Bargaining and Cooperative Games

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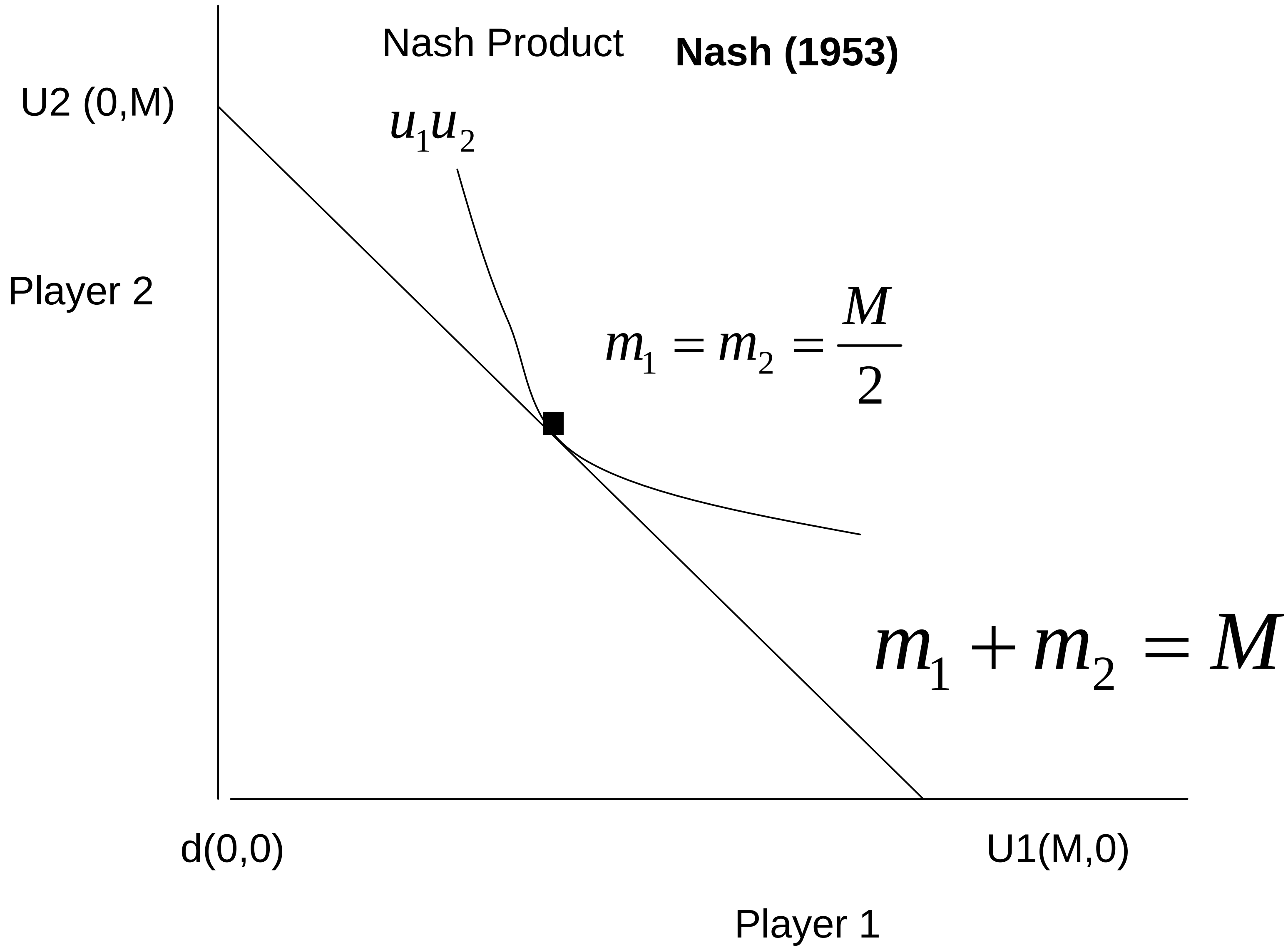
Nash Product: Utility Possibility Frontier



Bargaining on Splitting a Pie

The very common example for bargaining game is splitting a pie between two individuals.

The sum of the shares of the pie claimed by both cannot exceed more than 1, otherwise each will get zero.

If we denote these shares by θ_i and θ_j then $\theta_i + \theta_j \leq 1$ is required for a meaningful solution of the game where each get $\pi_i \geq 0$ and $\pi_i \geq 0$ payoff. When $\theta_i + \theta_j > 1$ then $\pi_i = 0$ and $\pi_i = 0$.

Standard technique to solve this problem is to use the concept of Nash Product .

Nash Product Solution of Splitting a Pie Game

$$\max \quad U = (\theta_i - 0)(\theta_j - 0)$$

subject to

$\theta_i + \theta_j \leq 1$ or by non-satiation property $\theta_i + \theta_j = 1$

Using a Lagrangian function

$$L(\theta_i, \theta_j, \lambda) = (\theta_i - 0)(\theta_j - 0) + \lambda[1 - \theta_i - \theta_j]$$

First order conditions

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_i} = \theta_j - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_j} = \theta_i - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \lambda} = 1 - \theta_i - \theta_j = 0$$

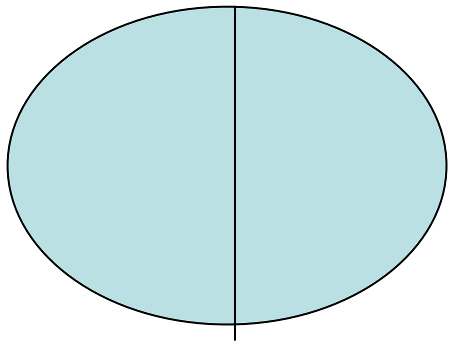
Nash Product Solution of Splitting a Pie Game

From the first two first order conditions

$\theta_j - \lambda = \theta_i - \lambda$ implies

$\theta_j = \theta_i$ and putting this into the third first order condition

$$\theta_j = \theta_i = \frac{1}{2}.$$



This is called focal point.

Thus Nash solution of this problem is to divide the pie symmetrically into two equal parts. Any other solution of this not stable.

Roy Gardner (2003) and Rasmusen (2007) have a number of interesting examples on bargaining game.

Nash Product Solution: A numerical Example

Suppose there is 1000 in the table to be split between two players.

What is the optimal solution from a symmetric bargaining game if the threat point is given by d(0,0)?

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda[1000 - u_1 - u_2]$$

Nash Product Solution A numerical Example

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 1000 - u_1 - u_2 = 0$$

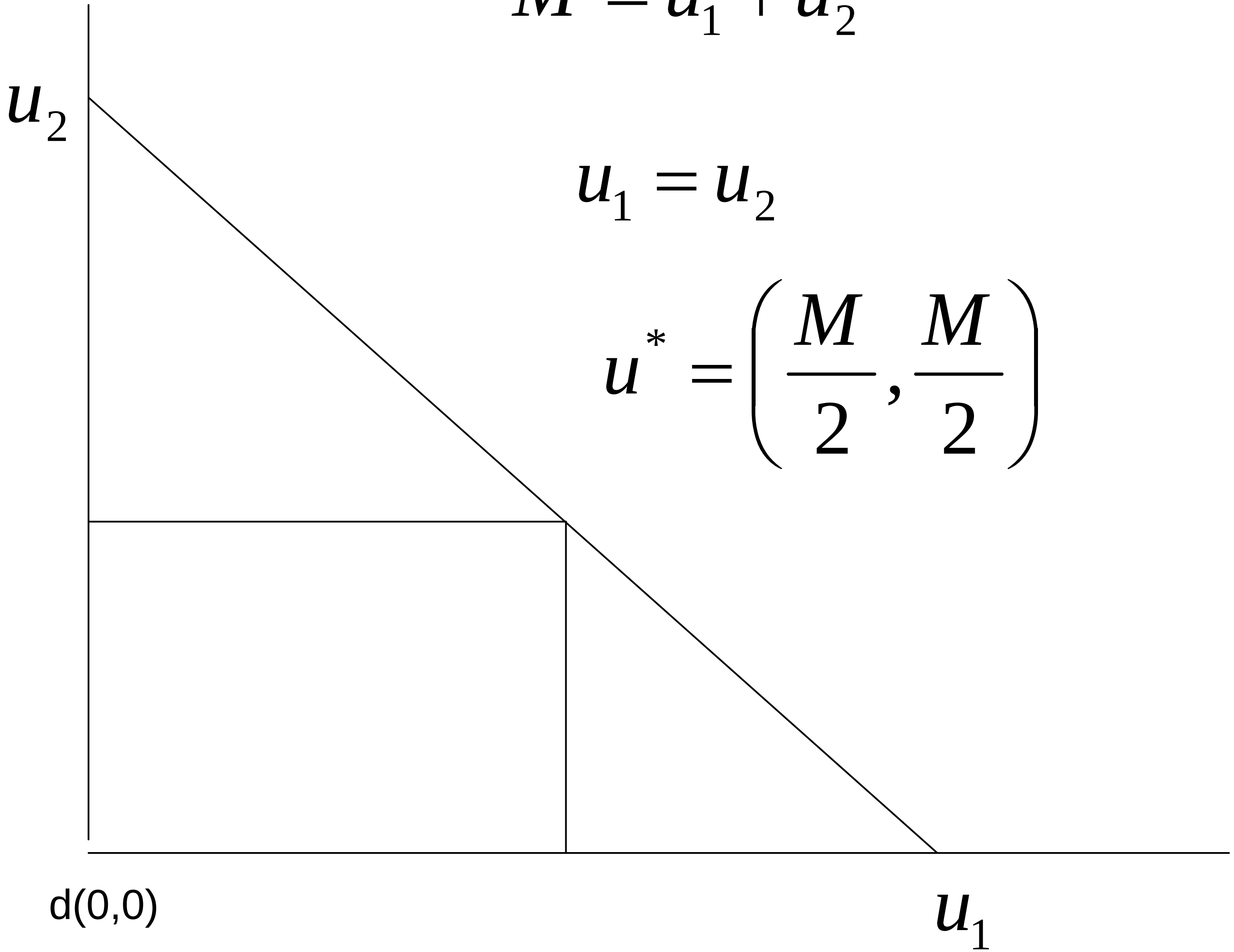
Thus $u_1 - \lambda = u_2 - \lambda$ implies $u_1 = u_2$ and putting this into the third first order condition $u_1 = u_2 = \frac{1000}{2} = 500$.

Splitting M amount between two players

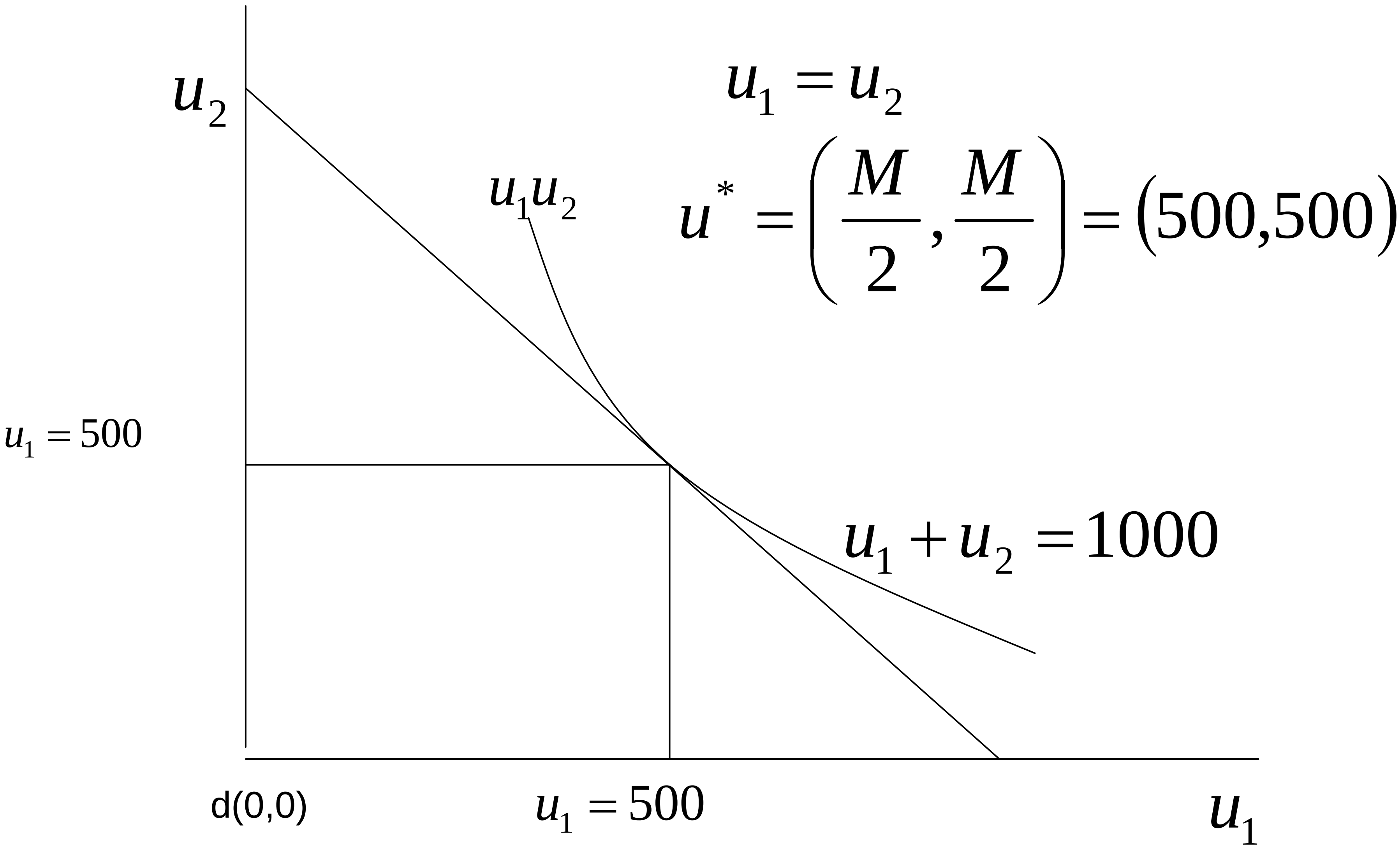
$$M = u_1 + u_2$$

$$u_1 = u_2$$

$$u^* = \left(\frac{M}{2}, \frac{M}{2} \right)$$



Symmetric Allocation of Amount 1000 Between Two Players



Linear Invariance in a Bargaining Game

$$L(u_1, u_2, \lambda) = (u_1 - d_1)(u_2 - d_2) + \lambda[1 - u_1 - u_2]$$

Suppose the player 1 has side payment $d_1 = 15000$

$$L(u_1, u_2, \lambda) = (u_1 - 15000)(u_2 - d_2) + \lambda[50000 - u_1 - u_2]$$

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 15000 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 50000 - u_1 - u_2 = 0$$

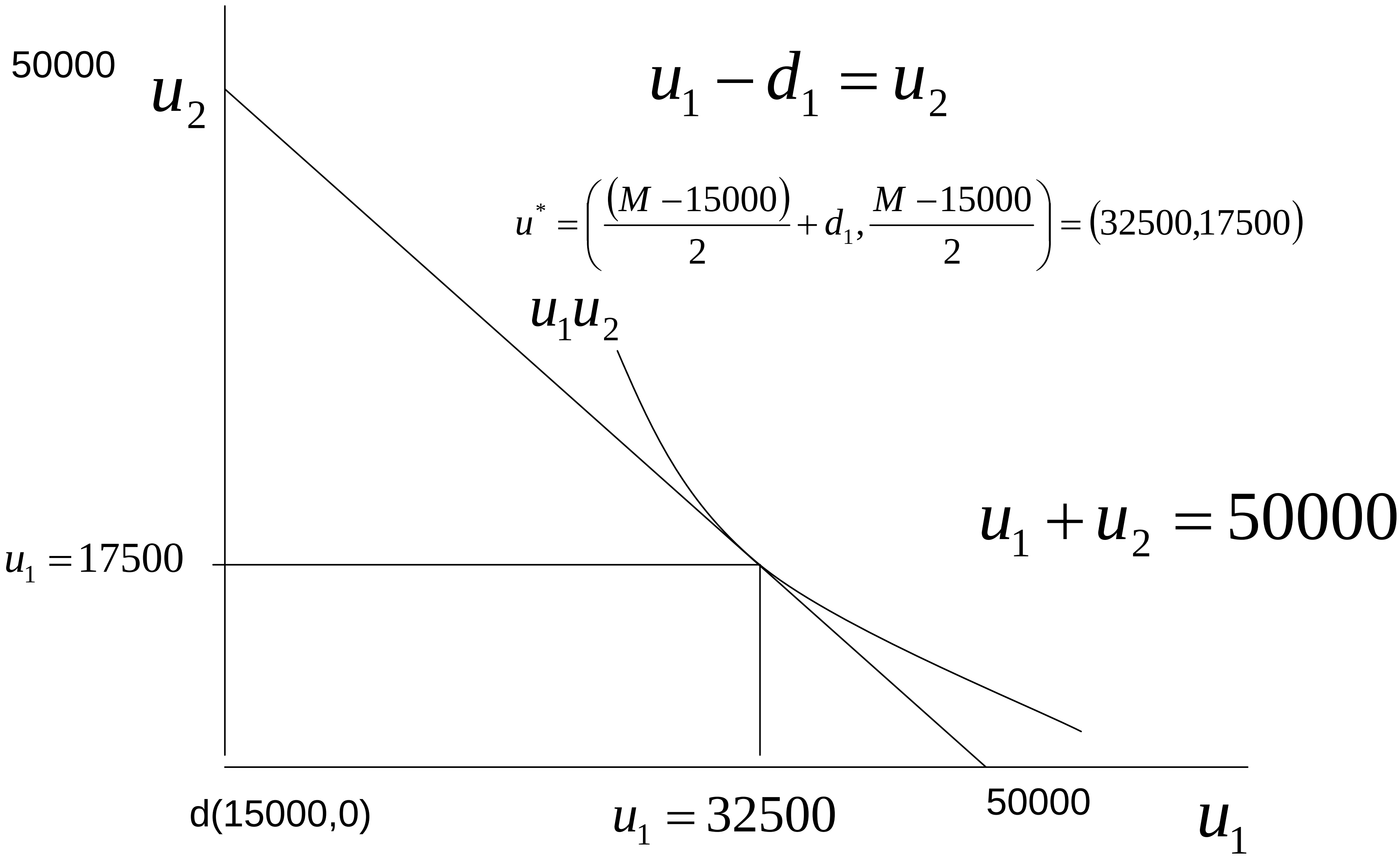
$u_1 - 15000 - \lambda = u_2 - \lambda$ implies $u_1 = 15000 + u_2$ and putting this into the third first order condition $u_2 + 15000 + u_2 = 50000$

$$2u_2 = 50000 - 15000$$

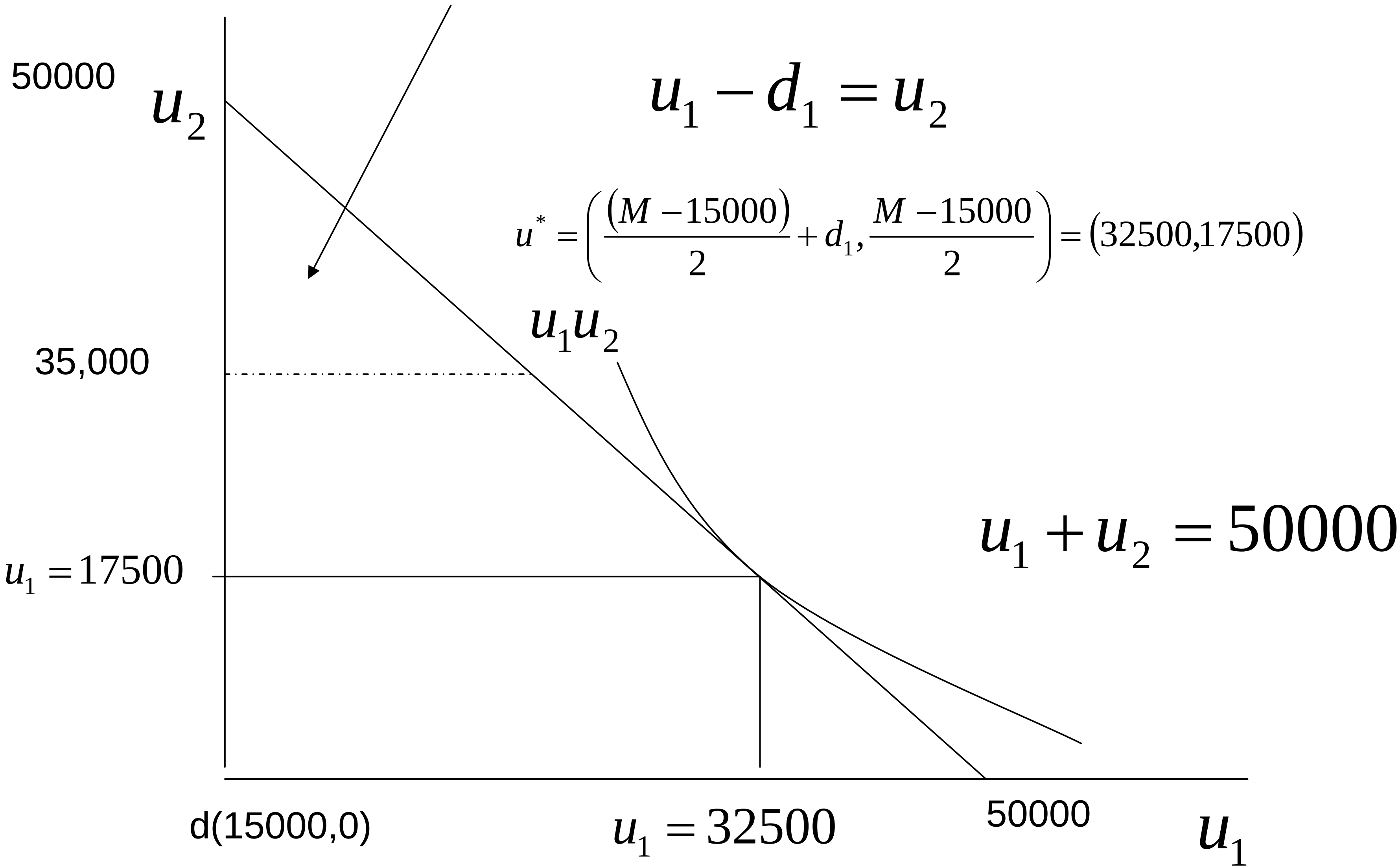
$$2u_2 = 35000 \quad u_2 = \frac{35000}{2} = 17500$$

$$u_1 = 15000 + 17500 = 32500$$

Symmetric Allocation of Amount 50,000 Between Two Players
With Linear Invariance



Independence of Irrelevant Alternative (IIA)



Division of Gains between Risk Averse and Risk Neutral Players

A risk averse person looses in bargaining but the risk neutral person gains.

Suppose the utility functions of risk averse person is given by

$u_2 = (m_2)^{0.5}$ but the risk neutral person has a linear utility

$u_1 = m_1$.

$$m_1 + m_2 = M$$

$$u_1 + u_2^2 = 100$$

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda\left[100 - u_1 - u_2^2\right]$$

Division of Gains between Risk Averse and Risk Neutral Players

First order conditions

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 2\lambda u_2 = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 100 - u_1 - u_2^2 = 0$$

From the first two first order conditions $\frac{u_2}{u_1} = \frac{\lambda}{2\lambda u_2}$ implies $u_1 = 2u_2^2$

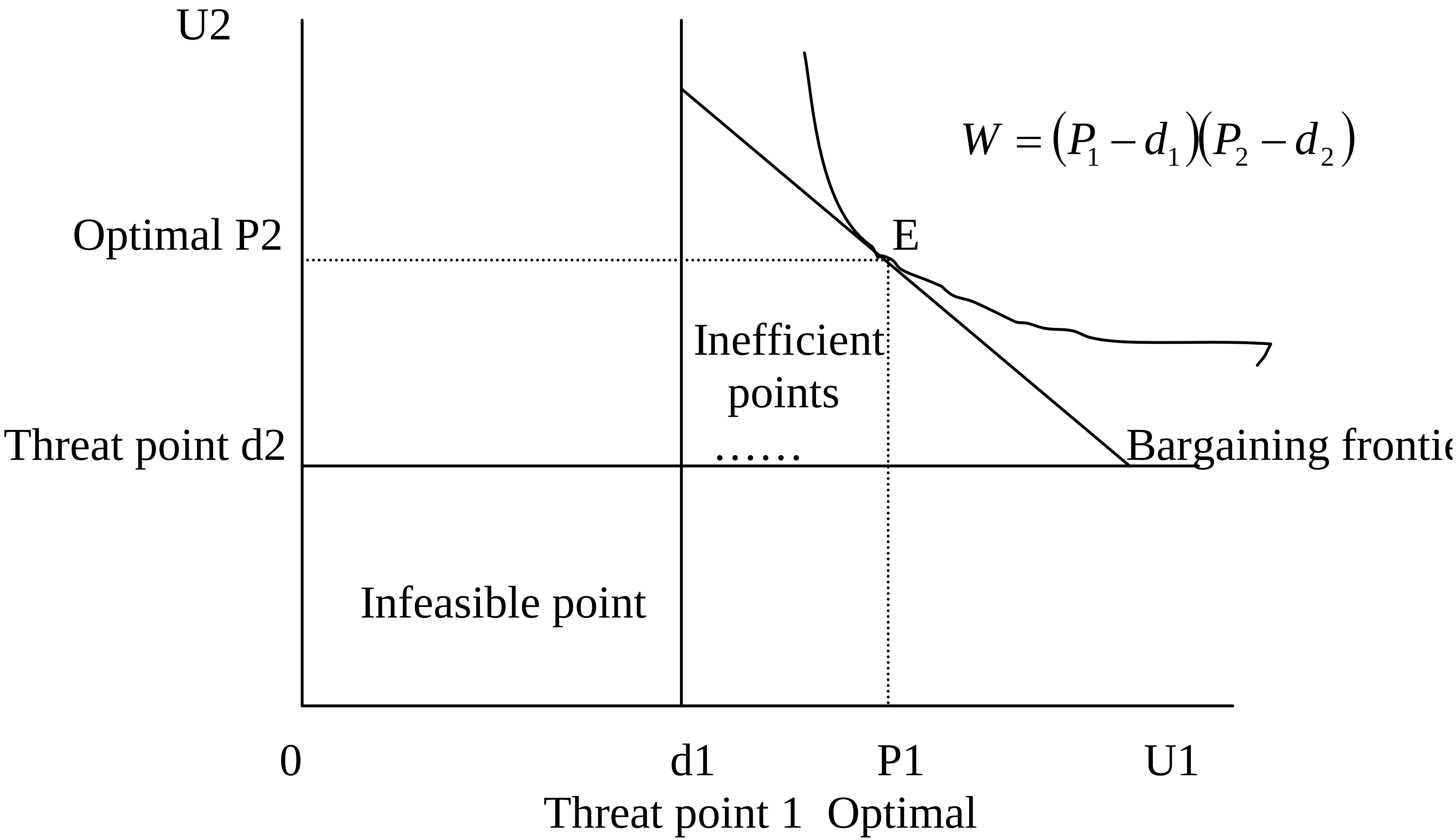
and putting this into the third first order condition $3u_2^2 = 100$.

$$u_2^2 = \frac{100}{3} = 33.33; \; u_2 = 5.77$$

$$u_1 = 2u_2^2 = 2(5.77)^2 = 66.6$$

$$u_1 + u_2^2 = 66.67 + 33.33 = 100$$

Efficient and Inefficient Bargaining Solutions



CA

Coalition Formation and Cooperation and Core

- 2^N -1 rule for possible coalition
- Consider Four Players A,B,C,D
- A, B, C, D
- AB, AC, AD
- BC, BD, CD
- ABC, ACD, BCD
- ABCD
- 2^4 -1=15

Recommended Texts for GAME Theory

- Carmichael F.(2005) A Guide to Game Theory, ISBN: 0273684965
- Gardner R. (2003) Games for Business and Economics, Wiley, ISBN 0471451754
- Rasmusen E (2007) Games and Information, Blackwell, ISBN 1-140513666-9.
- Varian HR (2003) Intermediate Microeconomics: Modern Approach, Norton.
- Pindyck R.S. and D.L. Rubinfeld (2005) Microeconomics, 6th Edition, Pearson; ISBN 0-13-191207-0.