Economic Modelling: Lecture 12
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Linear Programming problem
Duality
Non-linear Programming as a special case of Linear programme

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$$Max \quad R = 10x_1 + 5x_2$$

Subject to:

Skilled labour constraint: $25x_1 + 10x_2 \leq 1000$

Unskilled labour constraint: $20x_1 + 50x_2 \leq 1500$

Non-negativity constraints: $x_1 \geq 0 \quad x_2 \geq 0$

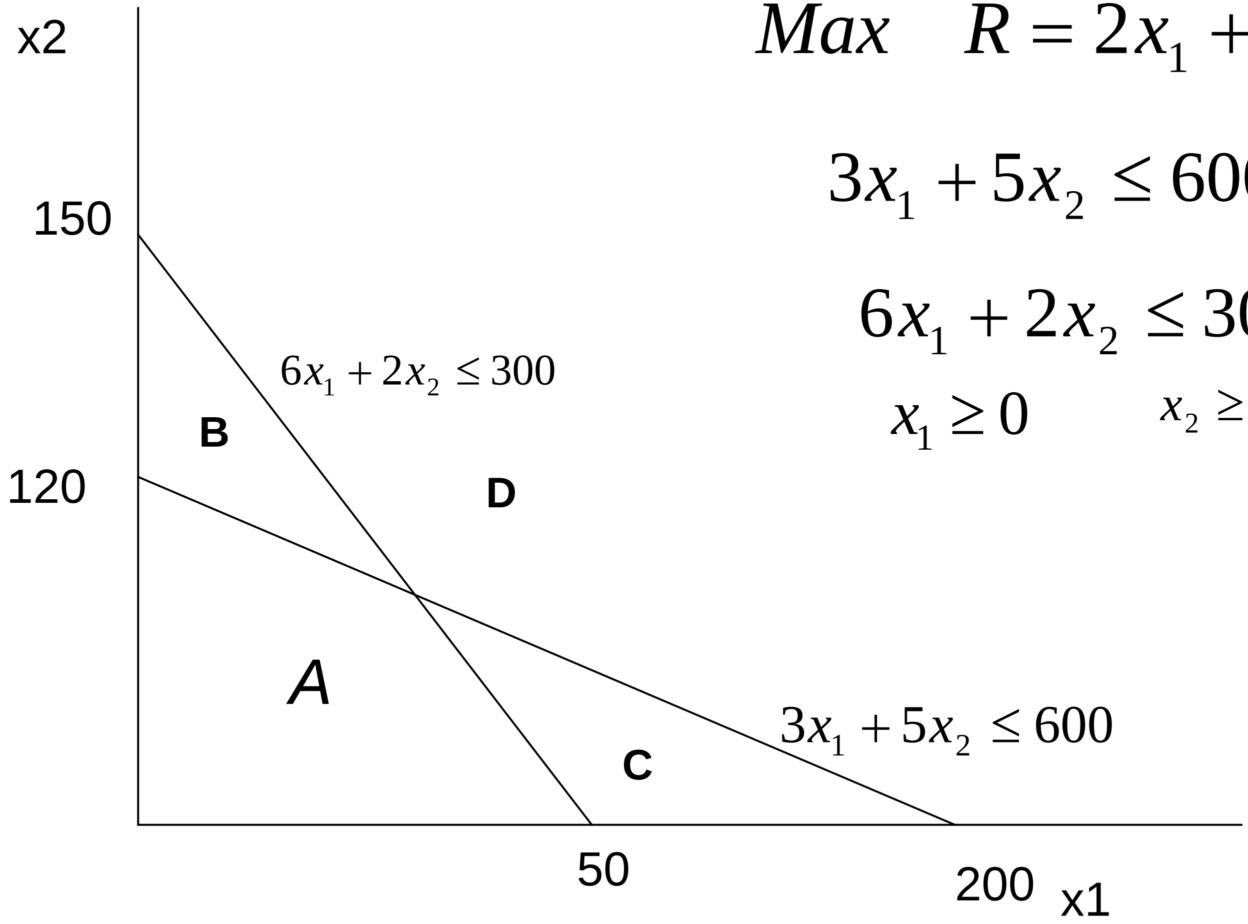
Representation of non-linear programme in a diagram

$$Max \quad R = 2x_1 + 3x_2$$

$$3x_1 + 5x_2 \le 600$$

$$6x_1 + 2x_2 \le 300$$

$$x_1 \ge 0 \qquad x_2 \ge 0$$



1. A producer wants to maximise revenues producing two goods x_1 and x_2 in the market. Market prices of goods are 10 and 5 respectively. Production of x_1 and x_2 requires 25 and 10 units of skilled labour and total endowment of skilled labour is 1000. Similarly production of x_1 and x_2 also requires 20 and 50 units of unskilled labour and whose total endowment is 1500. How much should this firm produce x_1 and x_2 in order to maximise the total revenue.

$Max \quad R = 10x_1 + 5x_2$

Subject to:

Skilled labour constraint: $25x_1 + 10x_2 \leq 1000$

Unskilled labour constraint: $20x_1 + 50x_2 \leq 1500$

Non-negativity constraints: $x_1 \geq 0 \quad x_2 \geq 0$

First add slack variables $s_1 \geq 0$ $s_2 \geq 0$ in order find the basis feasible solution. Above three equations can be written as:

$$R - 10x_1 - 5x_2 - 0.s_1 - 0.s_2 = 0$$
$$25x_1 + 10x_2 + s_1 + 0.s_2 = 1000$$
$$20x_1 + 50x_2 + s_1 + 0s_2 = 1500$$

It is easier to write above equations in the tabular format.

Table 1

Secondly write the coefficients in a tabular format (basic feasible solution)

	R	x_1	x_2	s_1	s_2	constant	Ratios
Row0	1	-10	-5	0	0	0	
Row1	0	25	10	1	0	1000	1000/24=40
Row2	0	20	50	0	1	1500	1500/20=75

Here the basic feasible solution is $(R, x_1, x_2, s_1, s_2) = (0,0,0,1000,1500)$; this is the point of origin and slacks make up all of the revenue.

Now bring the real economic variables to the programme by examining their relative contribution in the revenue and minimum requirement. Here x_1 contributes most to the revenue as shown in Row0 and therefore x_1 is the pivot column. Then Row1 is the pivot row based on minimal input ratio of 40. Now solve this model for x_1 by eliminating Row0 and Row2 element and making Row1 as one.

Table 2

	R	x_1	x_2	s_1	s_2	constant	Ratios
Row0	1	0	-1	2/5	0	400	
Row1	0	1	2/5	1/25	0	40	40/(2/5)=100
Row2	0	0	42	-4/5	1	700	700/42=16.7

Optimal Solution in the Linear Programme

Further operation on Table 2 to solve for x_2 .

Table 3

	R	x_1	x_2	s_1	s_2	constant	Ratios
Row0	1	0	0	8/21	1/42	17500/42	1250/3
Row1	0	1	0	1/25	-1/105	700/21	100/3
Row2	0	0	1	-2/105	1/42	700/42	50/30

The optimal solution is $(R, x_1, x_2, s_1, s_2) = (17500/42, 700/21, 700/42, 0, 0)$. The coefficients for slack variables in the objective function Row0 represent shadow prices for inputs.

Duality

Every maximisation has corresponding minimisation problem. In the revenue maximisation problem above can be equivalent to the cost minimisation problem.

Primal

$$Max \quad R = 10x_1 + 5x_2$$

Subject to:

$$\begin{bmatrix} 25 & 10 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1000 \\ 1500 \end{bmatrix} \text{ and } x_1 \geq 0 \quad x_2 \geq 0$$

This is equivalent to

$$Min \quad C = 1000y_1 + 1500y_2$$

subject to:

$$\begin{bmatrix} 25 & 20 \\ 10 & 50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 5 \end{bmatrix} \text{ and } y_1 \geq 0 \quad y_2 \geq 0$$

Two fundamental theorems of duality:

(1) Optimal values of the primal and the dual objective functions are always identical, provided that optimal feasible solution does exist.

(2) If a certain choice variable in a linear programme is optimally nonzero then the corresponding dummy variable should be equal to zero. Similarly if a certain choice variable in a linear programme is optimally zero then the corresponding choice variable in the linear programme should be non-zero.

the linear programming is a special case of the non-linear programming,

$$L = 10x_1 + 5x_2 + +\lambda_1\left[1000 - 25x_1 - 10x_2\right] + \lambda_2\left[1500 - 20x_1 - 50x_2\right]$$

The solution of this Langrangian problem will be equivalent to the Simplex solution found above. For instance taking the first order conditions:

$$\frac{\partial L}{\partial x_1} = 10 - 25\lambda_1 - 20\lambda_2 = 0 \;; \quad \frac{\partial L}{\partial x_2} = 2 - 10\lambda_1 - 50\lambda_2 = 0; \qquad \text{(I)}$$

$$\frac{\partial L}{\partial \lambda_1} = 1000 - 25x_1 - 10x_2 = 0; \quad \frac{\partial L}{\partial \lambda_2} = 1500 - 20x_1 - 50x_2 = 0; \quad \text{(II)}$$

From (II) $1000 = 25x_1 + 10x_2$ and $1500 = 20x_1 + 50x_2$

Or $200 = 5x_1 + 2x_2$ and $150 = 2x_1 + 5x_2$ solving these two $x_1 = \frac{100}{3}$ and $x_2 = \frac{50}{3}$.

Similarly from (I) $10 = 25\lambda_1 + 20\lambda_2$ and $2 = 10\lambda_1 + 50\lambda_2$

$\lambda_1 = \frac{8}{21}$ and $\lambda_2 = \frac{1}{42}$.

QED.