strong test: $\text{var}(y_t) = \text{var}(y_t^*) + \text{var}(\varepsilon_t)$

Many studies have rejected rationality based on strong test (Lovell (1986)).

There are a number of ways of doing weak test of rationality

$$y_t = \beta_0 + \beta_1 y_t^* + \varepsilon_t$$
$$H_0 : \beta_0 = 0; \beta_1 = 1$$

$$y_t - y_t^* = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$$

$H_0 : \alpha_0 = 0; \alpha_1 = 0$ since $y_{t-1}$ is already in the information set $I_{t-1}$, thus already used in forming expectation. It cannot explain any part of error.

$$y_t - y_t^* = \alpha_0 + \alpha_1 \left(y_{t-1} - y_{t-1}^*\right) + \varepsilon_t$$
$$H_0 : \alpha_1 = 0$$

Difference in coefficient procedure

$$y = \beta_1 z_1 + \beta_2 z_2 + u$$
$$y^* = \beta_1^* z_1 + \beta_2^* z_2 + u^*$$
$$y - y^* = \left(\beta_1 - \beta_1^*\right) z_1 + \left(\beta_2 - \beta_2^*\right) z_2 + u - u^*$$

$\left(\beta_1 - \beta_1^*\right) = 0$ and $\left(\beta_2 - \beta_2^*\right) = 0$ imply rational expectation.

## Application of rational expectation: Market model

Demand $q_t = \beta_1 p_t + \gamma_1 z_{1,t} + u_{1,t}$

Supply $q_t = \beta_2 p_t^* + \gamma_2 z_{1,t} + u_{2,t}$

when $z_{1,t}$ and $z_{2,t}$ are known it is simple to replace the $p_t = p_t^* + \varepsilon_t$ $E(\varepsilon_t) = 0$; $\text{cov}(\varepsilon_t, z_{1,t}) = 0$ and $\text{cov}(\varepsilon_t, z_{2,t}) = 0$. Now transform the supply function appropriately

$$q_t = \beta_2 p_t + \gamma_2 z_{2,t} + u_{1,t} - \beta_2 \varepsilon_t$$

when $z_{1,t}$ and $z_{2,t}$ are unknown then use an autoregressive process to form

$$z_{1,t} = \alpha_1 z_{1,t-1} + v_{1,t}$$

$$z_{2,t} = \alpha_2 z_{2,t-1} + v_{2,t}$$

Estimate the demand supply model with these lagged instruments of the dependent variables

## Substitution Method of Rational Expectation

Rational expectation: $p_t^* = E\left(p_t / I_{t-1}\right)$

Equilibrium in the market

$$\beta_1 p_t + \gamma_1 z_{1,t} + u_{1,t} = \beta_2 p_t^* + \gamma_2 z_{1,t} + u_{2,t}$$

Take conditional expectation on both sides

$$\beta_1 p_t^* + \gamma_1 z_{1,t}^* + u_{1,t} = \beta_2 p_t^* + \gamma_2 z_{2,t}^* + u_{2,t}$$

Assume the autoregressive process for $z_{1,t}^*$ and $z_{2,t}^*$ as

$z_{1,t}^* = z_{1,t-1} + w_{1,t}$ and $z_{2,t}^* = z_{2,t-1} + w_{2,t}$

then by differencing

$$\beta_1\left(p_t - p_t^*\right) + \gamma_1 w_{1,t} + u_{1,t} = \gamma_2 w_{2,t} + u_{2,t}$$

$$p_t^* = p_t + \frac{1}{\beta_1}\left(\gamma_1 w_{1,t} - \gamma_2 w_{2,t} + u_{1,t} - u_{2,t}\right)$$

Now substitute this into the supply function

$$q_t = \beta_2 p_t^* + \gamma_2 z_{1,t} + u_{2,t}$$

$$q_t = \beta_2\left[p_t + \frac{1}{\beta_1}\left(\gamma_1 w_{1,t} - \gamma_2 w_{2,t} + u_{1,t} - u_{2,t}\right)\right] + \gamma_2 z_{1,t} + u_{2,t}$$

$$q_t = \beta_2 p_t + \frac{1}{\beta_1}\left(\gamma_1 w_{1,t} - \gamma_2 w_{2,t} + u_{1,t} - u_{2,t}\right) + \gamma_2 z_{1,t} + u_{2,t}$$

$$q_t = \beta_2 p_t + \gamma_2 z_{1,t} + \frac{\gamma_1 \beta_2 w_{1,t}}{\beta_1} - \frac{\gamma_2 \beta_2}{\beta_1} w_{2,t} + u_{2,t} + \frac{\beta_2}{\beta_1}\left(u_{1,t} - u_{2,t}\right)$$

# Rational Expectation in Supply and Demand

- Aggregate supply

$$y_t^s = p_t - w_t = p_t - \underset{t-1}{E^*} p_t \quad (1)$$

wage rate depends on future price, with conditional expectation: $w_t = \underset{t-1}{E^*} p_t$

- Aggregate demand is given by the real money balances combined with a monetary shock:

$$y_t^d = m_t - p_t - \theta_t \qquad (2)$$

- Shock to monetary policy (policy rule) is an auto regressive process as following:

$$\theta_t = \lambda \theta_{t-1} + \mu_t \ (3) \qquad \mu_t \sim N\left(0, \sigma_\mu^2\right) \quad (3)$$

# Rational Expectation procedure

- Take conditional expectation of aggregate demand in (2)

$$E^{*}_{t-1} y_t = E_{t-1} m_t - E_{t-1} p_t - \lambda\theta_{t-1}$$

(4)

- Expected demand (1) and supply (4) are equal in equilibrium

$$E_{t-1} p_t - E^{*}_{t-1} p_t = E_{t-1} m_t - E_{t-1} p_t - \lambda\theta_{t-1} \qquad (5)$$

- This solves for $E_{t-1} p_t$

$$E_{t-1} p_t = \frac{1}{2} E^{*}_{t-1} p_t + \frac{1}{2} E_{t-1} m_t - \frac{1}{2}\lambda\theta_{t-1} \qquad (6)$$

- Use (6) in (4)
- $$E^{*}_{t-1} y_t = E_{t-1} m_t - \left[\frac{1}{2} E^{*}_{t-1} p_t + \frac{1}{2} E_{t-1} m_t - \frac{1}{2}\lambda\theta_{t-1}\right] - \lambda\theta$$

$$E^{*}_{t-1} y_t = \frac{1}{2} E_{t-1} m_t - \frac{1}{2} E^{*}_{t-1} p_t - \frac{1}{2}\lambda\theta_{t-1}$$

# Rational Expectation solution procedure

- Expected deviation of output from equilibrium is 0 ; $E^*_{t-1} y_t = 0$. This gives the unconditional expectation of money supply rule as $E_{t-1} m_t = E^*_{t-1} p_t + \lambda\theta_{t-1}$ (7)
- Equate supply and demand:
- $p_t - E^*_{t-1} p_t = m_t - p_t - \lambda\theta_{t-1} - \mu_t$ (8)
- solve for $p_t$
- $p_t = \frac{1}{2} E^*_{t-1} p_t + \frac{1}{2} m_t - \frac{1}{2}\lambda\theta_{t-1} - \frac{1}{2}\mu_t$ (9)

# Rational Expectation Model -1

- Take the supply equation
- $y^S = p_t - w_t = p_t - E^*_{t-1} p_t$ (1 again)
- Use $p_t$ from (9) in the above aggregate supply function to see that the monetary policy affects variance of output
- $$\mathrm{var}(y_t) = \mathrm{var}\left(p_t - E^*_{t-1} p_t\right) = E\left[\frac{1}{2} E^*_{t-1} p_t + \frac{1}{2} m_t - \frac{1}{2}\lambda\theta_{t-1} - \frac{1}{2}\mu_t - E^*_{t-1} p_t\right]^2$$

Stick the money supply rule (without expectation of (7))

$$E_{t-1} m_t = E^*_{t-1} p_t + \lambda\theta_{t-1} + \mu_t$$

# Rational Expectation and unanticipated shock

$$\mathrm{var}(y_t) = \mathrm{var}\left(p_t - {}_{E_{t-1}^*}p_t\right) = E\begin{bmatrix} \frac{1}{2}{}_{E_{t-1}^*}p_t + \frac{1}{2}\left({}_{E_{t-1}^*}p_t + \lambda\theta_{t-1} + \mu_t\right) \\ -\frac{1}{2}\lambda\theta_{t-1} - {}_{E_{t-1}^*}p_t \end{bmatrix}^2$$

$$\mathrm{var}(y_t) = \mathrm{var}\left(p_t - {}_{E_{t-1}^*}p_t\right) = E\left[\frac{1}{2}\lambda\theta_{t-1} + \frac{1}{2}\mu_t - \frac{1}{2}\lambda\theta_{t-1}\right]^2$$

$$= \frac{1}{4}\sigma_\mu^2$$

**Conclusion: Only unanticipated monetary shock $(\mu_t)$ affects variance of output. Anticipated increase or decrease in aggregate demand by changing money supply in the economy cannot have any impact on output.**

# Rational Expectation and unanticipated shock

$$\operatorname{var}(y_t) = \operatorname{var}\left(p_t - E^*_{t-1} p_t\right) = E\begin{bmatrix} \frac{1}{2} E^*_{t-1} p_t + \frac{1}{2}\left(E^*_{t-1} p_t + \lambda\theta_{t-1} + \mu_t\right) \\ -\frac{1}{2}\lambda\theta_{t-1} - E^*_{t-1} p_t \end{bmatrix}^2$$

$$\operatorname{var}(y_t) = \operatorname{var}\left(p_t - E^*_{t-1} p_t\right) = E\left[\frac{1}{2}\lambda\theta_{t-1} + \frac{1}{2}\mu_t - \frac{1}{2}\lambda\theta_{t-1}\right]^2$$

$$= \frac{1}{4}\sigma^2_\mu$$

**Conclusion: Only unanticipated monetary shock $(\mu_t)$ affects variance of output. Anticipated increase or decrease in aggregate demand by changing money supply in the economy cannot have any impact on output.**