Simultaneous Equation Model of Growth, Inflation and Interest Rates

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Abstract

Depreciation of Sterling has contributed to growth of UK output by enhancing the international competitiveness and so has the higher rate of growth of money supply while the higher interest rates have been contractionary. Model estimates do not support the neutrality of money hypothesis. There is some persistency in the interest rate. Depreciation in value of pound or appreciation of foreign currency has raised the rate of interest. According to these estimates higher the liquidity in the system higher is the interest rate. Inflation is driven up by growth rate of money as well as the depreciation of pounds and higher interest rates. High degree of non-linearity of the reduced form coefficients in a three equation simultaneous equation model is illustrated with analytical solutions for reduced form parameters which are estimated and analysed using ILS, 2SLS and 3SLS techniques.

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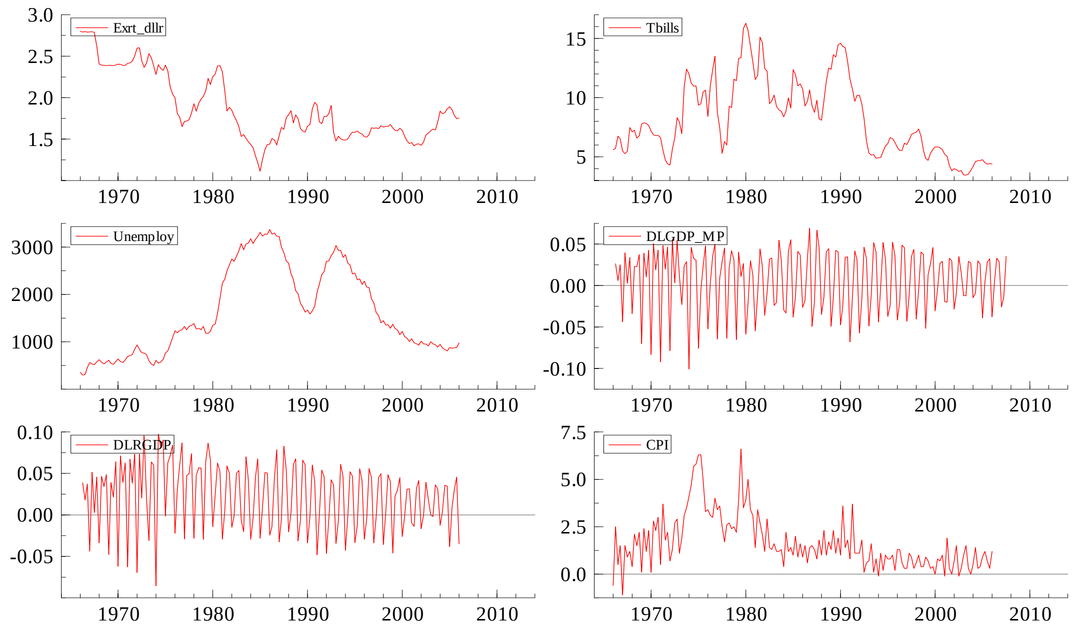
I. Introduction

There is a considerable controversy on whether the growth rates of output are influenced by demand side factors in the short run despite more unanimity among economists about the supply side factors of economic growth such a physical and human capital and technology in the long run. Classical and new classical economists emphasise neutrality of money whereas Keynesian and new Keynesian economists believe in positive impacts of money in the economy. Policy makers who look at time series as given in Figures 1 to close the gaps in output and employment using fiscal, monetary or exchange rate policies taking the interest rate, money supply and exchange rates as instruments to achieve those objectives need a model that can give a precise relation between these instruments and the objectives. A macroeconometric simultaneous equation models have been used in the literature for this purpose.

Figure 1
Exchange rate, Interest Rate, Unemployment rate, Growth rate of output and Money Supply and inflation in UK



On methodological side the superiority of estimators of a simultaneous equation model over single equation models is well known in the literature (George et. al (1985) Hendry (1997)). The explicit solutions of structural parameters from the reduced form coefficients and proofs of the above propositions are either limited to two equations model or are presented abstractly in matrix notations (Davidson (2000)). Applied econometricians, however, often have more than two equations in their models and need explicit solutions in order to explain the impacts of one variable on other variables. Since the reduced form coefficients are highly non-linear in the structural parameters, the explicit solution becomes more complex as more equations are added to a model and more so when estimated parameters have more than one solutions. This point is illustrated here using a structural macro-econometric model which involves endogenous relation among the growth rates, inflation, interest rate with the exchange rate and growth rate of money and the lagged interest rates in the set of exogenous or policy variables.

A three equation simultaneous equation model is outlined in the next section. Analytical solutions for twelve reduced form coefficients along with the rank and order conditions to retrieve those structural coefficients are presented in section III. Derivation and asymptotic properties the indirect least square (ILS), two stage least square (2SLS) and three stage least squares (3SLS) estimators based on the literature in econometric theory is in section IV. Analysis of results from the estimations of ILS, 2SLS and 3SLS is in Section V followed by conclusions and references and tables of results at the end.

II. A Simultaneous Equation Model of Growth, Inflation and Interest Rates

Objective of the model is to analyse interdependence among quarterly growth rates, interest rates and inflation in the UK taking growth rate of money supply, exchange rate and lagged interest rate among its determinants. The first equation relates growth rate of output to the interest rate in the current and previous periods and the current exchange rate. This can be interpreted as an IS curve for an open economy. Interest rates reflect the cost of capital; higher the cost of capital lower will be the growth rate. Higher growth rate of money is expansionary and should raise output. Similarly a higher exchange rate makes economy less competitive in the global economy and thus has negative impact on output. This is exactly what is seen in the empirical results. Second equation links current real interest rate to the previous interest rate, exchange rate and money supply. This can be considered a version of interest rate rule as explained in (Bhattarai(2008)). In a time of greater liquidity interest rate is expected to rise with a higher rate of growth of , fall with higher exchange rate and but and show a persistent pattern to the lagged interest rate as is observed in the empirical results for the interest rate equation. The third equation relates inflation to domestic supply factors represented by growth rates and foreign demand side factors captured in the exchange rates such as the growth rate of money supply. As expected inflation is higher with higher growth rate of money supply and higher rate of interest rate but lower with depreciation.

This three equation simultaneous equations model looks as following:

Output growth equation: $g_t = \alpha_0 + \alpha_2 R_t + \alpha_3 E_t + \alpha_5 R_{t-1} + e_{1,t}$ (1)

Interest rate equation: $R_t = \gamma_0 + \gamma_2 \pi_t + \gamma_3 M_t + \gamma_5 R_{t-1} + e_{3,t}$ (2)

Inflation rate equation: $\pi_t = \beta_0 + \beta_1 g_t + \beta_3 E_t + \beta_4 M_t + e_{2,t}$ (3)

where g_t is the rate of growth of output, π_t the inflation rate, R_t the nominal interest rate, E_t the nominal exchange rate, M_t the growth rate of money supply and R_{t-1} is the nominal interest rate in the previous period. Here g_t π_t and R_t are thee endogenous variables of the system and E_t M_t and R_{t-1} are three exogenous or pre-determined variables. The error terms, $e_{1,t}$, $e_{2,t}$ and $e_{3,t}$, are identically and normally distributed noises with zero mean and constant variance. Coefficients α_0, α_2, α_3, α_5, β_0, β_1, β_2, β_3, β_4, γ_0, γ_1, γ_2, γ_3 and γ_5 are twelve structural parameters of this model that remain to be estimated.

As the application of the OLS technique to each individual equation results in biased and inconsistent estimates (proof in section IV), the system of equations need to be estimated simultaneously. This involves four distinct steps: (1) forming a reduced form of the system; (2) estimating the reduced form coefficients; (3) retrieving the structural coefficients of the model; and (4) using those parameters in prediction and forecasts. Each variable used in regression is stationary (see appendix 2) Given that it is impossible to retrieve the structural coefficients of an equation if it is unidentified it is important to check that each of the model equations is identified using the rank and order conditions. Form a matrix of structural coefficients as:

Table 1
Matrix of coefficients of the system for rank condition

	constant	g_t	π_t	R_t	E_t	M_t	R_{t-1}
Growth	-α_0	1	0	-α_2	-α_3	0	-α_5
Interest rate	-γ_0	0	-γ_2	1	0	-γ_4	-γ_5
Inflation	-β_0	-β_1	1	0	-β_3	-β_4	0

Matrix of coefficients missing from each of the above equations is:

For growth equation: $A_1 = \begin{bmatrix} -\gamma_2 & -\gamma_4 \\ 1 & \beta_4 \end{bmatrix} \quad |A_1| = \begin{vmatrix} -\gamma_2 & -\gamma_4 \\ 1 & \beta_4 \end{vmatrix} = -\beta_4\gamma_2 + \gamma_4 \neq 0$

For interest rate: $A_2 = \begin{bmatrix} 1 & -\alpha_3 \\ -\beta_1 & \beta_3 \end{bmatrix} \quad |A_2| = 1\begin{vmatrix} 1 & -\alpha_3 \\ \beta_1 & \beta_3 \end{vmatrix} = -\beta_3 + \beta_1\alpha_3 \neq 0$

For inflation rate: $A_3 = \begin{bmatrix} -\alpha_2 & -\alpha_5 \\ 1 & -\gamma_5 \end{bmatrix} \quad |A_3| = \begin{vmatrix} -\alpha_2 & -\alpha_5 \\ 1 & -\gamma_5 \end{vmatrix} = \alpha_2\gamma_5 + \alpha_5 \neq 0$

Thus each of the above equation is identified by the rank condition. This means the structural coefficients can be retrieved from the estimates of the reduced form coefficients.

Order condition involves checking the identifiability condition $K - k \geq m - 1$ where K is the number of exogenous variable in the system including the intercept term, k is the number of exogenous variables in the equation under consideration, m is the number of endogenous variables in that equation. The number of exogenous variables in this system is six, K = 6, i.e. α_0, β_0, γ_0, E_t M_t and R_{t-1}. Order condition is satisfied for each of above equations $K-k=6-3=3\geq m-1=3-1=2$. Therefore each of the above equations is over identified. That means it is possible to retrieve more than one value of the structural coefficients from the estimates of the reduced form coefficients. The procedure on this will be shown in the next section.

III. Reduced form coefficients and retrieval of structural coefficients

Rearrange the above equations for endogenous and exogenous variables as following:

$$g_t-\alpha_2 R_t+0\pi_t=\alpha_0+\alpha_3 E_t+\alpha_5 R_{t-1}+e_{1,t} \tag{4}$$

$$-\beta_1 g_t+0R_t+\pi_t=\beta_0+\beta_3 E_t+\beta_4 M_t+e_{2,t} \tag{5}$$

$$-0g_t+R_t-\gamma_2\pi_t=\gamma_0+\gamma_4 M_t+\gamma_5 R_{t-1}+e_{3,t} \tag{6}$$

$$\begin{bmatrix}1 & -\alpha_2 & 0\\ -\beta_1 & 0 & 1\\ 0 & 1 & -\gamma_2\end{bmatrix}\begin{bmatrix}g_t\\ R_t\\ \pi_t\end{bmatrix}=\begin{bmatrix}\alpha_0 & \alpha_3 & 0 & \alpha_5\\ \beta_0 & \beta_3 & \beta_4 & 0\\ \gamma_0 & 0 & \gamma_4 & \gamma_5\end{bmatrix}\begin{bmatrix}1\\ E_t\\ M_t\\ R_{t-1}\end{bmatrix}+\begin{bmatrix}e_{1,t}\\ e_{2,t}\\ e_{3,t}\end{bmatrix} \tag{7}$$

$$\begin{bmatrix}g_t\\ R_t\\ \pi_t\end{bmatrix}=\begin{bmatrix}1 & -\alpha_2 & 0\\ -\beta_1 & 0 & 1\\ 0 & 1 & -\gamma_2\end{bmatrix}^{-1}\begin{bmatrix}\alpha_0 & \alpha_3 & 0 & \alpha_5\\ \beta_0 & \beta_3 & \beta_4 & 0\\ \gamma_0 & 0 & \gamma_4 & \gamma_5\end{bmatrix}\begin{bmatrix}1\\ E_t\\ M_t\\ R_{t-1}\end{bmatrix}+\begin{bmatrix}1 & -\alpha_2 & 0\\ -\beta_1 & 0 & 1\\ 0 & 1 & -\gamma_2\end{bmatrix}^{-1}\begin{bmatrix}e_{1,t}\\ e_{2,t}\\ e_{3,t}\end{bmatrix}$$

$$Y \quad = \quad A^{-1} \qquad B \qquad X \qquad A^{-1} \qquad e$$

$$\begin{bmatrix}g_t\\ R_t\\ \pi_t\end{bmatrix}=\frac{1}{(\alpha_2\beta_1\gamma_2-1)}\begin{bmatrix}\begin{vmatrix}0 & 1\\ 1 & -\gamma_2\end{vmatrix} & -\begin{vmatrix}-\beta_1 & 1\\ 0 & -\gamma_2\end{vmatrix} & \begin{vmatrix}-\beta_1 & 0\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-\alpha_2 & 0\\ 1 & -\gamma_2\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & -\gamma_2\end{vmatrix} & -\begin{vmatrix}1 & -\alpha_2\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-\alpha_2 & 0\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}1 & 0\\ -\beta_1 & 1\end{vmatrix} & \begin{vmatrix}1 & -\alpha_2\\ -\beta_1 & 0\end{vmatrix}\end{bmatrix}'\begin{bmatrix}\alpha_0 & \alpha_3 & 0 & \alpha_5\\ \beta_0 & \beta_3 & \beta_4 & 0\\ \gamma_0 & 0 & \gamma_4 & \gamma_5\end{bmatrix}\begin{bmatrix}1\\ E_t\\ M_t\\ R_{t-1}\end{bmatrix}$$

$$+\frac{1}{(\alpha_2\beta_1\gamma_2-1)}\begin{bmatrix}\begin{vmatrix}0 & 1\\ 1 & -\gamma_2\end{vmatrix} & -\begin{vmatrix}-\beta_1 & 1\\ 0 & -\gamma_2\end{vmatrix} & \begin{vmatrix}-\beta_1 & 0\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-\alpha_2 & 0\\ 1 & -\gamma_2\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & -\gamma_2\end{vmatrix} & -\begin{vmatrix}1 & -\alpha_2\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}-\alpha_2 & 0\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}1 & 0\\ -\beta_1 & 1\end{vmatrix} & \begin{vmatrix}1 & -\alpha_2\\ -\beta_1 & 0\end{vmatrix}\end{bmatrix}'\begin{bmatrix}e_{1,t}\\ e_{2,t}\\ e_{3,t}\end{bmatrix} \tag{8}$$

$$\begin{bmatrix} g_t \\ R_t \\ \pi_t \end{bmatrix} = \frac{1}{(\alpha_2\beta_1\gamma_2 - 1)} \begin{bmatrix} -1 & \alpha_2\gamma_2 & -\alpha_2 \\ -\beta_1\gamma_2 & -\gamma_2 & -1 \\ -\beta_1 & 1 & \beta_1\alpha_2 \end{bmatrix} \begin{bmatrix} \alpha_0 & \alpha_3 & 0 & \alpha_5 \\ \beta_0 & \beta_3 & \beta_4 & 0 \\ \gamma_0 & 0 & \gamma_4 & \gamma_5 \end{bmatrix} \begin{bmatrix} 1 \\ E_t \\ M_t \\ R_{t-1} \end{bmatrix}$$

$$+ \frac{1}{(\alpha_2\beta_1\gamma_2 - 1)} \begin{bmatrix} -1 & \alpha_2\gamma_2 & -\alpha_2 \\ -\beta_1\gamma_2 & -\gamma_2 & -1 \\ -\beta_1 & 1 & \beta_1\alpha_2 \end{bmatrix} \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \tag{9}$$

This is the reduced form of the above model. More explicitly it can be written as:

$$\begin{bmatrix} g_t \\ R_t \\ \pi_t \end{bmatrix} = \frac{1}{(\alpha_2\beta_1\gamma_2 - 1)} \begin{bmatrix} -\alpha_0 + \alpha_2\beta_0\gamma_0 - \alpha_2\gamma_{01} & -\alpha_3 + \alpha_2\beta_3\gamma_2 & -\gamma_2\beta_4 - \alpha_2\gamma_4 & -\alpha_5 - \alpha_2\gamma_5 \\ -\alpha_0\beta_1\gamma_2 - \gamma_2\beta_0 - \beta_1\gamma_0 & -\alpha_3\beta_1\gamma_2 - \gamma_2\beta_3 & \gamma_2\beta_4 - \gamma_4 & -\gamma_2\beta_1\alpha_5 - \gamma_4 \\ \alpha_0\beta_1 + \beta_0 + \gamma_0\beta_1\alpha_2 & \alpha_3\beta_1 - \beta_3 & \beta_4 + \gamma_4\beta_1\alpha_2 & -\beta_1\alpha_5 + \gamma_5\beta_1\alpha_2 \end{bmatrix} \begin{bmatrix} 1 \\ E_t \\ M_t \\ R_{t-1} \end{bmatrix}$$

$$+ \frac{1}{(\alpha_2\beta_1\gamma_2 - 1)} \begin{bmatrix} -e_{1,t} + \gamma_2 e_{2,t} - \alpha_2 e_{3,t} \\ -\beta_1\gamma_2 e_{1,t} - \gamma_2 e_{2,t} - \gamma_2 e_{3,t} \\ -\beta_1 e_{1,t} + e_{2,t} + \beta_1\alpha_2 e_{3,t} \end{bmatrix} \tag{10}$$

$$g_t = \frac{-\alpha_0 + \alpha_2\beta_0\gamma_0 - \alpha_2\gamma_{01}}{(\alpha_2\beta_1\gamma_2 - 1)} + \frac{-\alpha_3 + \alpha_2\beta_3\gamma_2}{(\alpha_2\beta_1\gamma_2 - 1)} E_t + \frac{-\gamma_2\beta_4 - \alpha_2\gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)} M_t + \frac{-\alpha_5 - \alpha_2\gamma_5}{(\alpha_2\beta_1\gamma_2 - 1)} R_{t-1} + v_{1t} \tag{11}$$

$$R_t = \frac{-\alpha_0\beta_1\gamma_2 - \gamma_2\beta_0 - \beta_1\gamma_0}{(\alpha_2\beta_1\gamma_2 - 1)} + \frac{-\alpha_3\beta_1\gamma_2 - \gamma_2\beta_3}{(\alpha_2\beta_1\gamma_2 - 1)} E_t + \frac{\gamma_2\beta_4 - \gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)} M_t + \frac{-\gamma_2\beta_1\alpha_5 - \gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)} R_{t-1} + v_{2t} \tag{12}$$

$$\pi_1 = \frac{\alpha_0\beta_1 + \beta_0 + \gamma_0\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)} + \frac{\alpha_3\beta_1 - \beta_3}{(\alpha_2\beta_1\gamma_2 - 1)} E_t + \frac{\beta_4 + \gamma_4\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)} M_t + \frac{-\beta_1\alpha_5 + \gamma_5\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)} R_{t-1} + v_{3t} \tag{13}$$

where $v_{1t} = \dfrac{-e_{1,t} + \gamma_2 e_{2,t} - \alpha_2 e_{3,t}}{(\alpha_2\beta_1\gamma_2 - 1)}$; $v_{2t} = \dfrac{-\beta_1\gamma_2 e_{1,t} - \gamma_2 e_{2,t} - \gamma_2 e_{3,t}}{(\alpha_2\beta_1\gamma_2 - 1)}$;

$$v_{3t} = \frac{-\beta_1 e_{1,t} + e_{2,t} + \beta_1\alpha_2 e_{3,t}}{(\alpha_2\beta_1\gamma_2 - 1)}$$

This is written more compactly in the reduced form system as:

$$g_t = \Pi_{10} + \Pi_{11} E_t + \Pi_{12} M_t + \Pi_{13} R_{t-1} + v_{1t} \tag{14}$$

$$R_t = \Pi_{20} + \Pi_{21} E_t + \Pi_{22} M_t + \Pi_{23} R_{t-1} + v_{2t} \tag{15}$$

$$\pi_t = \Pi_{30} + \Pi_{31} E_t + \Pi_{32} M_t + \Pi_{33} R_{t-1} + v_{3t} \tag{16}$$

where

$$\Pi_{10} = \frac{-\alpha_0 + \alpha_2\beta_0\gamma_0 - \alpha_2\gamma_{01}}{(\alpha_2\beta_1\gamma_2 - 1)}; \Pi_{11} = \frac{-\alpha_3 + \alpha_2\beta_3\gamma_2}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{12} = \frac{-\gamma_2\beta_4 - \alpha_2\gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)};$$

$$\Pi_{13} = \frac{-\alpha_5 - \alpha_2\gamma_5}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{20} = \frac{-\alpha_0\beta_1\gamma_2 - \gamma_2\beta_0 - \beta_1\gamma_0}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{21} = \frac{-\alpha_3\beta_1\gamma_2 - \gamma_2\beta_3}{(\alpha_2\beta_1\gamma_2 - 1)};$$

$$\Pi_{22} = \frac{\gamma_2\beta_4 - \gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{23} = \frac{-\gamma_2\beta_1\alpha_5 - \gamma_4}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{30} = \frac{\alpha_0\beta_1 + \beta_0 + \gamma_0\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{31} = \frac{\alpha_3\beta_1 - \beta_3}{(\alpha_2\beta_1\gamma_2 - 1)};$$

$$\Pi_{32} = \frac{\beta_4 + \gamma_4\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)}; \ \Pi_{33} = \frac{-\beta_1\alpha_5 + \gamma_5\beta_1\alpha_2}{(\alpha_2\beta_1\gamma_2 - 1)}; \tag{17}$$

It is possible to retrieve twelve structural coefficients from the twelve reduced form coefficients though they are highly non-linear in the structural parameters.

IV. Estimation Techniques

Parameters of a simultaneous equation model can be estimated by a number of single or multiple equation estimation techniques. Indirect least square(ILS), two stage least square (2SLS), three stage least square (3SLS) and full information maximum likelihood (FIML) estimators are single equation techniques are prominent ones in the literature (Judge et. al. (1985)).

Indirect least square method involves applying the least square method to the reduced form equations and retrieving the structural coefficients using equation (17). It the matrix algebra the estimates of the above reduced form equation g_t can be written as

$$\hat{\Pi} = (X'X)^{-1}(X'Y) \tag{18}$$

where

$$(X'X) = \begin{bmatrix} N & \sum_t E_t & \sum_t M_t & \sum_t R_{t-1} \\ \sum_t E_t & \sum_t E_t^2 & \sum_t E_t M_t & \sum_t E_t R_{t-1} \\ \sum_t M_t & \sum_t E_t M_t & \sum_t M_t^2 & \sum_t M_t R_{t-1} \\ \sum_t R_{t-1} & \sum_t E_t R_{t-1} & \sum_t M_t R_{t-1} & \sum_t R_{t-1}^2 \end{bmatrix}; \ (X'Y) = \begin{bmatrix} \sum_t Y_t \\ \sum_t Y_t E_t \\ \sum_t Y_t M_t \\ \sum_t Y_t R_{t-1} \end{bmatrix} \tag{19}$$

It is obvious that these reduced form coefficients can be used to estimate the structural coefficients of the growth equation if the determinant of $(X'X)$ is non zero. Relating it back to equation (9) it means

$$\hat{\Pi}A = -B \tag{20}$$

Now substituting the value of $\hat{\Pi}$ in (20) including the variables missing from the particular equation, the coefficients of equation *i* of the model can be given by

$$(X'X)^{-1}(X'Y_i)\begin{pmatrix}-1\\ \hat{a}_i\\ 0\end{pmatrix} = \begin{pmatrix}-\hat{b}_i\\ 0\end{pmatrix} \text{ or } (X'Y)\begin{pmatrix}-1\\ \hat{a}_i\end{pmatrix} = -(X'X)\hat{b}_i \tag{21}$$

$$X'y_i = X'y_i\hat{a}_i + (X'X)\hat{b}_i = X'Z_i\hat{\delta}_i + X'e_i \tag{22}$$

$$\hat{\delta}_i = (\hat{a}_i, \hat{b}_i) \text{ and } Z_i = (y_i, X_i)$$

$$X'Y = X'y_i\hat{a}_i + (X'X)\hat{b}_i = X'Z_i\hat{\delta}_i$$

$$\hat{\delta}_{i(ILS)} = (X'Z_i)^{-1}(X'Y) \tag{23}$$

The covariance structure of $X'e_i$ in (21) is $\sigma_{i,i}(X'X)^{-1}$. This need to be used to make this indirect least square estimator asymptotically efficient applying GLS procedure as (Judge et. al. p. 596):

$$\hat{\delta}_{i(ILS)} = \left[(X'Z_i)'(\sigma_{i,i}X'X)^{-1}(X'Z_i)\right]^{-1}\left[(X'Z_i)'(\sigma_{i,i}X'X)^{-1}(X'Y_i)\right]$$

$$= \left[(X'Z_i)'(X'X)^{-1}(X'Z_i)\right]^{-1}\left[(X'Z_i)'(X'X)^{-1}(X'Y_i)\right] \tag{24}$$

where $\sigma_{i,i} = (y_i - z_i\hat{\delta}_i)'(y_i - z_i\hat{\delta}_i)/T$

Thus $\hat{\delta}_{i(ILS)}$ exists whenever the inverse of term $\left[(X'Z_i)'(\sigma_{i,i}X'X)^{-1}(X'Z_i)\right]$ non-trivial; the determinant of $(X'X)$ should be non zero.

The two stage least square (2SLS) estimation is a single equation method. The first stage involves using OLS of Y on X to get the predicted values of the endogenous variables, $\hat{Y}$. The second stage involves using both $\hat{Y}$ and X to estimate parameters. In fact there is very little difference between the GLS method applied to the ILS above and the two stage least square method as seen from below:

$$\hat{Z}_i = \left[\hat{Y}_i, X_i\right] = \left[X\hat{\Pi} \quad X_i\right] = \left[X(X'X)^{-1}(X'Y_i) \quad X(X'X)^{-1}X'X_i\right] = X(X'X)^{-1}X'Z_i \tag{25}$$

$$\hat{\delta}_{i(2SLS)} = \left[Z'_i X(X'X)^{-1}X'Z_i\right]^{-1} Z'_i X(X'X)^{-1}X'Y_i = \left[Z'_i Z_i\right]^{-1} Z'_i Y_i \tag{26}$$

Three stage least square (3SLS) method is a system wise generalised least square technique where all equations are estimated simultaneously. It is used to correct the autocorrelation or heteroscedasticity existing in the model. Here 1..m model equations are written as (see Judge et. al. p. 601 for more derivations):

$$\begin{bmatrix} X'Y_1 \\ X'Y_2 \\ . \\ . \\ X'Y_m \end{bmatrix} = \begin{bmatrix} X'Z_1 & 0 & . & . & 0 \\ 0 & X'Z_2 & 0 & . & 0 \\ 0 & 0 & X'Z_3 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & X'Z_m \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ . \\ . \\ \delta_m \end{bmatrix} + \begin{bmatrix} X'e_1 \\ X'e_2 \\ . \\ . \\ X'e_m \end{bmatrix} \tag{27}$$

This system can be written in one equation using the Kronneker product as:

$$(I \otimes X')y = (I \otimes X')Z\delta + (I \otimes X')e \tag{28}$$

The covariance matrix of the error $(I \otimes X')e$ is $\sum \otimes E(X'X)$. Taking account of this

$$\hat{\delta}_{(3SLS)} = \left\{Z'_i (I \otimes X')' \left[\sum{}^{-1} \otimes (X'X)^{-1}\right](I \otimes X')' Z_i\right\}^{-1} Z'_i (I \otimes X')' \left[\sum{}^{-1} \otimes (X'X)^{-1}\right](I \otimes X')' Y_i \tag{29}$$

which is distributed normally as:

$$\sqrt{T}\left(\hat{\delta}_{(3SLS)} - \delta\right) \rightarrow N\left(0, p\lim\left[T^{-1}Z' \sum{}^{-1} \otimes (X'X)^{-1} Z\right]^{-1}\right) \tag{30}$$

where $\left[T^{-1}Z' \sum^{-1} \otimes (X'X)^{-1} Z\right]^{-1}$ represents the variance covariance matrix of $\hat{\delta}_{(3SLS)}$.

This is consistent and efficient estimator.

V. Analysis of Results from the ILS, 2SLS and 3SLS Estimations

Basic hypotheses regarding the determinants of growth rate, retail price index and the interest rate are supported by the estimates. There is a negative relation between growth rates and the exchange rates. Lower exchange rates are associated with higher growth rates. Continued depreciation of the Sterling Pounds from 2.5 in 1972 to 1.11 in 1984 and then a bit appreciation toward 1.5 in 2000 and 1.8 in 2008 has been one important factor influencing the rate of economic growth. Depreciation of pound in this manner has made the UK economy more competitive in the world and it has contributed to higher growth rate. Higher level of liquidity has promoted growth as reflected in positive contribution of growth rate of money supply in the growth rate of output. One percent change in the growth rate of money brings 0.6 percent increase in the growth rate of output. These estimates do not support the neutrality of money hypothesis. Higher interest rates in the previous period, which can be considered as an instruments of contractionary economic policy, has negative impact, but the coefficient is not significant.

Model estimates suggest persistency in the quarterly interest rates. About 15 percent of current interest rate is explained by previous interest rate. It is a bit surprising to note a positive relation between the interest rate and growth rate of money. Higher the liquidity in the system higher is the interest rate. Given the prominent place of London as a financial centre of the world this might signify that speculation demand dominates the transaction demand for money in the UK's financial system. Similarly the association between the exchange rate and the interest rate is negative and significant. Depreciation in value of pound or appreciation of foreign currency results in higher rate of interest rate in the UK. This may indicate to capital outflow likely to take place in the wake of depreciation of pounds.

There seem to be a significant relation between the growth rate of money supply and inflation. Higher rate of growth of money leads to higher inflation. This is sensible in light of the quantity theory of money. Depreciation of pound has positive impact on inflation. This is also sensible as the depreciation makes imports more expensive and prices of imported commodity rise leading to further increase in inflation. Similarly higher interest rates have positive impact on prices. Higher interest rate raise the cost of capital and hence the inflation. All three estimation techniques suggest that the money supply has remained a significant determinant of the interest rate during the study period.

Comparison of prediction errors and AIC criteria suggest that 2SLS and 3SLS estimates are better than ILS estimator though the estimates of coefficients in all three methods are quite close to each other.

VI. Conclusion

Depreciation of Sterling has contributed to growth by enhancing the international competitiveness and so has the higher rate of growth of money supply. Higher interest rates have been contractionary. Model estimates do not support the neutrality of money hypothesis. There is some persistency in the interest rate. Depreciation in value of pound or appreciation of foreign currency raises the rate of interest. According to these estimates higher the liquidity in the system higher is the interest rate. Inflation is driven up by growth rate of money as well as the depreciation of pounds and higher interest rates. High degree of non-linearity of the reduced form coefficients in a three equation simultaneous equation model is illustrated with analytical solutions for reduced form parameters which are estimated and analysed using ILS, 2SLS and 3SLS techniques.

VII. References

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Table A1

Estimations of Reduced Form Coefficients by ILS, 2SLS and 3SLS Methods

(Quarterly Series of UK 1967:2-2006:1)

Indirect Least Square

Growth equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	-0.0032	0.0258	-0.1230	0.9020
DLExrt_dllr	-0.0187	0.0725	-0.2580	0.7970
DLM4	0.6349	0.1940	3.2700	0.0010
Constant	0.0047	0.0062	0.7610	0.4480

Interest rate Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.1503	0.0752	2.0000	0.0470
DLExrt_dllr	-0.6302	0.2114	-2.9800	0.0030
DLM4	1.9670	0.5655	3.4800	0.0010
Constant	-0.0568	0.0179	-3.1700	0.0020

Inflation Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.6128	0.8585	0.7140	0.4760
DLExrt_dllr	-2.7670	2.4130	-1.1500	0.2530
DLM4	30.6555	6.4530	4.7500	0.0000
Constant	0.8351	0.2047	4.0800	0.0000

AIC	-10.1402	SC	-9.9086
HQ	-10.0461	FPE	0.0000

Two Stage Least Square Estimates

Growth equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	-0.0032	0.0258	-0.1230	0.9020
DLExrt_dllr	-0.0187	0.0725	-0.2580	0.7970
DLM4	0.6349	0.1940	3.2700	0.0010
Constant	0.0047	0.0062	0.7610	0.4480

Interest rate Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.1503	0.0752	2.0000	0.0470
DLExrt_dllr	-0.6302	0.2114	-2.9800	0.0030
DLM4	1.9670	0.5655	3.4800	0.0010
Constant	-0.0568	0.0179	-3.1700	0.0020

Inflation Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.6128	0.8585	0.7140	0.4760
DLExrt_dllr	-2.7670	2.4130	-1.1500	0.2530
DLM4	30.6555	6.4530	4.7500	0.0000
Constant	0.8351	0.2047	4.0800	0.0000

AIC	-10.2569	SC	-10.0253
HQ	-10.1629	FPE	0.0000

Table A2

Estimations of Reduced Form Coefficients by ILS, 2SLS and 3SLS Methods

(Quarterly Series of UK 1967:2-2006:1)

Three stage least square

Growth equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	-0.0032	0.0258	-0.1230	0.9020
DLExrt_dllr	-0.0187	0.0725	-0.2580	0.7970
DLM4	0.6349	0.1940	3.2700	0.0010
Constant	0.0047	0.0062	0.7610	0.4480

Interest rate Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.1503	0.0752	2.0000	0.0470
DLExrt_dllr	-0.6302	0.2114	-2.9800	0.0030
DLM4	1.9670	0.5655	3.4800	0.0010
Constant	-0.0568	0.0179	-3.1700	0.0020

Inflation Equation

	Coefficient	Std.Error	t-value	t-prob
DLTbills_1	0.6128	0.8585	0.7140	0.4760
DLExrt_dllr	-2.7670	2.4130	-1.1500	0.2530
DLM4	30.6555	6.4530	4.7500	0.0000
Constant	0.8351	0.2047	4.0800	0.0000

AIC	-10.0094	SC	-9.8357
HQ	-9.9389	FPE	0.0000

Note: Modes is estimated using quarterly data 1970:2 to 2000:1 obtained from the Office of the National Statistics and each variable was properly checked for stationarity (Dickey and Fuller (1979), Johansen (1988)). Only variables cointegrated were used if there were nonstationary. All above models were estimated by PcGive (Doornik and Hendry ((2003)).

Unit-root tests (using macro_uk.csv)
The sample is 1967 (2) - 2006 (1)

DLRGDP: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-3.256*	0.49641	0.02237	-14.66	0.0000	-7.568	
2	-13.10**	-1.1546	0.03471	3.970	0.0001	-6.696	0.0000
1	-15.05**	-0.65147	0.03634	6.057	0.0000	-6.610	0.0000
0	-14.44**	-0.14804	0.04034			-6.408	0.0000

DLM4: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-2.629	0.76351	0.01276	-7.063	0.0000	-8.691	
2	-4.696**	0.54428	0.01467	0.009788	0.9922	-8.419	0.0000
1	-5.073**	0.54463	0.01462	-5.037	0.0000	-8.432	0.0000
0	-9.450**	0.27188	0.01574			-8.291	0.0000

Tbills: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-2.183	0.93751	1.086	-1.315	0.1905	0.1969	
2	-2.510	0.92960	1.089	0.9859	0.3258	0.1955	0.1905
1	-2.358	0.93525	1.089	1.715	0.0883	0.1891	0.2616
0	-2.068	0.94382	1.096			0.1953	0.1342

Exrt_dllr: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-2.579	0.95813	0.07713	0.7501	0.4543	-5.093	
2	-2.549	0.95873	0.07702	-0.7942	0.4283	-5.102	0.4543
1	-2.607	0.95792	0.07692	3.138	0.0020	-5.111	0.5524
0	-2.488	0.95871	0.07910			-5.061	0.0138

CPI: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-1.787	0.90181	0.8382	-5.288	0.0000	-0.3214	
2	-2.778	0.83832	0.9095	-0.9907	0.3234	-0.1644	0.0000
1	-3.100*	0.82472	0.9095	-6.734	0.0000	- 0.1707	0.0000
0	-5.653**	0.66898	1.032			0.07600	0.0000

DLTbills: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-6.340**	0.086375	0.1203	-0.4912	0.6240	-4.204	
2	-7.849**	0.048364	0.1200	2.489	0.0139	-4.215	0.6240
1	-7.549**	0.20553	0.1220	-1.031	0.3041	-4.188	0.0435
0	-10.91**	0.13423	0.1221			-4.194	0.0618

DLExrt_dllr: ADF tests (T=156, Constant; 5%=-2.88 1%=-3.47)

D-lag	t-adf	beta Y_1	sigma	t-DY_lag	t-prob	AIC	F-prob
3	-5.563**	0.23968	0.04347	-0.2538	0.8000	-6.240	
2	-6.436**	0.22356	0.04334	-0.6590	0.5109	-6.252	0.8000
1	-8.173**	0.17964	0.04326	0.7509	0.4539	-6.262	0.7806
0	-9.855**	0.22652	0.04319			-6.271	0.7881