Advanced Economic Theory

Definition of a univariate time series process
Trends, cycles, seasons and random elements
Box-Jenkis (1976) approach AR, MA, ARMA, ARIMA
Forecasting and forecast errors with univariate TS models
Stationarity and spuriousness
Unit root test and transformations of data for stationarity
Granger representation theorem, cointegration and error correction
Vector Autoregression
Eigen values and Eigen vectors
Johansen's tests of coitegration
Impulse Response
Forecasting

Four Components of a Time Series

Time Series

$$y_t = \mu_t + \gamma_t + \psi_t + \nu_t + \varepsilon_t \qquad \varepsilon_t \sim NID\left(0, \sigma_\varepsilon^2\right)$$

Trend $$\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t \qquad \varepsilon_t \sim NID\left(0, \sigma_\eta^2\right)$$

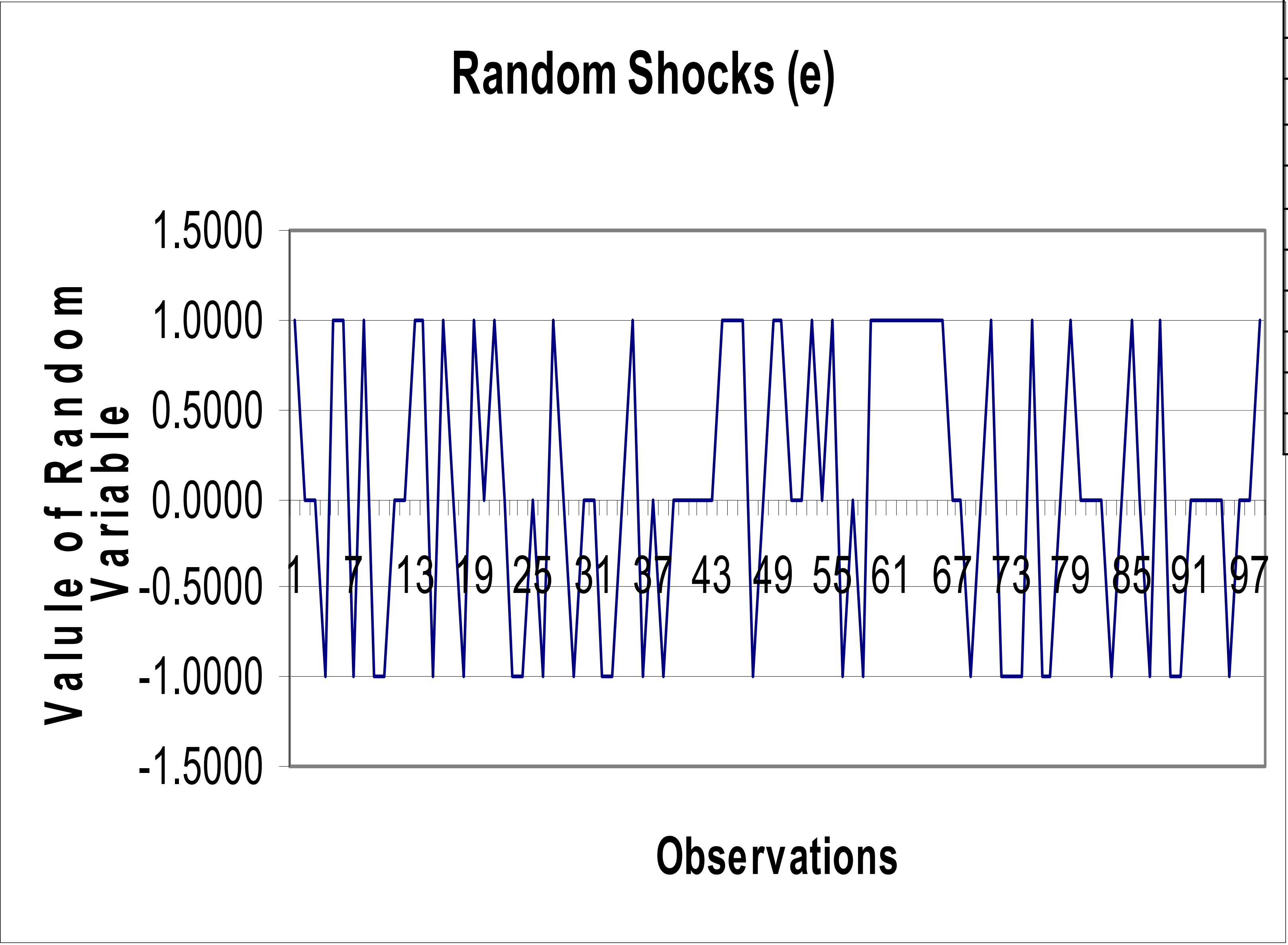
$$\beta_t = \beta_{t-1} + \zeta_t \qquad \zeta_t \sim NID\left(0, \sigma_\zeta^2\right)$$

eason $$\gamma_t = -\gamma_{t-1} - .. - \gamma_{t-s+1} + \omega_t \qquad \omega_t \sim NID\left(0, \sigma_w^2\right)$$

$$\begin{bmatrix} \gamma_{j,t} \\ \gamma_{j,t}^* \end{bmatrix} = \begin{bmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{bmatrix} \begin{bmatrix} \lambda_{j,t-1} \\ \lambda_{j,t-1}^* \end{bmatrix} + \begin{bmatrix} \omega_{j,t} \\ \omega_{j,t}^* \end{bmatrix} \qquad \begin{matrix} j = 1,\dots,[s/2] \\ t = 1,\dots.,T \end{matrix}$$

ycle $$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho_\psi \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{j,t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix} \qquad t = 1,\dots.,T \qquad 0 < \rho_\psi < 1$$

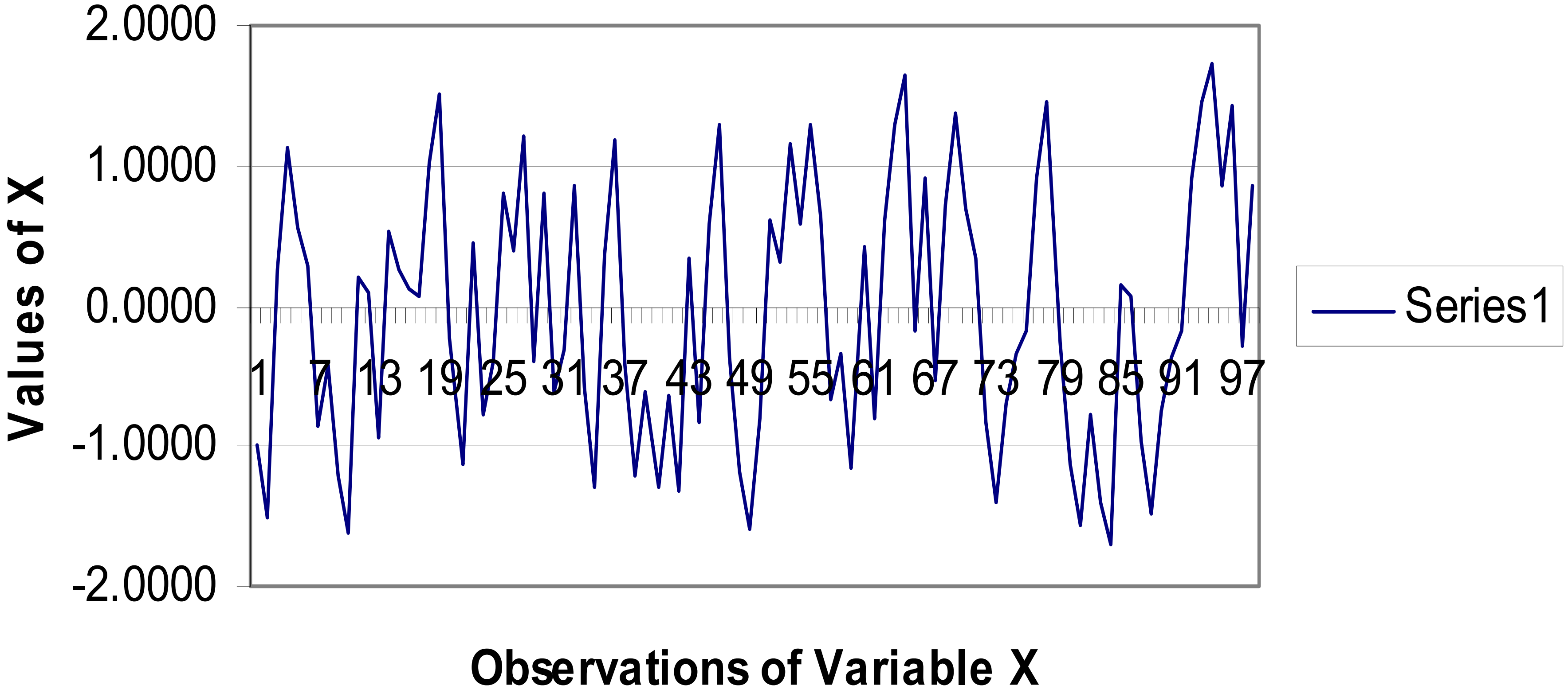
Random $$\nu_t = \rho_\nu \nu_{t-1} + \xi_t \qquad \xi_t \sim NID\left(0, \sigma_\zeta^2\right)$$



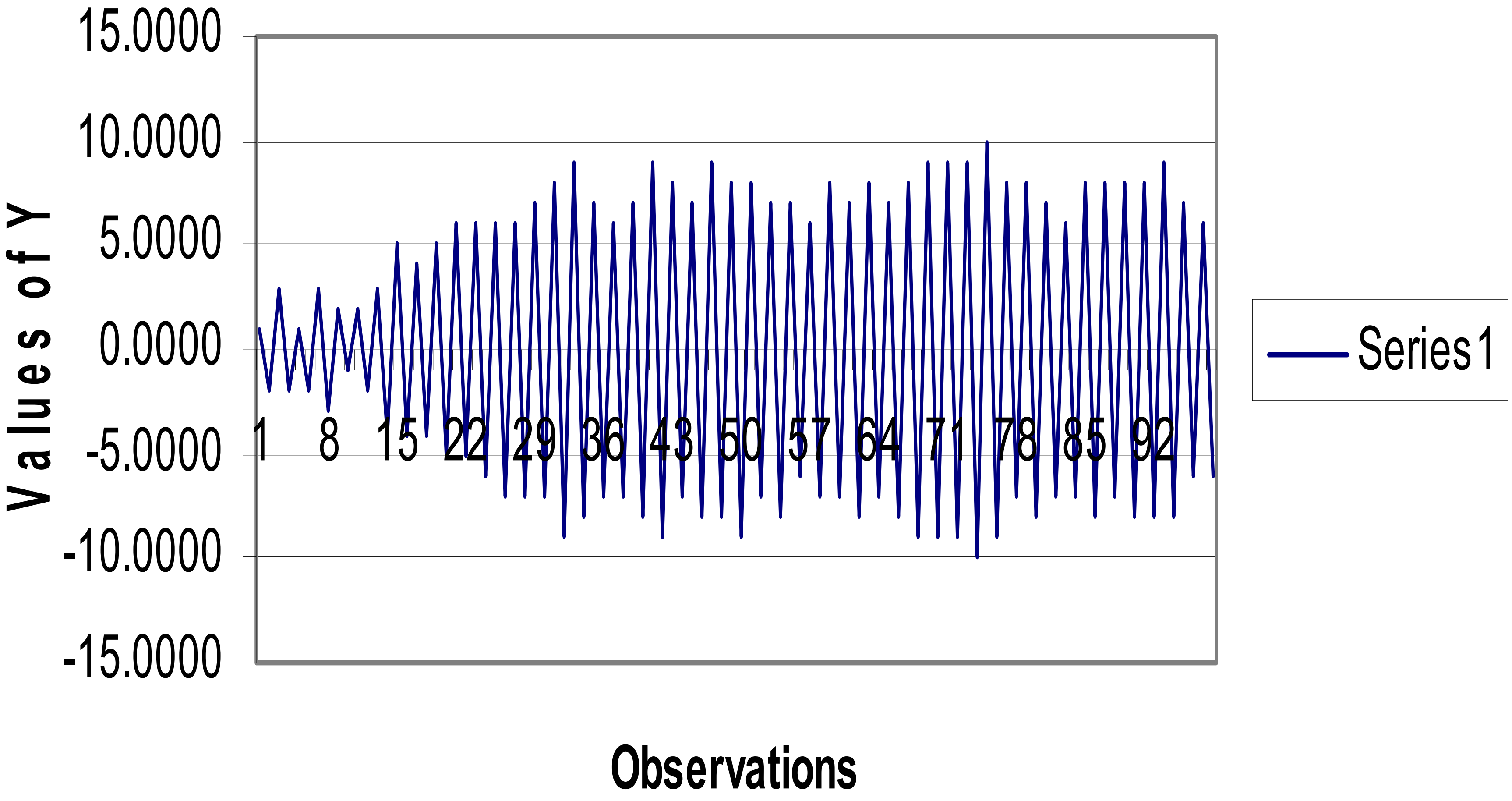
Time	e
0	-1.00
1	0.00
2	1.00
3	1.00
4	1.00
5	-1.00
6	1.00
7	-1.00
8	-1.00
9	1.00
10	-1.00

See: Time_series_Numerical Examples.xls

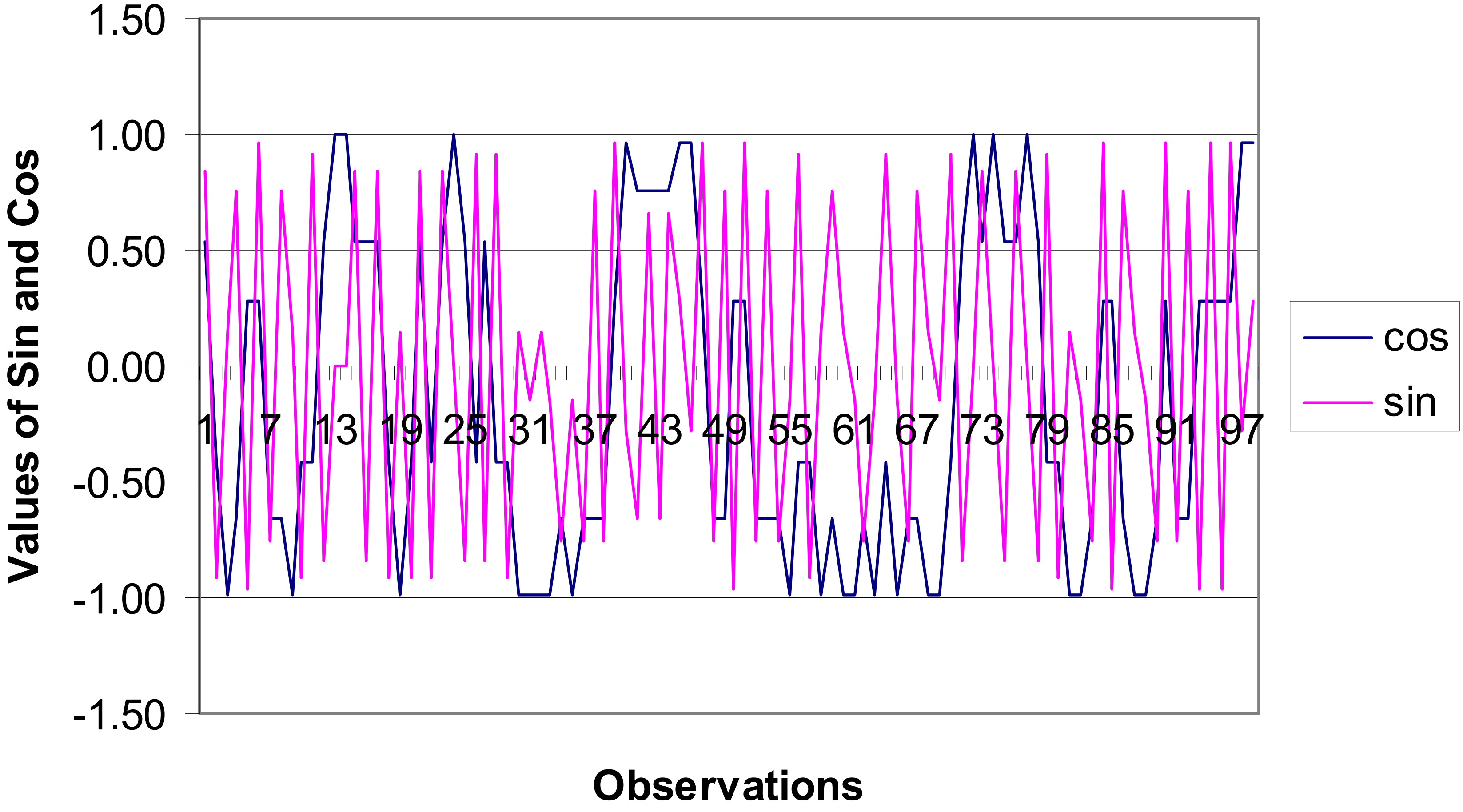
Convergent Stationary Series (x(T)= e(t) +0.5*x(t-1))



Non-Stationary Cyclical Fluctuation (X(t) = e(t)-x(t-1))



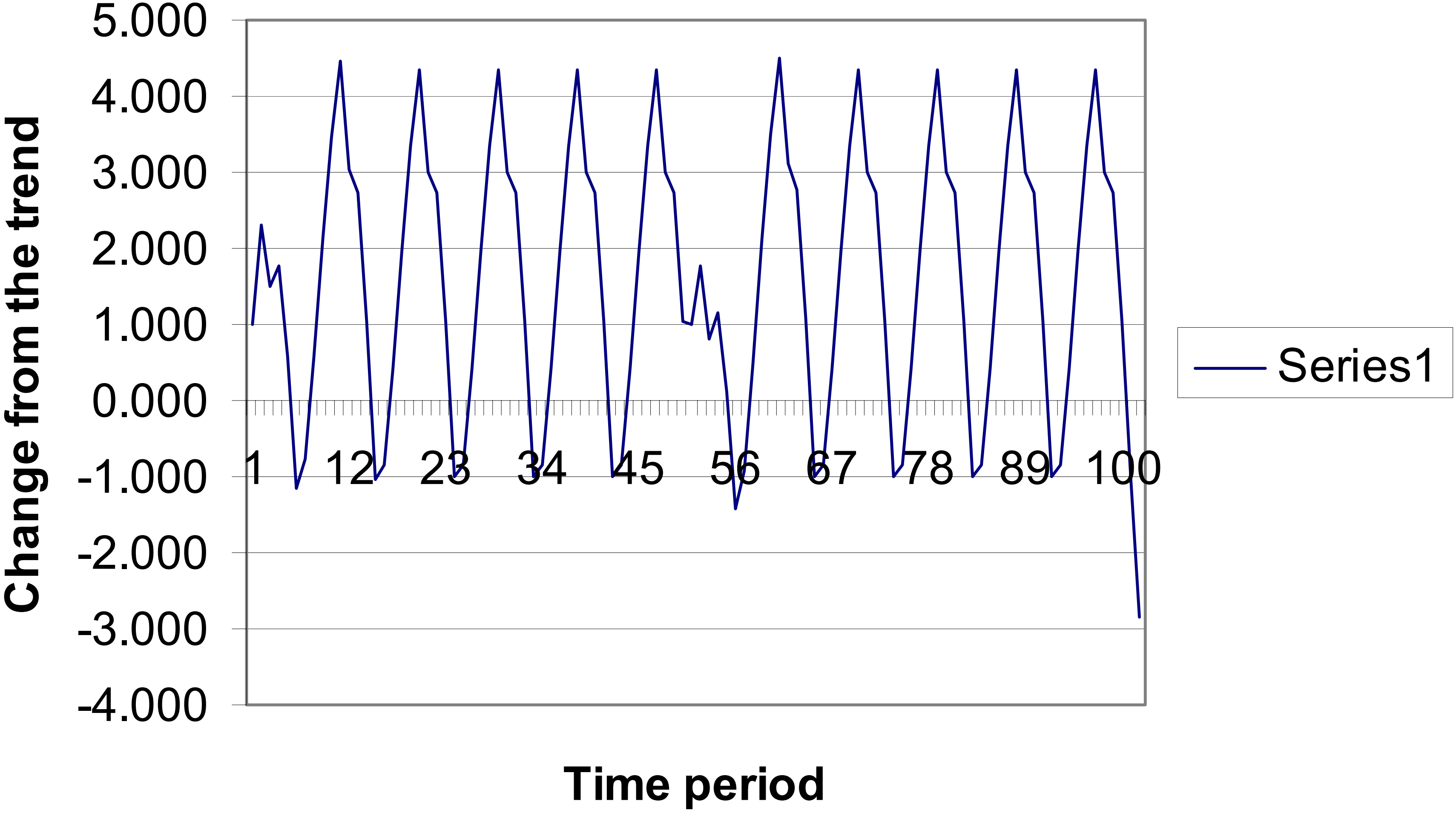
Periodic Functions: sin(x3) and cos(x3)



a=1.3
b=0.5

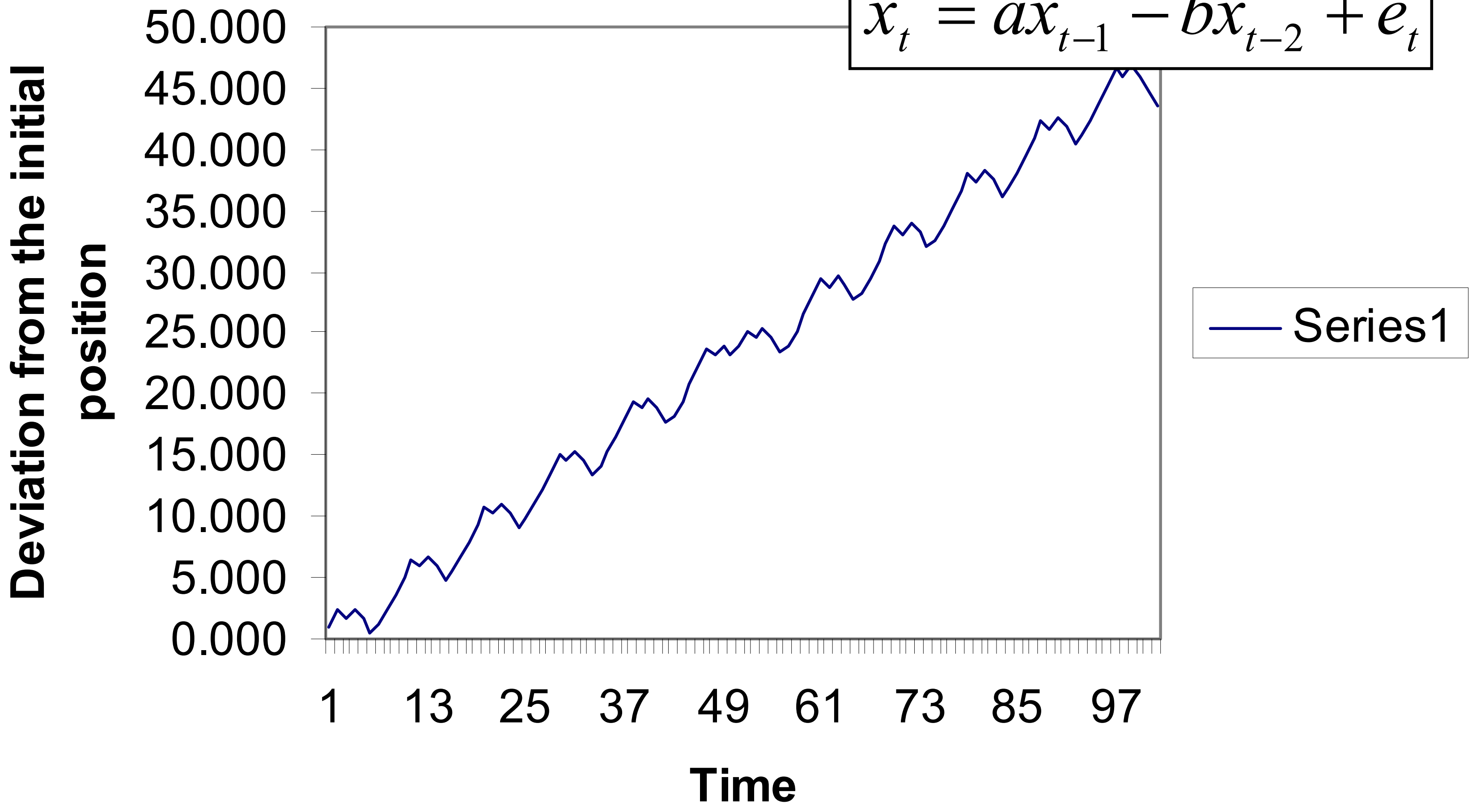
Business Cycle from Technical Shock

$$x_t = ax_{t-1} - bx_{t-2} + e_t$$



Trend with Cycle

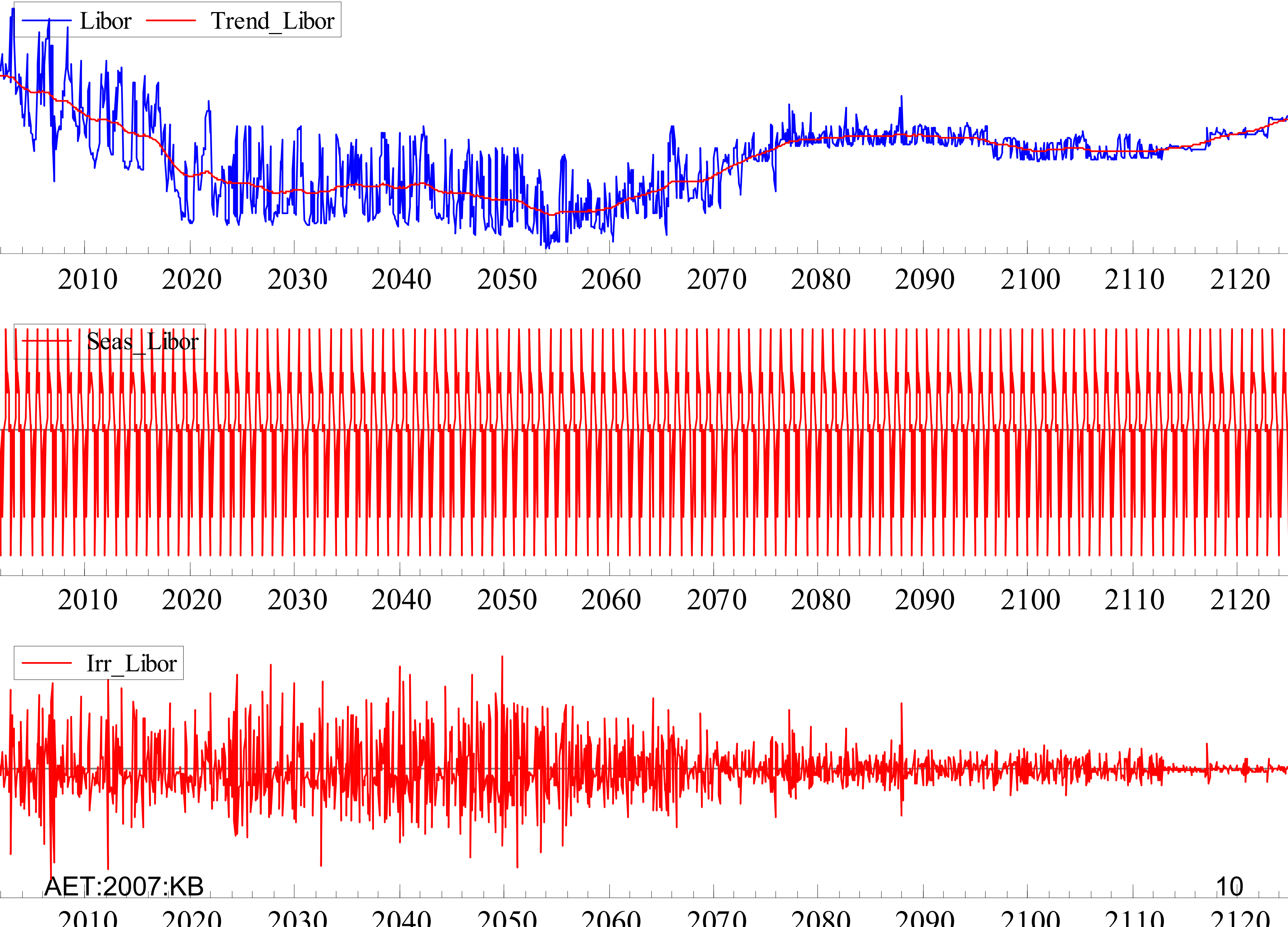
$$x_t = a x_{t-1} - b x_{t-2} + e_t$$



Stamp Program for Time Series Analysis

```
stimation sample is 1971. 2 - 2000. 1. (T =  116, n =  111).
og-Likelihood is 250.781 (-2 LogL = -501.563).
ediction error variance is 0.0106071
ummary statistics
                    ER
td.Error       0.10299
ormality        9.9490
( 37)          0.58124
 1)         0.0039775
 9)          -0.10584
W               1.9721
( 9, 6)         7.2307
s^2            -0.41360
R = Trend + Trigo seasonal + Expl vars + Irregular
```

```
Eq  3 : Estimated coefficients of final state vector.
Variable      Coefficient      R.m.s.e.       t-value
Lvl              1.2519       0.26875       4.6583  [ 0.0000]
Slp          -0.0056233     0.0082428      -0.68221  [ 0.496
Sea_ 1        0.0026817     0.0081813       0.32779  [ 0.74
Sea_ 2        0.0017017     0.0081876       0.20784  [ 0.83
Sea_ 3      -0.00036460     0.0040829     -0.089298  [ 0.
Eq  3 : Estimated coefficients of explanatory variables.
Variable      Coefficient      R.m.s.e.       t-value
ER_1            0.22213      0.096664        2.298  [ 0.0234
ER_2          -0.070019      0.099049      -0.70692  [ 0.48
ER_3           0.027285      0.099054       0.27545  [ 0.783
ER_4           0.035090      0.096723       0.36279  [ 0.717
Eq  3 : Seasonal analysis (at end of period).
Seasonal Chi^2( 3) test is 0.173462  [0.9818].
              Seas  1     Seas  2     Seas  3     Seas  4
 Value        0.0023171   0.0020663  -0.0030463  -0.001
```


Libor
Trend_Libor
2010
2020
2030
2040
2050
2060
2070
2080
2090
2100
2110
2120
Seas_Libor
2010
2020
2030
2040
2050
2060
2070
2080
2090
2100
2110
2120
Irr_Libor
2010
2020
2030
2040
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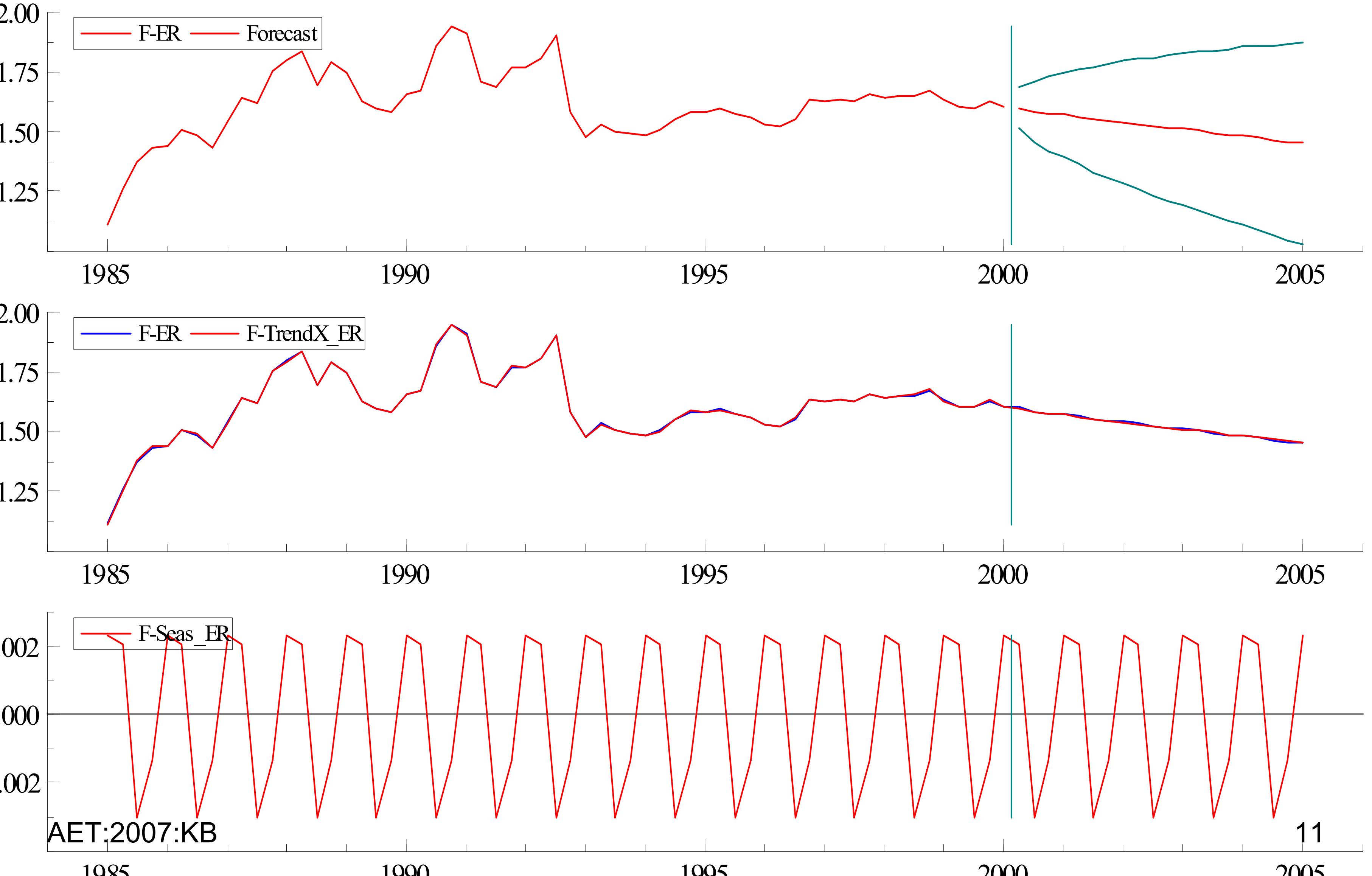
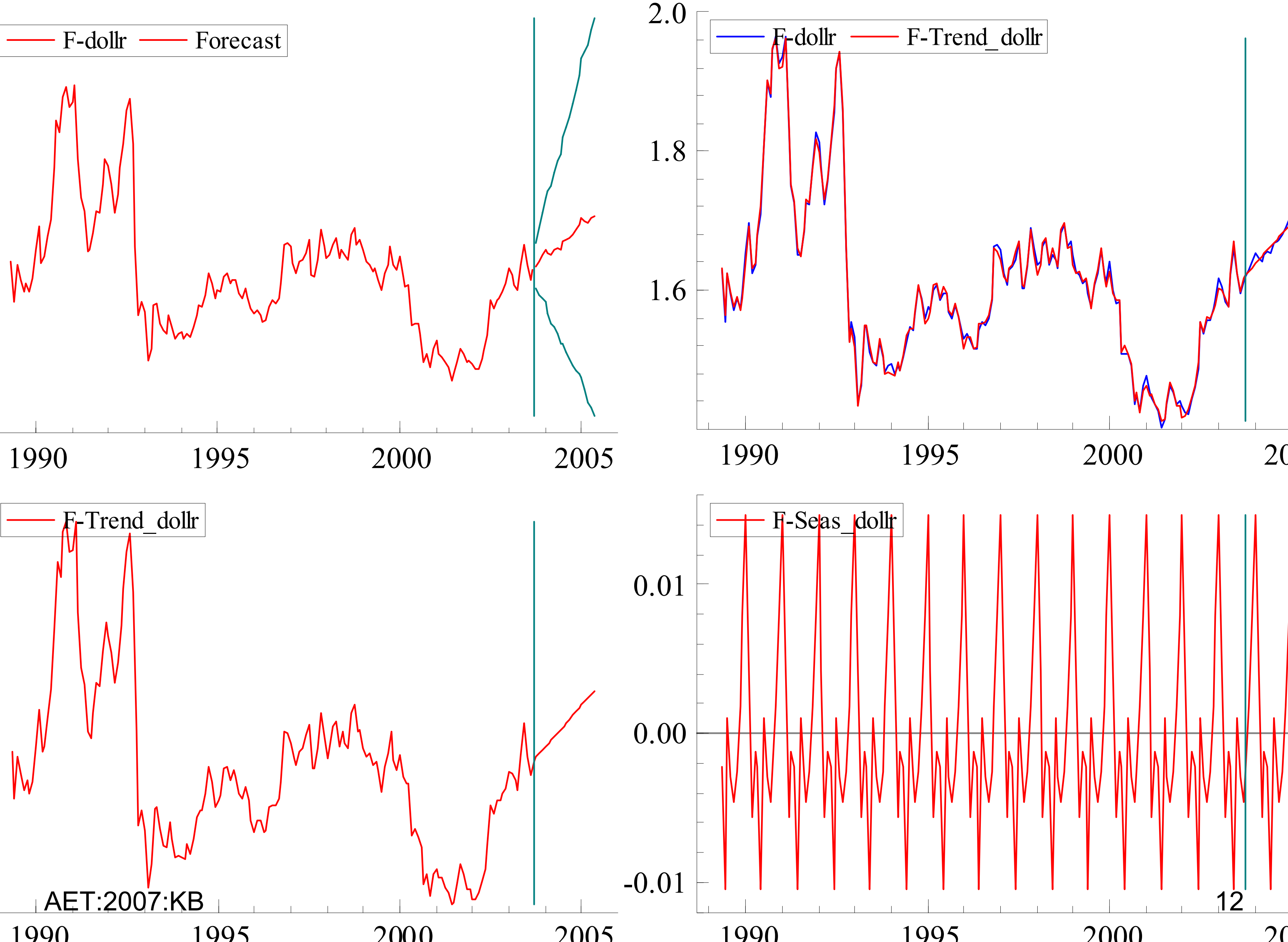
2060
2070
2080
2090
2100
2110
2120

Forecasting of the Exchange Rate



Forecasting the components of Time Series-Stamp



Box-Jenkins Approach (1976)

- An example of data generating process (dgp)
- Autoregressive model AR(p)
  - Mean, variance and autocorrelation functions
- Moving average models MA(q)
  - Mean, variance and autocorrelation functions
- Autoregressive moving average ARMA(p,q)
- ARIMA (p,d,q)
- Importance of Stationarity
- Test for Unit rood

An AR(1) Data Generating Process

$$y_t = \rho y_{t-1} + v_t$$
$$y_{t-1} = \rho y_{t-2} + v_{t-1}$$
$$y_{t-2} = \rho y_{t-3} + v_{t-3}$$
$$y_{t-3} = \rho y_{t-4} + v_{t-4}$$
$$. \quad \cdots \qquad\qquad \cdots$$
$$y_{t-n} = \rho y_{t-n-1} + v_{t-n} \qquad (5)$$

From substitution

$$y_t = \rho y_{t-1} + v_t =$$

$$y_t = \rho\left[\rho y_{t-2} + v_{t-1}\right] + v_t = \rho^2 y_{t-2} + \rho v_{t-1} + v_t$$

$$y_t = \rho\left[\rho y_{t-3} + v_{t-2}\right] + v_t$$

$$y_t = \rho^2\left[\rho y_{t-3} + v_{t-2}\right] + \rho v_{t-1} + v_t = \rho^3 y_{t-3} + \rho^2 v_{t-2} + \rho v_{t-1} + v_t$$

AR(1) Time Series as a Function of Past Innovations (Impulses or Shock

$$y_t = \rho^n y_{t-n} + \rho^{n-1} v_{t-n-1} + \ldots + \rho^3 y_{t-3} + \rho^2 v_{t-2} + \rho$$

(6)

In the limit the term $\rho^n y_{t-n}$ becomes close to zero as $n \to \infty$.

Rearranging (6) we can write $\{y_t\}$ in terms of current and past values of error terms

$$y_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \rho^3 v_{t-3} + \rho^4 v_{t-4} + \ldots + \rho^{n-1} v$$

(7)

Mean and time Dependent Variance

What is the mean of $\{y_t\}$ in (7)?

$$E(y_t) = E(v_t) + \rho E(v_{t-1}) + \rho^2 E(v_{t-2}) + \rho^3 E(v_{t-3}) + \rho^4 E(v_{t-4}) + \ldots + \rho^{n-1} E(v_{t-n})$$

Because of assumption $v_t \sim N\left(0, \sigma_v^2\right)$; $E(y_t) = 0$

What is the variance of $\{y_t\}$?

$$Var(y_t) = Var\left[(v_t) + \rho(v_{t-1}) + \rho^2(v_{t-2}) + \rho^3(v_{t-3}) + \rho^4(v_{t-4}) + \ldots + \rho^{n-1}(v_{t-n})\right] \text{if } \rho = 1$$

and

if there is no autocorrelation among the random terms $E(v_t v_{t-1}) = 0$

$$Var(y_t) = \sigma_v^2 + \sigma_v^2 + \sigma_v^2 + \ldots.. + \sigma_v^2 = t.\sigma_v^2$$

Thus the variance of Y term increases with time. This makes this series non stationary.

Rule of thumb : A series is non-stationary if $|\rho| \geq 1$.

A series is stationary if $|\rho| \leq 1$.

Moving Average-MA Process

$$Y_t = \mu + e_t + \alpha_1 e_{t-1}$$
$$E(y_t) = \mu$$
$$\operatorname{var}(y_t) = \operatorname{var}(\mu + e_t + \alpha_1 e_{t-1}) = \sigma_e^2 (1 + \alpha_1)$$
$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - \mu)(y_{t-1} - \mu) = \sigma_e^2 \alpha_1$$

Autocorrelation function: it tapers off after k lags

$$\rho_k = \frac{\operatorname{cov}(y_t y_{t-1})}{\operatorname{var}(y_t)} = \frac{\alpha_1 \sigma_e^2}{\sigma_e^2 (1 + \alpha_1^2)}$$

Some examples of MA (1) process:

$$Y_t = \mu + e_t + 0.8 e_{t-1}$$
$$Y_t = \mu + e_t - 0.8 e_{t-1}$$

MA(2) Process

$$Y_t = \mu + e_t + \alpha_1 e_{t-1} + \alpha_2 e_{t-2}$$
$$E(y_t) = \mu$$

$$\operatorname{var}(y_t) = \operatorname{var}(\mu + e_t + \alpha_1 e_{t-1} + \alpha_2 e_{t-2}) = \sigma_e^2 \left(1 + \alpha_1^2 + \alpha_2^2\right)$$

$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - \mu)(y_{t-1} - \mu) = \sigma_e^2 \left(\alpha_1 + \alpha_1 \alpha_2\right)$$

$$\operatorname{cov}(Y_t Y_{t-2}) = E(y_t - \mu)(y_{t-2} - \mu) = \alpha_2 \sigma_e^2$$
$$\operatorname{cov}(Y_t Y_{t-3}) = 0$$

$$\rho_1 = \frac{\operatorname{cov}\left(y_t y_{t-1}\right)}{\operatorname{var}(y_t)} = \frac{\alpha_1 \left(1 + \alpha_2\right)}{\left(1 + \alpha_1^2 + \alpha_2^2\right)}$$

$$\rho_2 = \frac{\operatorname{cov}\left(y_t y_{t-2}\right)}{\operatorname{var}(y_t)} = \frac{\alpha_1}{\left(1 + \alpha_1^2 + \alpha_2^2\right)} \quad ; \; \rho_k = 0$$

MA(2) process has tow period long memory.

Autoregressive Process

$$Y_t = \delta + \theta_1 y_{t-1} + e_t$$

$$E(y_t) = E(y_{t-1}) = ... = E(y_{t-k}) = \mu$$

$$E(Y_y) = E(\delta + \theta_1 y_{t-1} + e_t)$$

$$\mu = \delta + \theta_1 \mu \ \ ; \ \mu = \frac{\delta}{1-\theta_1}$$

$$\operatorname{var}(y_t) = \operatorname{var}(\theta_1 y_{t-1} + e_t) \ \ => \ \sigma_y^2 = \frac{\sigma_e^2}{1-\theta_1^2}$$

$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - E(y_t))(y_{t-1} - E(y_t)) = \theta_1 \sigma_y^2$$

Some examples:

$$Y_t = 0.8 y_{t-1} + e_t$$
$$Y_t = -0.8 y_{t-1} + e_t$$

Convergence occurs if $|\theta_1| < 1$. The series is called stationary.

ARMA(1,1) Process

$$Y_t = \delta + \theta_1 y_{t-1} + e_t + \alpha_1 e_{t-1}$$

$$\operatorname{var}(y_t) = E\left[(y_t - \mu)^2\right] = E\left[\left(\delta + \theta_1 y_{t-1} + e_t + \alpha_1 e_{t-1}\right)^2\right]$$

$$= \theta_1^2 \gamma_0 + \sigma_e^2 + \alpha_1^2 \sigma_e^2 + 2\theta_1 \alpha_1 E\left[y_{t-1} e_{t-1}\right]$$

Steps for estimating AR, MA and ARIMA model in PcGive

- Prepare a time series data in excel and open it with GiveWin (stockprice.xls, Diesel_weeklyUS.xls)
- Start PcGive and Select Packages Time series Model and ARFIMA
- Formulate Model/Choose variables for analysis
- Select constant/trend/seasonal elements as necessary
- Choose order of AR or MA in model settings
- Choose maximum likelihood or non-linear method
- Perform estimation and study results
- Do graphics of actual and fitted values
- Choose the forecasting horizon and settings for the error bands
- Study the graphs and compare to the hypothesis
- These issues can be studied also in X12 ARIMA

Autoregressive AR(1) Process

$$Y_t = \delta + \theta_1 y_{t-1} + e_t$$

$$E(y_t) = E(y_{t-1}) = \ldots\ldots = E(y_{t-k}) = \mu$$

$$E(Y_y) = E(\delta + \theta_1 y_{t-1} + e_t)$$

$\mu = \delta + \theta_1 \mu$; $\mu = \dfrac{\delta}{1-\theta_1}$; Assume $\delta = 0$ means $E(y) = 0$

$$\operatorname{var}(y_t) = \operatorname{var}(\theta_1 y_{t-1} + e_t) \Rightarrow \sigma_y^2 = \frac{\sigma_e^2}{1-\theta_1^2}$$

$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - E(y_t))(y_{t-1} - E(y_t)) = \theta_1 \sigma_y^2$$

Some examples:

$$Y_t = 0.8 y_{t-1} + e_t$$
$$Y_t = -0.8 y_{t-1} + e_t$$

Convergence occurs if $|\theta_1| < 1$. The series is called stationary.

Moving Average-MA(1) Process

$$Y_t = \mu + e_t + \alpha_1 e_{t-1}$$

$$E(y_t) = \mu$$

$$\operatorname{var}(y_t) = \operatorname{var}(\mu + e_t + \alpha_1 e_{t-1}) = \sigma_e^2 \left(1 + \alpha_1\right)$$

$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - \mu)(y_{t-1} - \mu) = \sigma_e^2 \alpha_1$$

Autocorrelation function: it tapers off after k lags

$$\rho_1 = \frac{\operatorname{cov}\left(y_t y_{t-1}\right)}{\operatorname{var}\left(y_t\right)} = \frac{\alpha_1 \sigma_e^2}{\sigma_e^2 \left(1 + \alpha_1^2\right)}$$

Some examples of MA (1) process:

$$Y_t = \mu + e_t + 0.8 e_{t-1}$$

$$Y_t = \mu + e_t - 0.8 e_{t-1}$$

ARMA(1,1) Process: Mean and Variance

$$Y_t = \delta + \theta_1 y_{t-1} + e_t + \alpha_1 e_{t-1}$$

$$\gamma_0 = \mathrm{var}(y_t) = E\left[(y_t - \mu)^2\right] = E\left[\left(\delta + \theta_1 y_{t-1} + e_t + \alpha_1 e_{t-1}\right)^2\right]$$

$$= \theta_1^2 \gamma_0 + \sigma_e^2 + \alpha_1^2 \sigma_e^2 + 2\theta_1 \alpha_1 E\left[y_{t-1} e_{t-1}\right]$$

$$E\left[y_{t-1} e_{t-1}\right] = E\left(\delta + \theta_1 y_{t-2} + e_{t-1} + \alpha_1 e_{t-2}\right) e_{t-1} = E\left(e_{t-1}\right)^2 = \sigma_e^2$$

$$\gamma_0 = \frac{\left(1 + \alpha_1^2 + 2\theta_1 \alpha_1\right)}{1 - \theta_1^2} \sigma_e^2$$

ARMA(1,1) Process: Covariance

$$\left[y_{t-1}y_t\right]=E\left[\left(\delta+\theta_1 y_{t-1}+e_t+\alpha_1 e_{t-1}\right)y_{t-1}\right]=\theta_1\gamma_0+\alpha\sigma$$

$$\left[y_{t-1}y_t\right]=E\left[\theta_1\left(y_{t-1}y_{t-1}\right)\right]+E\alpha_1\left[y_{t-1}e_{t-1}\right]=\theta_1\frac{\left(1+\alpha_1^2+2\theta_1\alpha_1\right)}{1-\theta_1^2}\sigma_e^2+\alpha_1\sigma_e^2$$

$$\left[y_{t-1}y_t\right]=\theta_1\frac{\left(1+\alpha_1^2+2\theta_1\alpha_1\right)}{1-\theta_1^2}\sigma_e^2+\alpha_1\sigma_e^2=\frac{\theta_1\left(1+\alpha_1^2+2\theta_1\alpha_1\right)+\alpha_1-\alpha_1\theta_1^2}{1-\theta_1^2}\sigma_e^2$$

$$=\frac{\theta_1+\alpha_1^2\theta_1+2\theta_1^2\alpha_1+\alpha_1-\alpha_1\theta_1^2}{1-\theta_1^2}\sigma_e^2=\frac{\left(\theta_1+\alpha_1^2\theta_1+\theta_1^2\alpha_1+\alpha_1\right)}{1-\theta_1^2}\sigma_e^2=\frac{\left(\theta_1+\alpha_1\right)\left(1+\theta_1\alpha_1\right)}{1-\theta_1^2}$$

$$\gamma_k=\theta_1\gamma_{k-1} \qquad k\geq 2$$

ARMA(1,1) Process: Autocorrelation Coefficients

$$\rho_1 = \frac{\mathrm{cov}\left(y_t y_{t-1}\right)}{\mathrm{var}\left(y_t\right)} = \frac{\gamma_1}{\gamma_0} = \frac{\left(1+\alpha_1^2+2\theta_1\alpha_1\right)}{\left(\theta_1+\alpha_1\right)\left(1+\theta_1\alpha_1\right)}$$

$$\rho_k = \theta_1 \rho_{k-1}$$

This process converges when $0 \le \rho_i \le 1$

Partial autocorrelation function and Ljung and Box statistics

$$\rho^S = \frac{\operatorname{cov}\left(y_t y_{t+1}\right)}{\operatorname{var}\left(y_t\right)} = \frac{\gamma_s}{\gamma_0}$$

where $\hat{\gamma}_S = \frac{\sum\left(y_t - \bar{y}\right)\left(y_{t+s} - \bar{y}\right)}{T}$ $\quad \hat{\gamma}_0 = \frac{\sum\left(y_t - \bar{y}\right)^2}{T}$

Use Ljung and Box statistics to test whether all ρ are equal to zero

$$Q = T\left(T+2\right)\sum_{s=1}^{m}\frac{\hat{\rho}_s^2}{T-s}$$

This statistic has a χ_m^2 distribution.

AR(2) Process: Mean

$$Y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-1} + e_t$$

$$E(y_t) = E\left(y_{t-1}\right) = \ldots.. = E\left(y_{t-k}\right) = \mu$$

$$E(Y_y) = E\left(\delta + \theta_1 y_{t-1} + \theta_2 y_{t-1} + e_t\right)$$

$$\mu = \delta + \theta_1 \mu + \theta_2 \mu \qquad \mu = \frac{\delta}{1 - \theta_1 - \theta_2}$$

AR(2) Process: Variance and Covariance

$$\text{var}\left(y_t\right) = E\left[\left(\theta_1 y_{t-1} + \theta_2 y_{t-2} + e_t\right) y_t\right]$$

$$\gamma_0 = \theta_1 \gamma_1 + \theta_2 \gamma_2 + \sigma_e^2$$

$$\text{ov}(Y_t Y_{t-k}) = \gamma_k = E\left[\left(\theta_1 y_{t-1} + \theta_2 y_{t-2} + e_t\right) y_{t-k}\right] = \theta_1 \gamma_{k-1} + \theta_2 \gamma_{k-2}$$

$$\gamma_k = \theta_1 \gamma_{k-1} + \theta_2 \gamma_{k-2}$$

Dividing both sides by γ_0

$$\rho_k = \theta_1 \rho_{k-1} + \theta_2 \rho_{k-2}$$

AR(2) Process: Partial autocorrelation

$$\rho_k = \theta_1 \rho_{k-1} + \theta_2 \rho_{k-2}$$

$$\rho_k = \rho_{-k}$$

$$\rho_0 = 1$$

$$\rho_1 = \frac{\theta_2}{(1-\theta_1)}$$

$$\rho_2 = \theta_2 + \frac{\theta_1^2}{(1-\theta_2)}$$

MA(2) Process

$$Y_t = \mu + e_t + \alpha_1 e_{t-1} + \alpha_2 e_{t-2}$$

$$E(y_t) = \mu$$

$$\operatorname{var}(y_t) = \operatorname{var}(\mu + e_t + \alpha_1 e_{t-1} + \alpha_2 e_{t-2}) = \sigma_e^2\left(1 + \alpha_1^2 + \alpha_2^2\right)$$

$$\operatorname{cov}(Y_t Y_{t-1}) = E(y_t - \mu)(y_{t-1} - \mu) = \sigma_e^2\left(\alpha_1 + \alpha_1\alpha_2\right)$$

$$\operatorname{cov}(Y_t Y_{t-2}) = E(y_t - \mu)(y_{t-2} - \mu) = \alpha_2\sigma_e^2$$

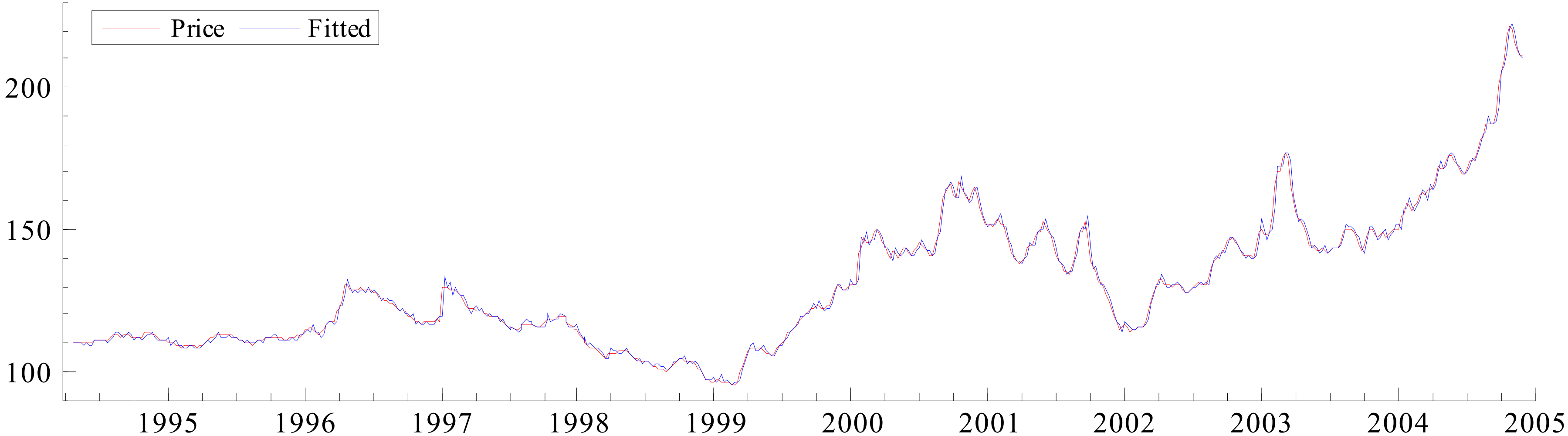
$$\operatorname{cov}(Y_t Y_{t-3}) = 0$$

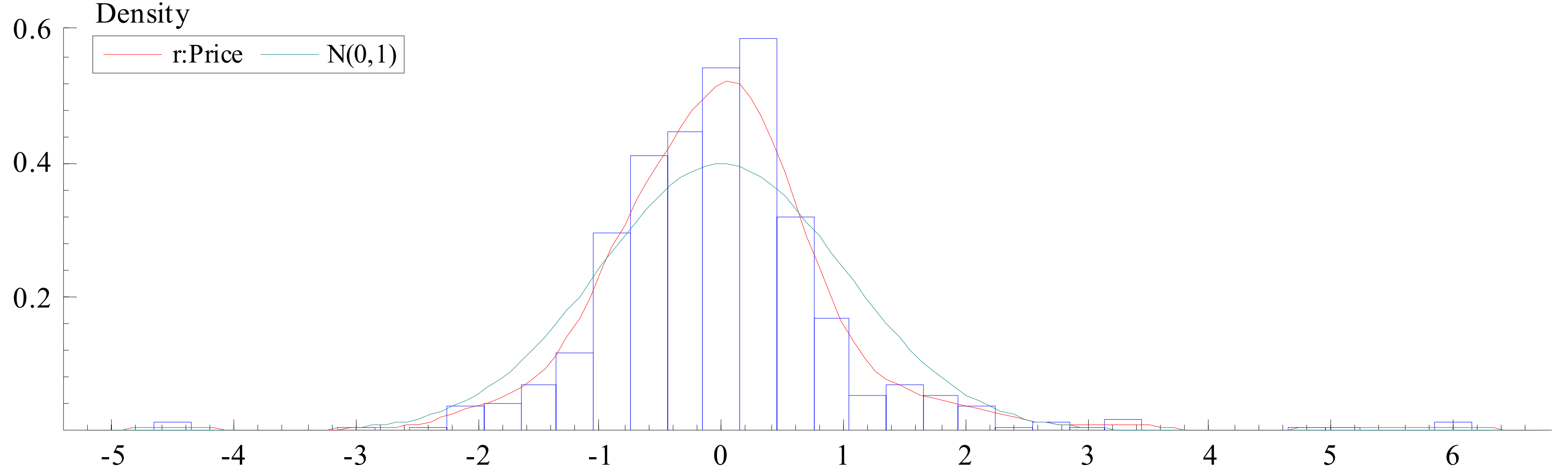
$$\rho_1 = \frac{\operatorname{cov}\left(y_t y_{t-1}\right)}{\operatorname{var}(y_t)} = \frac{\alpha_1\left(1 + \alpha_2\right)}{\left(1 + \alpha_1^2 + \alpha_2^2\right)}$$

$$\rho_2 = \frac{\operatorname{cov}\left(y_t y_{t-2}\right)}{\operatorname{var}(y_t)} = \frac{\alpha_1}{\left(1 + \alpha_1^2 + \alpha_2^2\right)} \quad ; \ \rho_k = 0$$

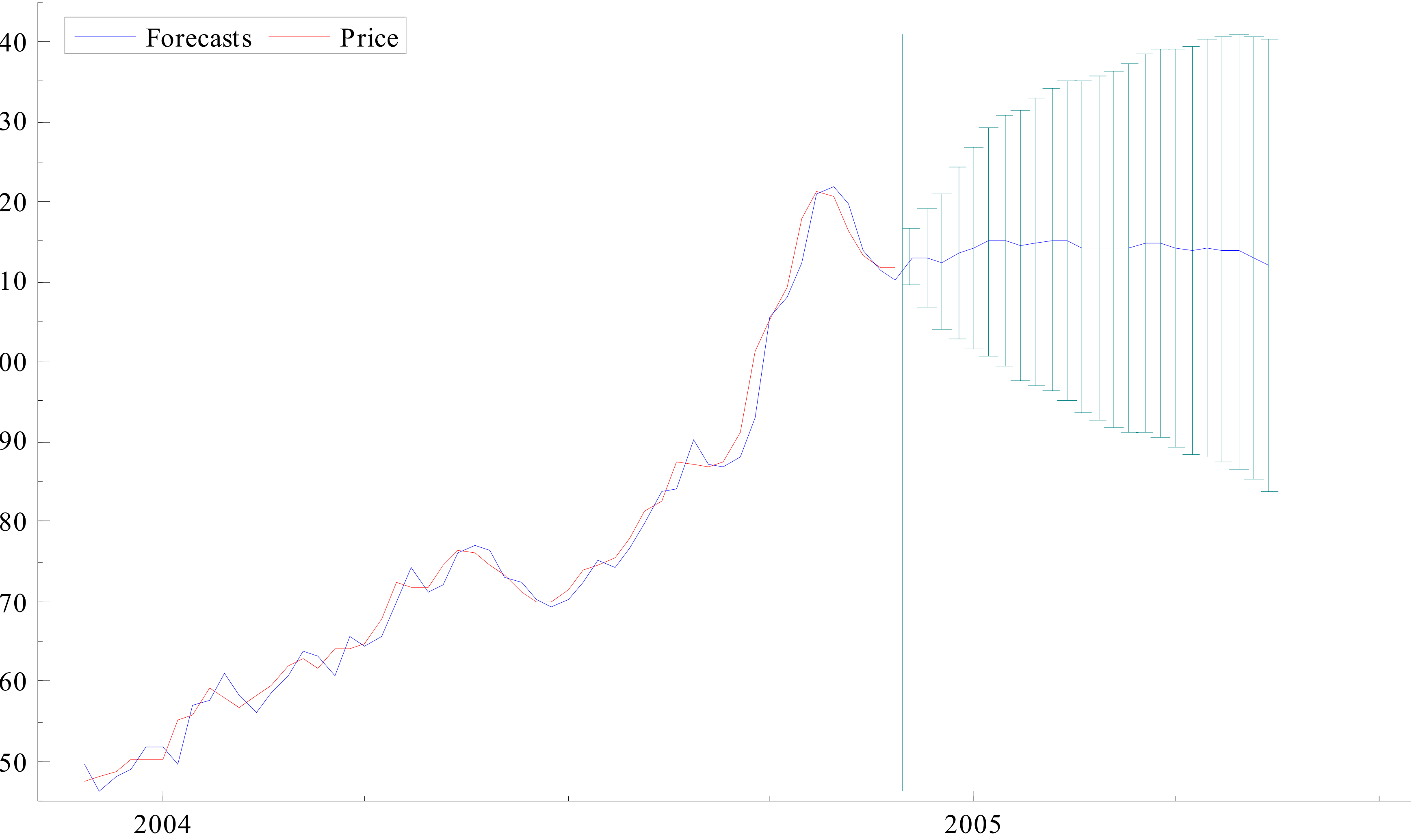
MA(2) process has tow period long memory.

eakly Prices of Diesel in the US AR(6) Model: PcGive Graphics





Weakly Prices of Diesel in the US



Data source: Economagic.com

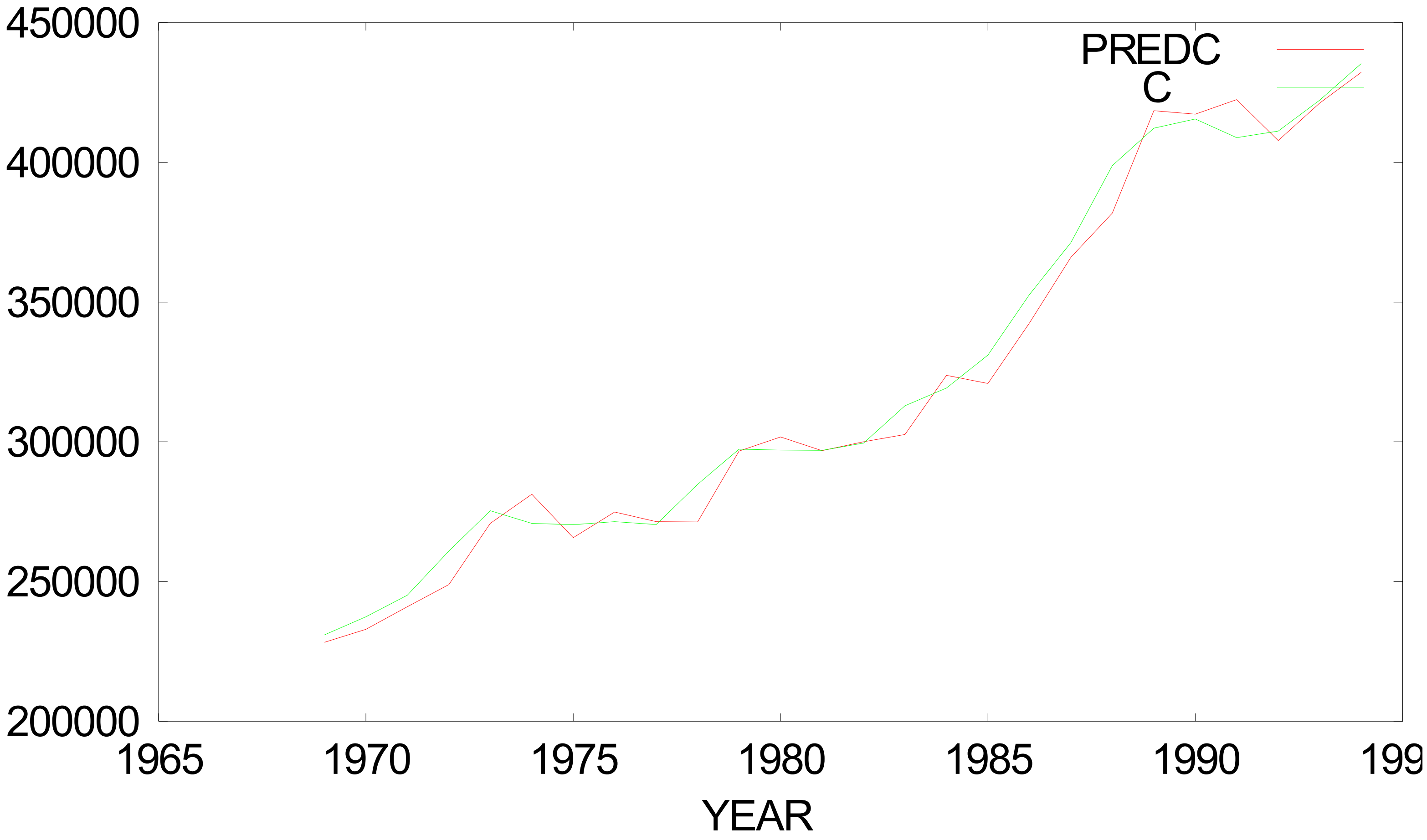
B-Given AR(6) F

Commands for ARMA (2,2) forecasting model

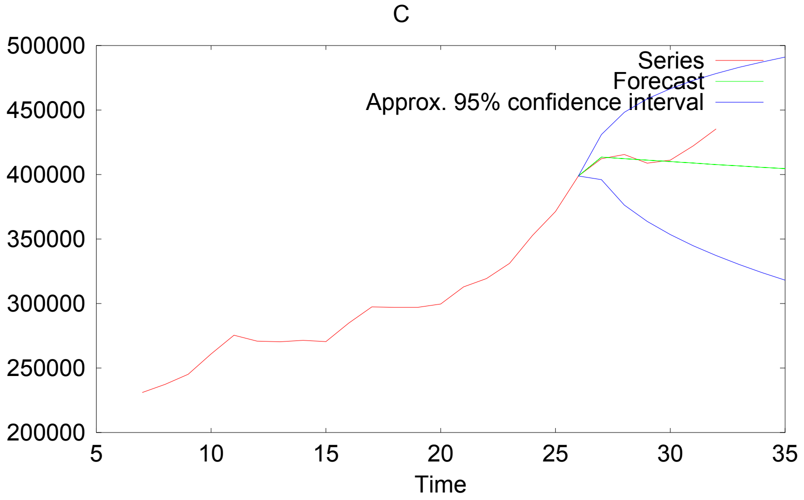
```
Arima C / NAR=2 NMA=2  Predict=predc plotac plotpac acf=cacf

plot predc c year /gnu lineonly

dim alpha 3
gen1 alpha:1=0.5
gen1 alpha:2=-0.2
gen1 alpha:3=100

arima c /NAR=1 NMA=1 coef=beta start=alpha
gen1 S=sqrt($sig2)
arima c/NAR=1 NMA=1 coef=beta fbeg=26 fend=30 sigma=s gnu
arima c/NAR=1 NMA=1 coef=beta fbeg=26 fend=35 sigma=s gnu

arima c/NAR=1 NMA=1 coef=beta fbeg=26 fend=40 sigma=s gnu

arima c/NAR=1 NMA=1 coef=beta fbeg=26 fend=50 sigma=s gnu

arima c/NAR=1 NMA=1 coef=beta fbeg=26 fend=100 sigma=s gnu
```

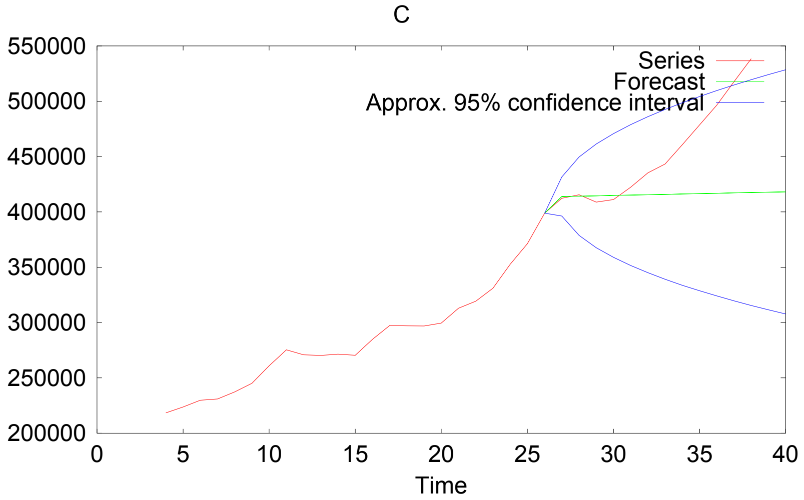
Plot of predected and actual investment series. It is a very good prediction.

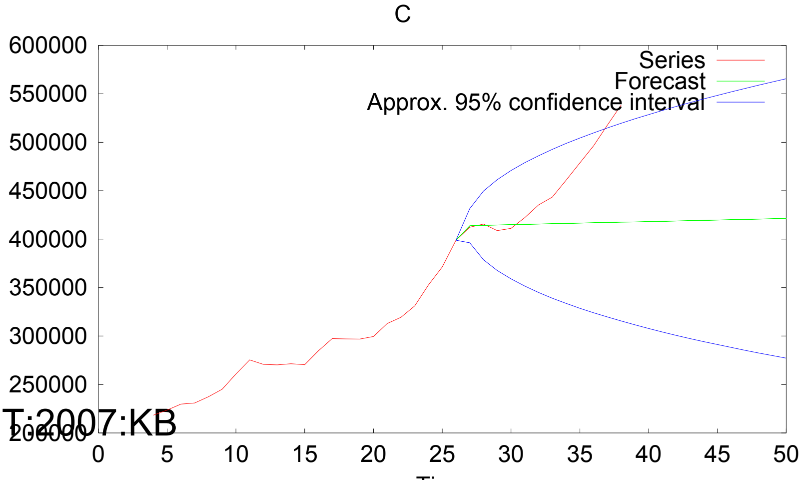
Prediction with an ARIMA Model

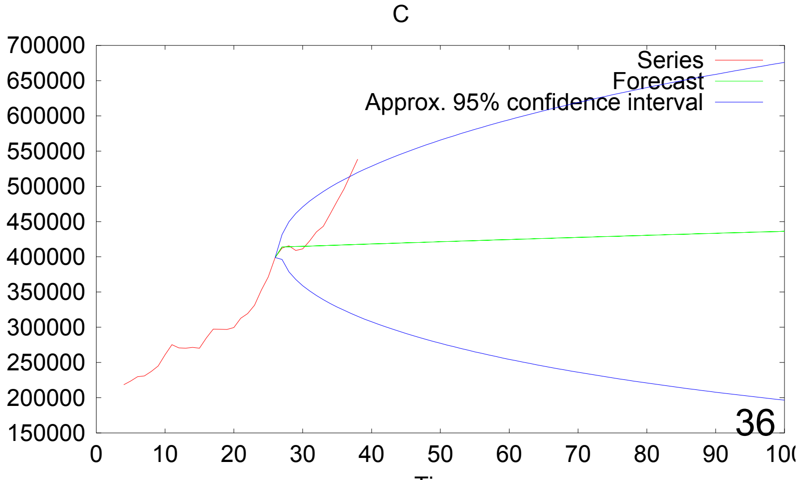


Forecast from an ARIMA Model

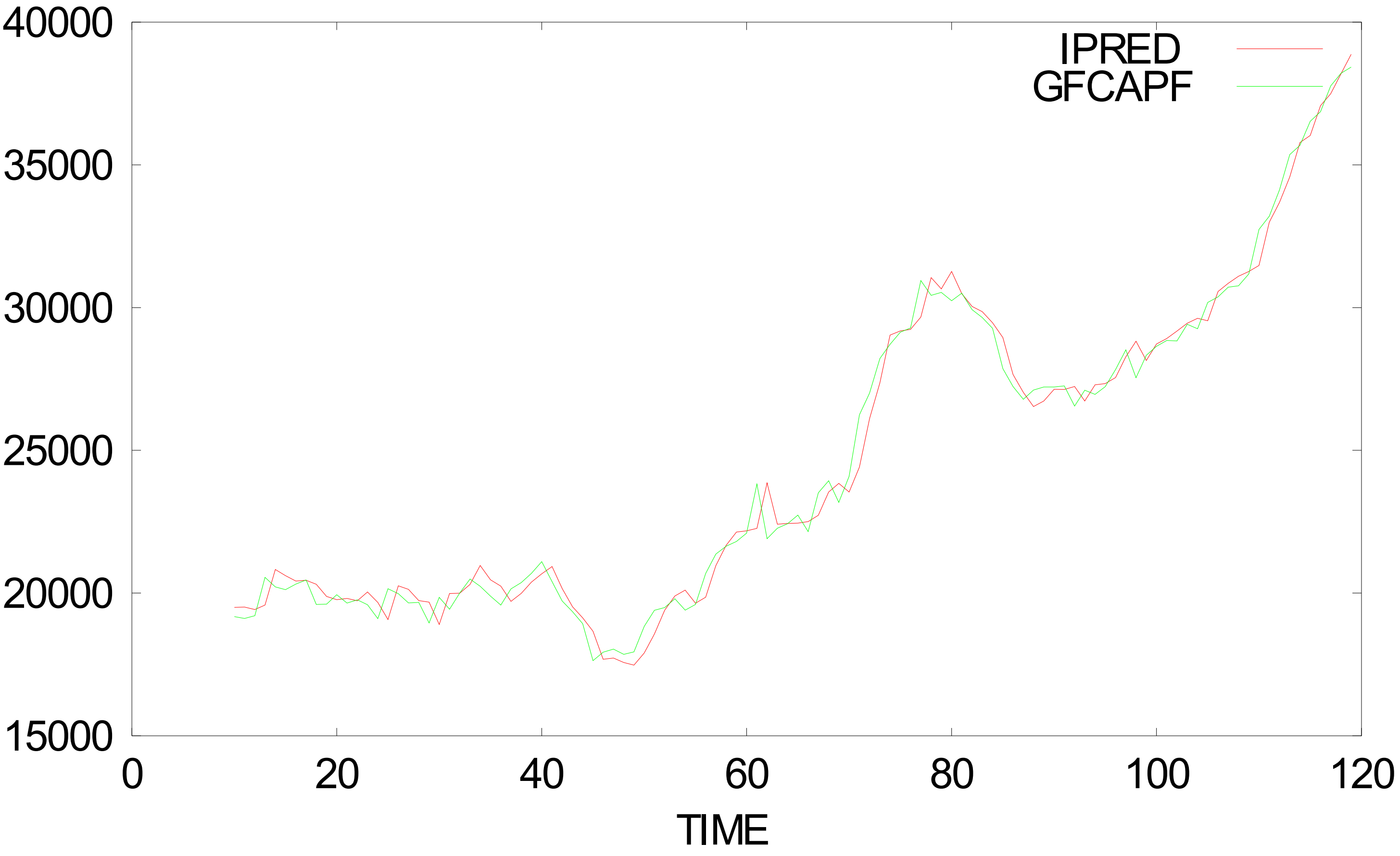








Prediction with AR or MA or ARMA Processes



Economic Forecasting

Mean and Variance Forecast in AR, MA and ARMA Model

(see more on CGH Chapter 20)

h =1 period ahead Forecast in AR(1) Model

$$y_t = \delta + \theta_1 y_{t-1} + e_t$$

$$y_{T+1} = \delta + \theta_1 y_T + e_{T+1} \qquad e_{T+1} \sim N(0,1)$$

$$\hat{Y}_{T+1} = E(Y_{T+1}) = \delta + \theta_1 y_T$$

Error of Forecast

$${}_{T+1} = Y_{T+1} - \hat{Y}_{T+1} = \delta + \theta_1 y_T + e_{T+1} - \delta - \theta_1 y_T = e_{T+}$$

$$\operatorname{var}\left(\hat{e}_{T+1}\right) = \sigma_e^2$$

h =2 periods ahead Forecast in AR(1) Model

$$Y_{T+2} = \delta + \theta_1 y_{T+1} + e_{T+2} \qquad e_{T+2} \sim N(0,1)$$

$$\hat{Y}_{T+2} = E\left(Y_{T+1}\right) = \delta + \theta_1 \hat{y}_{T+1}$$

$${}_{+2} = y_{T+2} - \hat{y}_{T+2} = \delta + \theta_1 y_{T+1} + e_{T+2} - \delta - \theta_1 \hat{y}_{T+1} = e_{T+1} + \theta_1\left(y_{T+1} - \hat{y}_{T+1}\right)$$

$$= e_{T+1} + \theta_1\left(e_{T+1}\right)$$

$$\operatorname{var}\left(\hat{e}_{T+1}\right) = \sigma_e^2\left(1 + \theta_1^2\right)$$

h period ahead Forecast in AR(1) Model

$$y_{T+h} = \delta + \theta_1 y_{T+h-1} + e_{T+h}$$

$$\hat{y}_{T+h} = E\left(y_{T+h}\right) = \delta + \theta_1 \hat{y}_{T+h-1}$$

$$_{h} = y_{T+h} - \hat{y}_{T+h} = \delta + \theta_1 y_{T+h-1} + e_{T+h} - \delta - \theta_1 \hat{y}_{T+h-1} = e_{T+h} + \theta_1 \Big(y_{T+h-1} - \hat{y}_{T+h-}$$

$$\operatorname{var}\left(\hat{e}_{T+h}\right) = \sigma_e^2 \left(1 + \theta_1^2 + \theta_1^4 + \ldots + \theta_1^{2(h-1)}\right)$$

h =1 period ahead Forecast in MA(1) Model

$$y_t = \mu + e_t + \alpha_1 e_{t-1}$$

$$y_{T+1} = \mu + e_{T+1} + \alpha_1 e_T$$

$$E\left(y_{T+1}\right) = \hat{y}_{T+1} = \mu + \alpha_1 e_T$$

$$\left(y_{T+1} - \hat{y}_{T+1}\right) = \mu + e_{T+1} + \alpha_1 e_T - \mu - \alpha_1 e_T = e_{T+1}$$

$$E\left(y_{T+1} - \hat{y}_{T+1}\right)^2 = \mathrm{var}\left(e_{T+1}\right)^2 = \sigma_e^2$$

h =2 periods ahead Forecast in MA(1) Model

$$y_{T+2} = \mu + e_{T+2} + \alpha_1 e_{T+1}$$

$$E\left(y_{T+2}\right) = \hat{y}_{T+2} = \mu$$

$$\hat{e}_{T+2} = \left(y_{T+2} - \hat{y}_{T+2}\right) = \mu + e_{T+2} + \alpha_1 e_{T+1} - \mu = e_{T+2} + \alpha_1 e_{T}$$

$$\operatorname{var}\left(\hat{e}_{T+2}\right) = \operatorname{var}\left(e_{T+2} + \alpha_1 e_{T+1}\right) = \sigma_e^2\left(1 + \alpha_1^2\right)$$

h period ahead forecasts

$$E\left(y_{T+h}\right) = \hat{y}_{T+h} = \mu$$

$$\operatorname{var}\left(\hat{e}_{T+h}\right) = \operatorname{var}\left(e_{T+h} + \alpha_1 e_{T+h-1}\right) = \sigma_e^2\left(1 + \alpha_1^2\right)$$

h =1 period ahead Forecast in ARMA(1,1) Mod

$$Y_t = \delta + \theta_1 y_{t-1} + e_t + \alpha_1 e_{t-1}$$

$$y_{T+1} = \delta + \theta_1 y_{t-1} + e_{T+1} + \alpha_1 e_T$$

$$E\left(y_{T+1}\right) = \hat{y}_{T+1} = \delta + \theta_1 y_{t-1} + \alpha_1 e_T$$

$$\hat{e}_{T+1} = \left(y_{T+1} - \hat{y}_{T+1}\right) = \delta + \theta_1 y_{t-1} + e_{T+1} + \alpha_1 e_T - \delta - \theta_1 y_{t-1} - \alpha_1 e_T = e_{T+1}$$

$$\operatorname{var}\left(\hat{e}_{T+1}\right) = E\left(y_{T+1} - \hat{y}_{T+1}\right)^2 = \operatorname{var}\left(e_{T+1}\right) = \sigma_e^2$$

h =2 period ahead Forecast in ARMA(1,1) Mod

$$y_{T+2} = \delta + \theta_1 y_{t+1} + e_{T+2} + \alpha_1 e_{T+1}$$

$$E\left(y_{T+2}\right) = \hat{y}_{T+2} = \delta + \theta_1 \hat{y}_{t+1}$$

$$_{T+2} = \left(y_{T+2} - \hat{y}_{T+2}\right) = \delta + \theta_1 y_{t+1} + e_{T+2} + \alpha_1 e_{T+1} - \delta - \theta_1 \hat{y}_{t+1}$$

$$\hat{e}_{T+2} = \theta_1\left(y_{t+1} - \hat{y}_{t+1}\right) + e_{T+2} + \alpha_1 e_{T+1} = \left(\theta_1 + \alpha_1\right) e_{T+1} + e_{T+}$$

$$\operatorname{var}\left(\hat{e}_{T+1}\right) = \operatorname{var}\left[\left(\theta_1 + \alpha_1\right) e_{T+1} + e_{T+2}\right] = \sigma_e^2\left[\left(\theta_1 + \alpha_1\right)^2 + 1\right]$$

h =3 periods ahead Forecast in ARMA(1,1) Model

$$y_{T+3} = \delta + \theta_1 y_{t+2} + e_{T+3} + \alpha_1 e_{T+2}$$

$$E\left(y_{T+3}\right) = \hat{y}_{T+3} = \delta + \theta_1 \hat{y}_{t+2}$$

$$\hat{e}_{T+3} = \left(y_{T+3} - \hat{y}_{T+3}\right) = \delta + \theta_1 y_{t+2} + e_{T+3} + \alpha_1 e_{T+2} - \delta - \theta_1 \hat{y}_{t+2}$$

$$_{+3} = e_{T+3} + \alpha_1 e_{T+2} + \theta_1\left(y_{t+2} - \hat{y}_{t+2}\right) = e_{T+3} + \alpha_1 e_{T+2} + (\theta_1 + \alpha_1) e_{T+1} + e_{T+2}$$

$$_{+3} = e_{T+3} + \alpha_1 e_{T+2} + \theta_1\left(y_{t+2} - \hat{y}_{t+2}\right) = e_{T+3} + \alpha_1 e_{T+2} + (\theta_1 + \alpha_1) e_{T+1} + e_{T+2}$$

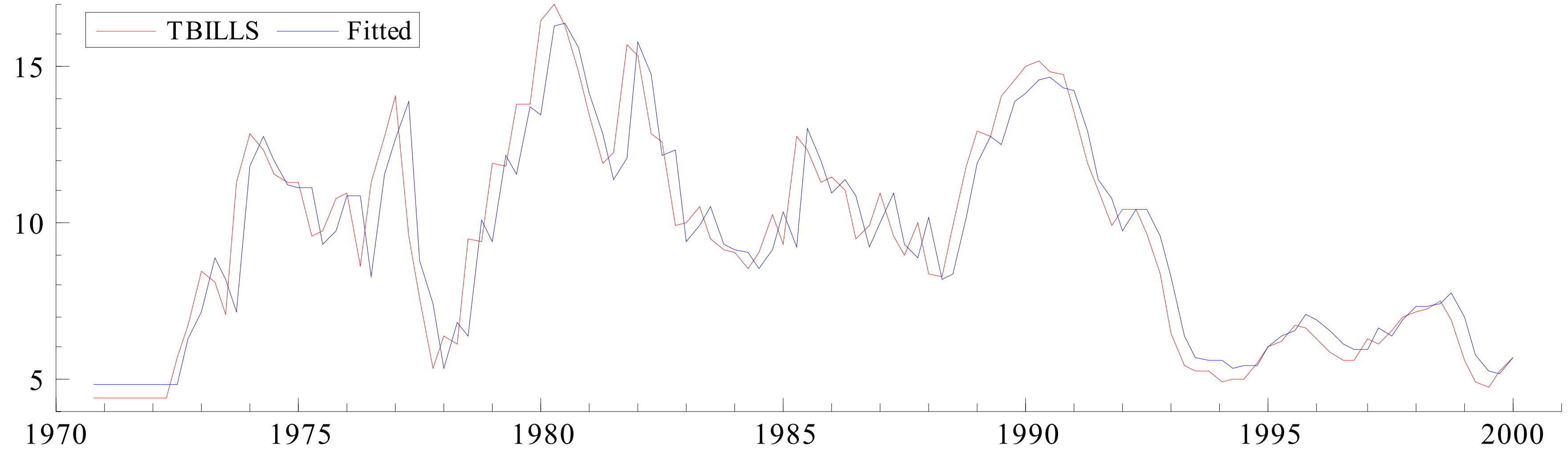
$$\operatorname{var}\left(\hat{e}_{T+3}\right) = \operatorname{var}\left(e_{T+3} + \alpha_1 e_{T+2} + \left(\theta_1 + \alpha_1\right) e_{T+1} + e_{T+2}\right) = \sigma_e^2\left[1 = \left(\left(1 + \alpha_1\right)^2 + \left(\theta_1 + \alpha_1\right)^2\right)\right]$$

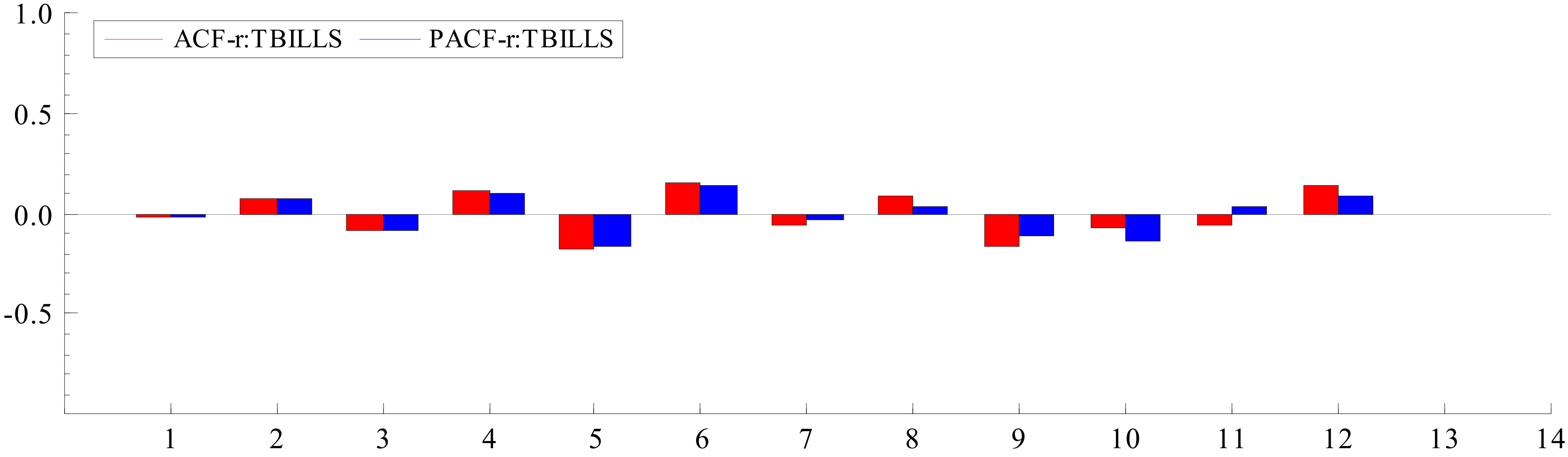
Estimates, Tests and Forecasts using AR(1) Model

Modelling TBILLS by OLS (using uk_r.xls)
The estimation sample is: 1970 (3) to 2000 (1)

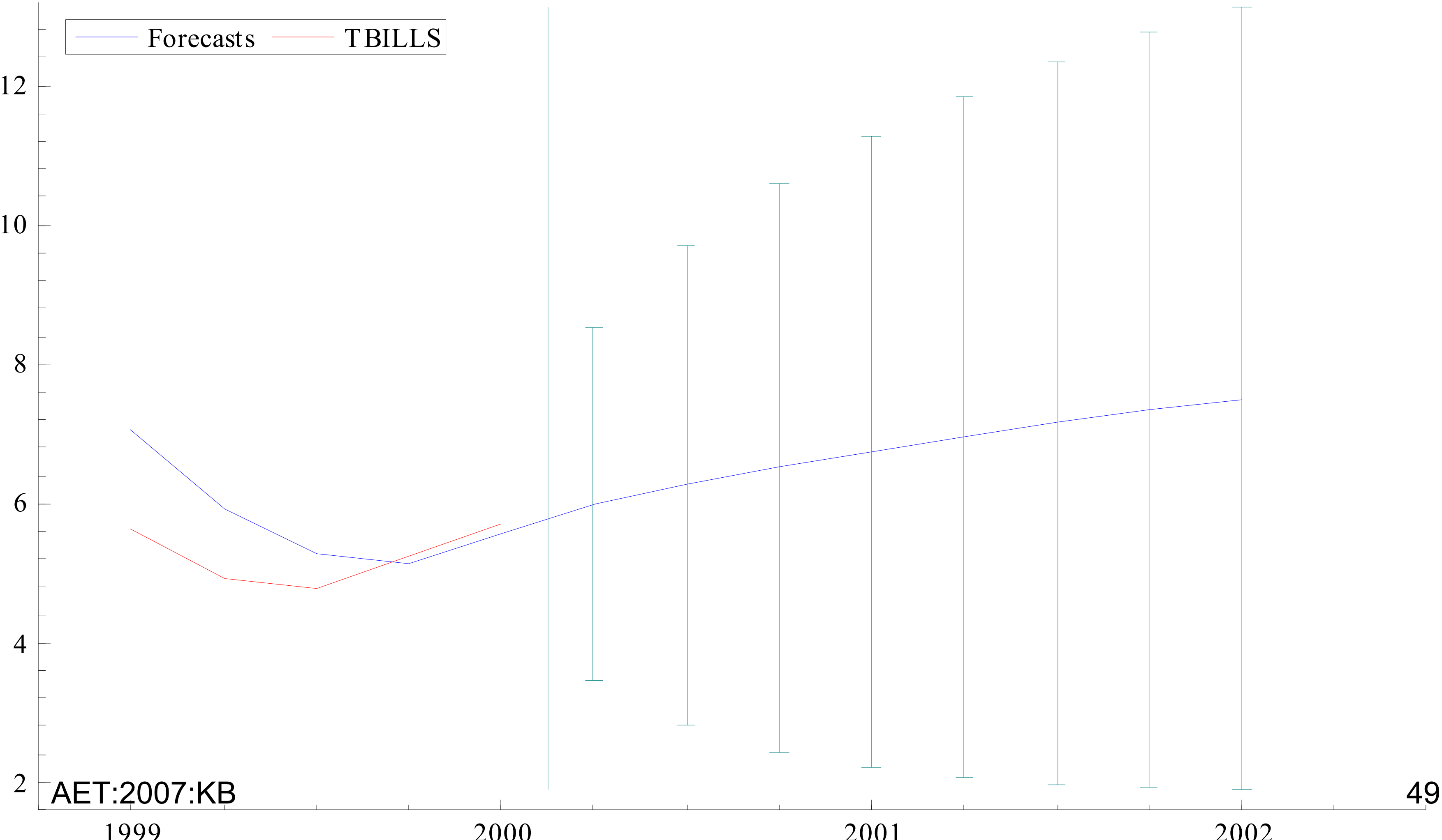
```
TBILLS =  + 0.9232*TBILLS_1 + 0.7311
(SE)      (0.0345)         (0.344)
sigma                 1.26921  RSS              188.473476
R^2                 0.859749  F(1,117) =    717.2 [0.000]**
log-likelihood       -196.214  DW                        1.69
no. of observations       119  no. of parameters            2
mean(TBILLS)          9.38748  var(TBILLS)     11.2927
AR 1-5 test:      F(5,112) =   1.6483 [0.1531]
ARCH 1-4 test:    F(4,109) =   1.3557 [0.2541]
Normality test:   Chi^2(2) =   12.114 [0.0023]**
hetero test:      F(2,114) =   2.5382 [0.0835]
hetero-X test:    F(2,114) =   2.5382 [0.0835]
RESET test:       F(1,116) =  0.96693 [0.3275]

      Horizon     Forecast      (SE)
        2000-2     5.99349     1.269
        2000-3     6.26445     1.727
        2000-4     6.51460     2.038
        2001-1     6.74555     2.270
        2001-2     6.95877     2.450
        2001-3     7.15562     2.594
        2001-4     7.33736     2.710
        2002-1     7.50515     2.806
```

Autocorrelation and Partial Autocorrelation in an AR(1) Model





Forecast of Treasury Bills using AR(1) Mode



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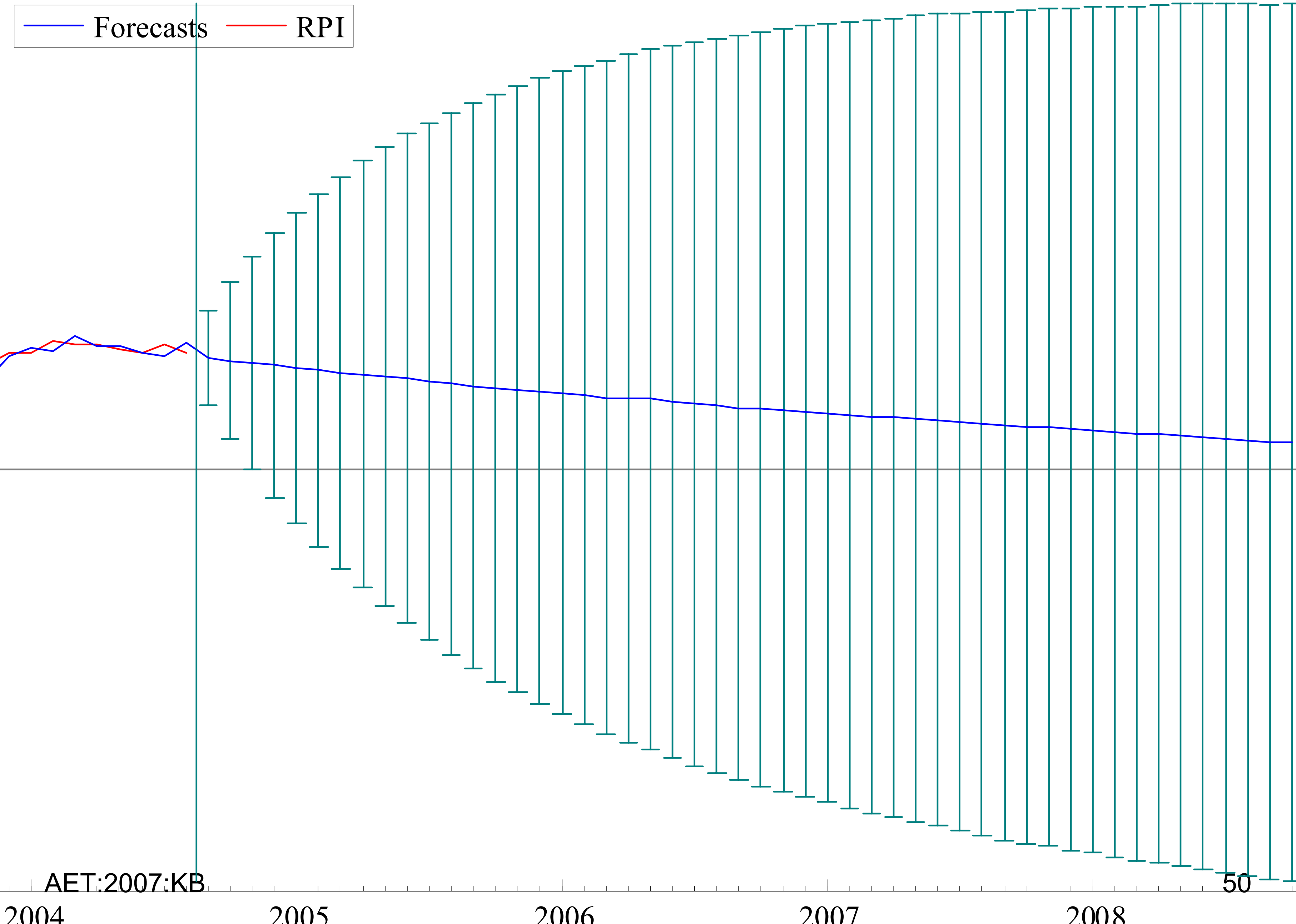

Forecasts
RPI
2004
2005
2006
2007
2008

Spurious Regression

Consider a stochastic autoregressive AR(1) series:

$$y_t = \rho y_{t-1} + v_t$$

Application of OLS in this may generate a spurious regression, with a high R^2 and very low Durbin-Watson statistics (R^2>d). OLS generates spurious regression if variables involved are non stationary.

Stationary and Nonstationary Series

A given time series $\{y_t\}$ is stationary when mean and variance are constant or independent of time.

$$E\left(y_t\right) = \mu$$ constant mean

$$\operatorname{var}\left(y_t\right) = \sigma^2$$ constant variance

$$\operatorname{cov}\left(y_t y_{t-s}\right) = \operatorname{cov}\left(y_t y_{t+s}\right) = \gamma_s$$ time independent covariance

Time series y_t is non-stationary if the mean and variance is not constant or is changing over time.

Many economic variables such as GDP, GDP components, inflation, exchange rates, labour force evolve over time. It is important to check whether these series are stationary or non stationary before any econometric estimation because estimation using non-stationary variables may generate a spurious relationship: reported to have relationship when there is no relationship.

Unit Root Test and Cointegration, VAR

Avoiding Spurious Regression

Stationary Time Series

y_t is stationary when

mean is constant

$$E\left(y_t\right)=\mu$$

and

variance is constant :

$$\operatorname{var}\left(y_t\right)=\sigma^2$$

or independent of time.

Covariance is time independent

$$\operatorname{cov}\left(y_t y_{t-s}\right)=\operatorname{cov}\left(y_t y_{t+s}\right)=\gamma_s$$

Non-Stationarity and Test for Unit Roots

- Many economic variables are non-stationary. They include:
- GDP its components C, I, M, T, G, X
- Inflation and exchange rates
- Labour force and unemployment rate
- Money supply and interest rates
- Are non-stationary.

- Estimates from non-stationary variable may generate a **spurious regression** unless they are cointegrated.
- Dicky-Fuller and Augmented Dicky Fuller tests for unit root.
- Hypotheses

  Null: Unit root exists

  Alternative:

  Unit root does not exist

Four ways of checking stationarity

- Partial autocorrelation function and Ljung and Box statistics
- Unit Root Test: Dicky-Fuller Test
- Cointegration Test
- Test for Error Correction

Partial autocorrelation function and Ljung and Box statistics

$$\rho^S = \frac{\operatorname{cov}\left(y_t y_{t+1}\right)}{\operatorname{var}\left(y_t\right)} = \frac{\gamma_s}{\gamma_0}$$

where $\hat{\gamma}_S = \frac{\sum\left(y_t - \overline{y}\right)\left(y_{t+s} - \overline{y}\right)}{T}$ $\quad \hat{\gamma}_0 = \frac{\sum\left(y_t - \overline{y}\right)^2}{T}$

Use Ljung and Box statistics to test whether all ρ are equal to zero

$$Q = T\left(T+2\right)\sum_{s=1}^{m} \frac{\hat{\rho}_s^2}{T-s}$$

This statistic has a χ_m^2 distribution.

Unit Root and Non-stationarity

Like autocorrelated errors consider a stochastic autoregressive AR(1) series:

$$y_t = \rho y_{t-1} + v_t \qquad (4)$$

Here $1 \leq \rho \leq 1$. It is a unit-root process if $\rho = 1$. Then (4) becomes a random walk $y_t = y_{t-1} + v_t$.

Application of OLS in this may generate a spurious regression, with a high R^2 and very low Durbin-Watson statistics (R^2>d). To see how (4) is non stationary , let us assume that

$$v_t \sim N\left(0, \sigma_v^2\right).$$

Dicky-Fuller and Augmented Dicky-Fuller Tests

$$y_t = \rho y_{t-1} + v_t$$

$$\Delta y_t = (1-\rho) y_{t-1} + v_t \,;$$

Random Walk:

$$\Delta y_t = \gamma y_{t-1} + v_t$$

Random Walk with a drift (intercept):

$$\Delta y_t = \alpha_0 + \gamma y_{t-1} + v_t$$

Trend stationary process

$$\Delta y_t = \alpha_0 + \alpha_1 t + \gamma y_{t-1} + v_t$$

Augmented Dicky Fuller Test

$$\Delta y_t = \alpha_0 + \alpha_1 t + \gamma y_{t-1} + \sum_{i=1}^{m} a_i \Delta y_{t-i} + v_t$$

Null hypotheses:
There is unit root and time
series in non-stationary
K=0 ➔ (1-Ψ)=0
Alternative hypothesis:
There is no unit root and
time series is stationary
K<0 ➔ (1-Ψ)<0 ➔ Ψ<1

Unit Root Test of Banana in the first difference in logs

$$Y_t = \rho Y_{t-1} + e_t$$

```
prodcam_L1: ADF tests (T=39, Constant; 5%=-2.94 1%=-3.61)
D-lag   t-adf     beta Y_1    sigma   t-DY_lag  t-prob      AIC  F-prob
 2    -3.181*    0.099137  0.04863     -1.807  0.0794   -5.950
 1    -6.235**    -0.26798  0.05013      2.123  0.0407   -5.912  0.0794
 0    -6.060**   0.0077992  0.05246                        -5.846  0.0266

prodch_L1: ADF tests (T=39, Constant; 5%=-2.94 1%=-3.61)
D-lag   t-adf     beta Y_1    sigma   t-DY_lag  t-prob      AIC  F-prob
 2    -3.564*     0.33520  0.08494    0.2420  0.8102   -4.835
 1    -4.104**     0.35912  0.08382    0.7727  0.4447   -4.884  0.8102
 0    -4.426**     0.42529  0.08336                        -4.919  0.7282
```

Prodcam_L1: production in Cambodia in the first difference

Unit Root Tests of Unemployment Rate

Unit root exists in the level of unemployment rate

```
URT: ADF tests (T=373, Constant; 5%=-2.87 1%=-3.45)
D-lag    t-adf     beta Y_1    sigma   t-DY_lag  t-prob       AIC  F-prob
  3     -1.143      0.99595   0.1969    -0.4586  0.6468    -3.237
  2     -1.165      0.99588   0.1967    -0.9016  0.3679    -3.242   0.6468
  1     -1.209      0.99573   0.1966      6.955  0.0000    -3.245   0.6005
  0    -0.9868      0.99630   0.2088                       -3.127   0.0000
```

There is no unit root in the first difference

```
DURT: ADF tests (T=372, Constant; 5%=-2.87 1%=-3.45)

D-lag    t-adf     beta Y_1    sigma   t-DY_lag  t-prob       AIC  F-prob
  3     -7.625**    0.40441   0.1948     -3.144  0.0018    -3.258
  2     -10.17**    0.28958   0.1971     0.4874  0.6263    -3.237  0.0018
  1     -11.59**    0.30717   0.1969     0.9270  0.3545    -3.242  0.0068
  0     -13.52**    0.33903   0.1969                       -3.245  0.0125
```

Output from the PcGive

How to make a Non-Stationary Series to a Stationary Series?

- Logs
- ratios
- First difference
- Second difference
- Third or higher order difference
- Cointegration
- Error Correction

Unit Root Test for RPII

```
 RPI: ADF tests (T=373, Constant; 5%=-2.87 1%=-3.45)

-lag    t-adf      beta Y_1    sigma   t-DY_lag  t-prob      AIC  F-prob
3     -2.175       0.98866   0.5856     0.8751  0.3821    -1.057
2     -2.089       0.98918   0.5854      3.513  0.0005   -1.060  0.3821
1     -1.729       0.99095   0.5943      10.04  0.0000   -1.033  0.0016
0    -0.9820       0.99422   0.6695                     -0.7971  0.0000


RPI: ADF tests (T=372, Constant; 5%=-2.87 1%=-3.45)

-lag    t-adf      beta Y_1    sigma   t-DY_lag  t-prob      AIC  F-prob
3     -6.943**    0.56614   0.5882  -8.517e    -005      0.9999  -1.048
2     -7.458**    0.56614   0.5874    -0.6487  0.5170    -1.054  0.9999
1     -8.435**    0.55099   0.5869     -3.297  0.0011    -1.058  0.8108
0     -11.73**    0.45967   0.5947                      -1.034  0.0112
```

Cointegration, ARCH (p), GARCH(p,q) and Vector Autoregressive Models

Importance of Error Terms in Regressions

Estimation, Impulse Response Analysis

Co-integration

If two economic variables have long-run equilibrium relationship linear combination of these variables may be stationary even if the individual series may be non stationary. These two variables are said to be co-integrated to each other.

Suppose y_t is consumption and X_t is disposable income.

$$e_t = Y_t - \beta_1 - \beta_2 X_t$$

Even if y_t and X_t are I(1) e_t is I(0).

$$e_t = \alpha_0 + \gamma e_{t-1} + v_t$$

If γ is zero then series e_t is stationary and y_t and X_t are I(1).

Engle-Granger Approach to Co-integration

$$Y_t = \beta_1 + \beta_2 X_t + e_t$$

$$e_t = Y_t - \beta_1 - \beta_2 X_t$$

If Y_t and X_t I(1) but e_t I(0)

$$e_t = \alpha_0 + \gamma e_{t-1} + v_t$$

$\gamma = 0 \longrightarrow$ Cointegration

Long-Run Multiplier in ARDL(1,1)

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \beta_3 X_t + \beta_4 X_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0,1)$$

$$Y_t = Y_{t-1} = \overline{Y} \qquad X_t = X_{t-1} = \overline{X}$$

$$\overline{Y} - \beta_2 \overline{Y} = \beta_1 + \beta_3 \overline{X} + \beta_4 \overline{X}$$

$$\overline{Y} = \frac{\beta_1}{(1-\beta_2)} + \left(\frac{\beta_3 + \beta_4}{(1-\beta_2)}\right)\overline{X}$$

Adjustment towards the long run in ARDL(1,1)

$$Y_t - Y_{t-1} = \beta_1 + \beta_2 Y_{t-1} - Y_{t-1} + \beta_3 X_t - \beta_3 X_{t-1} + \beta_3 X_{t-1} + \beta_4 X_{t-1} + \varepsilon_t$$

$$\Delta Y_t = \beta_1 + (\beta_2 - 1) Y_{t-1} + \beta_3 (X_t - X_{t-1}) + (\beta_3 + \beta_4) X_{t-1} + \varepsilon_t$$

$$\Delta Y_t = (\beta_2 - 1)\left[Y_{t-1} - \frac{\beta_1}{(1-\beta_2)} - \frac{(\beta_3 + \beta_4)}{(1-\beta_2)} X_{t-1} \right] + \beta_3 \Delta X_t + \varepsilon_t$$

Deviation from the long-term trend:

$$\left[Y_{t-1} - \frac{\beta_1}{(1-\beta_2)} - \frac{(\beta_3 + \beta_4)}{(1-\beta_2)} X_{t-1} \right]$$

$(\beta_2 - 1)$ Adjustment to the long run.

Error Correction Model

$$\Delta y_t = \alpha_1 + \alpha_2 \left(Y_t - \beta_1 - \beta_2 X_t \right) + v_t$$

The term in the parenthesis is the error term and the coefficient α_2 governs the speed of adjustment towards long-run equilibrium.

Structure of a VAR Model

$${}_{,t} = a_{10} + \sum_{j=1}^{P} a_{11j} y_{1,t-j} + .. + \sum_{j=1}^{P} a_{1nj} y_{n,t-j} + \sum_{j=1}^{r} b_{11j} x_{1,t-j} + .. + \sum_{j=1}^{P} b_{1mj} x_{m,t-j} + e_{1t}$$

$$\vdots$$

$$y_{n,t} = a_{n0} + \sum_{j=1}^{P} a_{n1j} y_{1,t-j} + .. + \sum_{j=1}^{P} a_{nnj} y_{n,t-j} + \sum_{j=1}^{r} b_{11j} x_{1,t-j} + .. + \sum_{j=1}^{P} b_{1mj} x_{m,t-j} + e_{nt}$$

Simple Example

$$y_t = a_{10} + a_{11} y_{1t-1} + a_{12} y_{2,t-2} + b_{11} x_{t-1} + b_{12} x_{t-2} + e_{1t}$$

$$x_t = a_{20} + a_{21} y_{1t-1} + a_{22} y_{2,t-2} + b_{21} x_{1,t-1} + b_{22} x_{t-2} + e_{2t}$$

Impulse Response Analysis in a VAR Model

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-2} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}\, b_{11} & b_{12} \\ a_{21} & a_{22}\, b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-2} \\ x_{1t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

$$\overline{Y} = (I - A)^{-1} BX + (I - A)^{-1} U$$

$$\overline{X} = (I - B)^{-1} AY + (I - B)^{-1} U$$

$$Y_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Y_1 = (I - A)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Y_2 = (I - A)^{-1} Y_1 = (I - A)^{-1} (I - A)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Roots of A matrix are called characteristic roots or eigenvalues of A. Corresponding to each solution of λ_i there is a characteristic vector x_i called latent or eigenvector. A simple example from the matrix of order two.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\left| A - \lambda I \right| = \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} (4-\lambda) & 2 \\ 2 & (1-\lambda) \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 4 = 4 - \lambda - 4\lambda + \lambda^2 - 4 = \lambda^2 - 5\lambda = 0$$

two roots $\lambda_1 = 5 \quad \lambda_2 = 0$

Understanding this is Crucial for cointegration test

Eigenvector corresponding to roots $\lambda = 5$ should satisfy:

$$\begin{bmatrix} (4-\lambda) & 2 \\ 2 & (1-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (4-5) & 2 \\ 2 & (1-5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_1 + 2x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 2x_2$$

Normalisation or Euclidian distance condition implies $x_1^2 + x_2^2 = 1$

$(2x_2)^2 + x_2^2 = 1 \quad x_2^2 = \frac{1}{5} \quad x_2 = \pm\frac{1}{\sqrt{5}}$ and when $x_2 = \frac{1}{\sqrt{5}} \quad x_1 = \frac{2}{\sqrt{5}}$

The Eigenvector corresponding to $\lambda_1 = 5$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Derivation of Eigen Values and Eigen Vectors

For $\lambda_2 = 0 \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_2 = -2x_1 \quad (-2x_1)^2 + x_1^2 =$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \text{ Orthogonal matrix } X_1^T X_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{2}{5} - \frac{2}{5} = 0$$

If two eigenvectors are put side by side as

$$X = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \quad X^T X = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Specification of Vector Autoregression Model

$$Y_t = A_1 Y_{t-1} + \varepsilon_t$$

Y_t **vector of variables**

interest rate,

output gap

inflation gap

ε_t **normally and identically distributed random error te**

$$\Delta Y_t = \left(A_1 - I\right) Y_{t-1} + \varepsilon_t$$

$$\Pi = \left(A_1 - I\right)$$

Decomposition of Long and Short Run Responses

Π **Long run response**

$$\Pi = \alpha\beta'$$

α **Dynamic process of adjustment**

β **Long run Steady State Relation**

Trace and Max Test for Cointegration Rank

$$\lambda_{trace(r)} = -T \sum_{i=r+1}^{n} \ln\left(1 - \hat{\lambda}_i\right)$$

$$\lambda_{\max(r,r+1)} = -T \ln\left(1 - \hat{\lambda}_{r+1}\right)$$

λ_i **Eigenvalues of the characteristic matrix**

rank	Trace test [Prob]	Max test [Prob]	Trace test (T-nm)	Max test (T-nm)
0	56.86 [0.000]**	34.38 [0.000]**	55.43 [0.000]**	33.52 [0.000]**
1	22.48 [0.003]**	12.68 [0.087]	21.91 [0.004]**	12.36 [0.097]
2	9.80 [0.002]**	9.80 [0.002]**	9.55 [0.002]**	9.55 [0.002]**

Technical Aspects of Eigen-Values and Eigen-Vectors Finding to Root of Coefficient Matrix

Take $A_{n\times n}$ matrix.

$$Min \quad X'AX \quad sub \quad to \quad X'X \qquad L = X'AX - \lambda\left[X'X - I\right]$$

$$\frac{\partial L}{\partial X} = 2AX - 2\lambda X = 0 \quad AX - \lambda X = 0 \quad \left(A - \lambda I\right)X = 0$$

$$\left|A - \lambda I\right| = 0$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \left(a_{1,1} - \lambda\right) & a_{1,2} & a_{1,3} \\ a_{2,1} & \left(a_{2,2} - \lambda\right) & a_{2,3} \\ a_{3,1} & a_{3,2} & \left(a_{3,3} - \lambda\right) \end{vmatrix}$$

This determinant is cubic in λ and has three solutions.

Numerical Example:

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & -4 & 4 \end{bmatrix}$$

Finding Roots

$$|A-\lambda I| = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & -4 & 4 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} (4-\lambda) & 2 & 2 \\ 2 & (1-\lambda) & 1 \\ 3 & -4 & (4-\lambda) \end{vmatrix} = 0$$

$$(4-\lambda)\begin{vmatrix} (1-\lambda) & 1 \\ -4 & (4-\lambda) \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & (4-\lambda) \end{vmatrix} + 2\begin{vmatrix} 2 & (1-\lambda) \\ 3 & -4 \end{vmatrix}$$

$$(4-\lambda)(1-\lambda)(4-\lambda)+4(4-\lambda)-4(4-\lambda)+6-16-6(1-\lambda) = (4-\lambda)^2(1-\lambda)-6(1-\lambda$$

$$= (1-\lambda)\left[(4-\lambda)^2-6\right]-10 = (1-\lambda)\left[16-8\lambda+\lambda^2-6\right]-10$$

$$= \left[16-8\lambda+\lambda^2-6-16\lambda+8\lambda^2-\lambda^3+6\lambda\right]-10 = -\lambda^3+9\lambda^2-18\lambda = -\lambda^2+9\lambda-18 = 0$$

$$= -\lambda^2+3\lambda+6\lambda-18 = 0 \quad = -\lambda(\lambda-3)+6(\lambda-3) = 0$$

$$= (\lambda-3)(6-\lambda) = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = -6 \quad \lambda_3 = 0$$

Conditions for Getting Eigen Vectors

There are three Characteristic vectors for these three roots; for $\lambda_3 = 0$

$$X_1 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda_3 = 3$

$$X_2 = \begin{bmatrix} (4-3) & 2 & 2 \\ 2 & (1-3) & 1 \\ 3 & -4 & (4-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_3 = -6$

$$X_3 = \begin{bmatrix} (4+6) & 2 & 2 \\ 2 & (1+6) & 1 \\ 3 & -4 & (4+6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In each case Normalisation or Euclidian distance condition implies $x_1^2 + x_2^2 + x_3^2 = 1$

Eigen Vector 1 For A matrix

For $\lambda_3 = 0$

$2x_1 + x_2 + x_3 = 0$

$3x_1 - 4x_2 + 4x_3 = 0$ or

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & -4 & 4 \end{bmatrix}$$

$8x_1 + 4x_2 + 4x_3 = 0$ (1)

$3x_1 - 4x_2 + 4x_3 = 0$ (2)

Eliminating x2 $11x_1 + 8x_3 = 0$ $x_1 = -\frac{8}{11}x_3$ $x_3 = -\frac{11}{8}x_1$ (3)

Putting the value of x1 in (3)

$$3x_1 - 4x_2 + 4x_3 = 3\left(-\frac{8}{11}x_3\right) - 4x_2 + 4x_3 = 0$$

$\left(-\frac{24}{11}x_3\right) - 4x_2 + 4x_3 = 0$ $\frac{20}{11}x_3 = 4x_2$ $x_2 = \frac{5}{11}x_3$ $x_3 = \frac{11}{5}x_2$ (4)

From (3) and (4) $-\frac{11}{8}x_1 = \frac{11}{5}x_2$ $x_1 = -\frac{8}{5}x_2$ or $x_2 = -\frac{5}{8}x_1$

Now using the normalisation $x_1^2 + x_2^2 + x_3^2 = 1$

$$x_1^2 + \left(-\frac{5}{8}x_1\right)^2 + \left(-\frac{11}{8}x_1\right)^2 = 1 \quad x_1^2\left(1 + \frac{25}{64} + \frac{121}{64}\right) = 1 \quad x_1^2\left(\frac{64+25+121}{64}\right) = 1$$

$x_1^2 = \frac{64}{210}$ $x_1 = \pm\frac{8}{\sqrt{210}}$ then $x_2 = -\frac{5}{8}x_1 = -\frac{5}{8}\frac{8}{\sqrt{210}} = -\frac{5}{\sqrt{210}}$ and

$$x_3 = -\frac{11}{8}x_1 = -\frac{11}{8}\frac{8}{\sqrt{210}} = -\frac{11}{\sqrt{210}}$$

Characteristic vector: $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{210}} & -\frac{5}{\sqrt{210}} & -\frac{11}{\sqrt{210}} \end{pmatrix}$

Similar characteristic vector can be found for roots $\lambda_3 = 3$ and $\lambda_3 = -6$.

Application of VAR Analysis

$$y_t - y_t^* = d\left(i_{t-1} - i_{t-1}^*\right) + \varepsilon_{1,t}$$

$$\pi_t = \pi_t^* + c\left(y_{t-1} - y_{t-1}^*\right) + \varepsilon_{2,t}$$

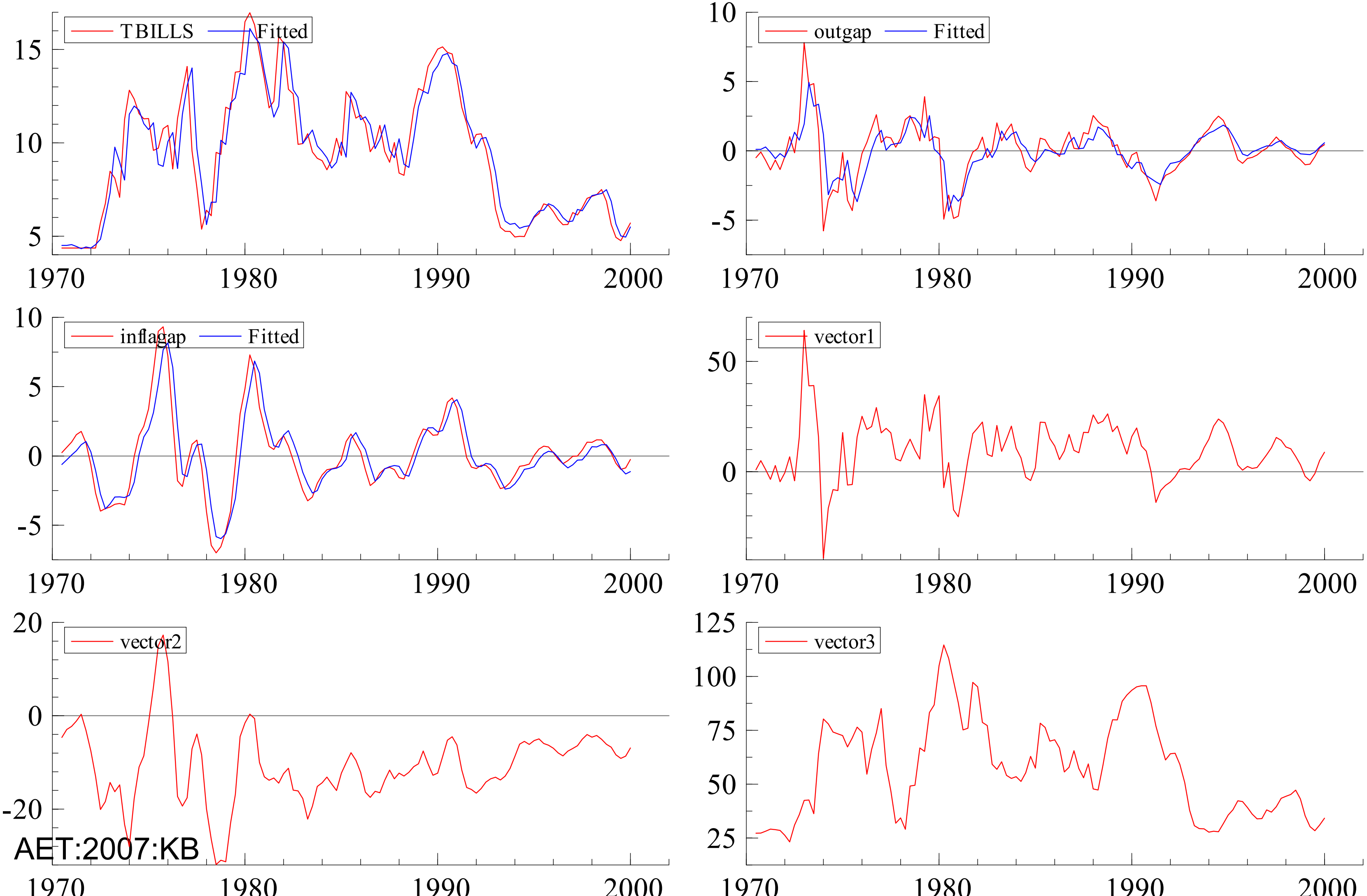
$$i_t = i_t^* + a\left(y_t - y_t^*\right) + b\left(\pi_t - \pi_t^*\right) + \varepsilon_{3,t}$$

Stationarity of variables in the model

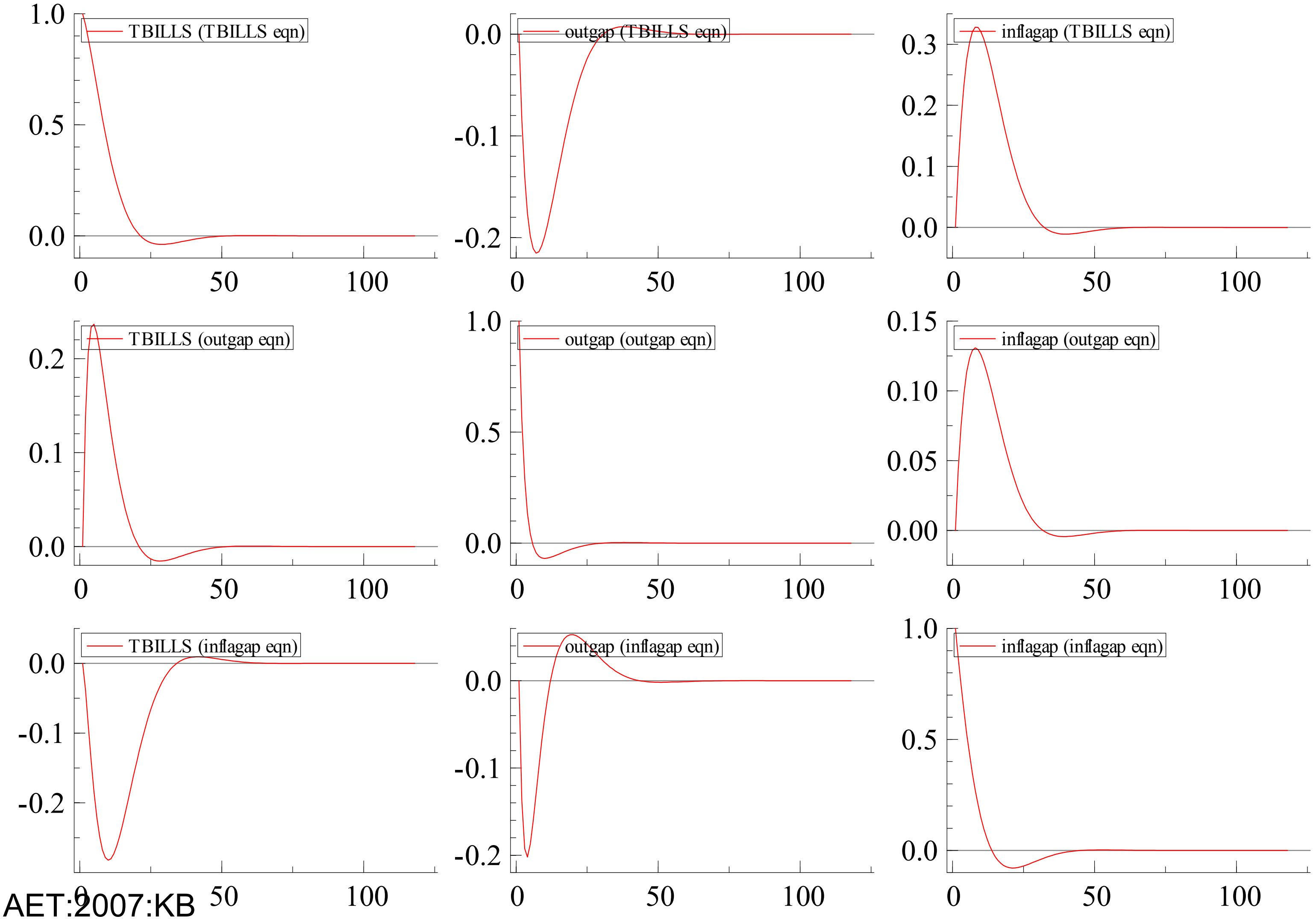
ADF tests (T=116, Constant; 5%=-2.89 1%=-3.49)

	Interest rate	Difference of Interest rate	Output gap	Inflation gap
Coefficient	-2.723	-6.463**	-6.160**	-7.428**
Lags	2	2	3	1

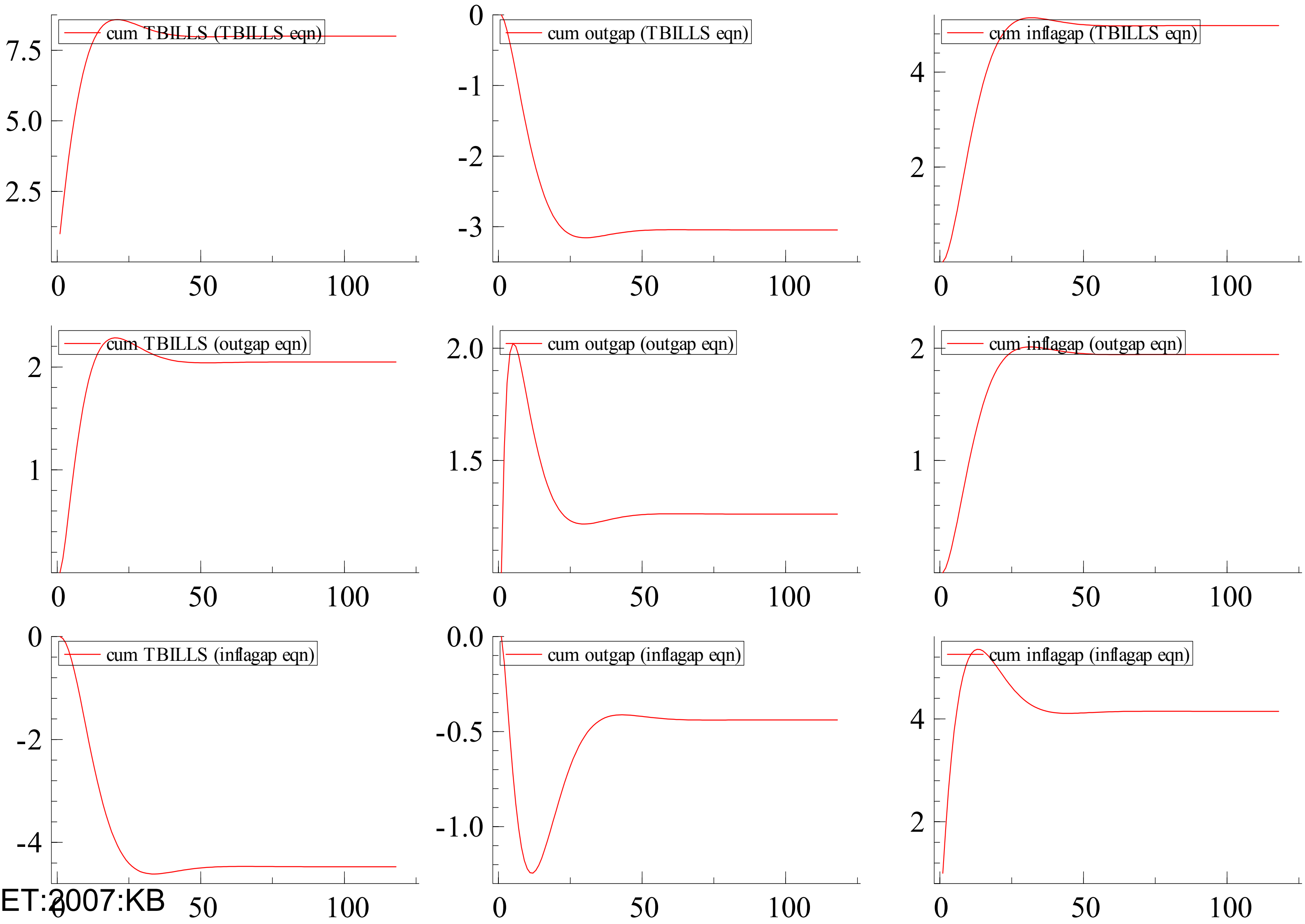
Prediction from Cointegrating VAR Model



Impulse Response Analysis



Cumulative Impulse Response Analysis



Steps for Granger Causality Test

oes increase in money supply increase GDP or an increase
 money supply increase GDP?

$$Y_t = \sum_{i=1}^{n} \alpha_i M_{t-i} + \sum_{j=1}^{n} \beta_j Y_{t-j} + u_{1,t}$$

$$M_t = \sum_{i=1}^{n} \lambda_i M_{t-i} + \sum_{j=1}^{n} \delta_j Y_{t-j} + u_{2,t}$$

Four possible cases for Granger Causality

1. Uni-directional causality from money and GDP

$$\sum_{i}^{n} \alpha_i \neq 0 \qquad \sum_{i}^{n} \delta_i = 0$$

2. Uni-directional causality from GDP tp money

$$\sum_{i}^{n} \alpha_i = 0 \qquad \sum_{i}^{n} \delta_i \neq 0$$

3. Bilateral causality

$$\sum_{i}^{n} \alpha_i \neq 0 \qquad \sum_{i}^{n} \delta_i \neq 0$$

3. Independence from each other

$$\sum^{n} \alpha_i = 0 \qquad \sum^{n} \delta_i = 0$$

Steps for Granger Causality Test

- Regress GDP to all lagged GDP and other variables but not to the lagged money terms and get error some square of the restricted model
- Include lags for GDP as well as money and get the unrestricted residual sum square

$$F_{m,(n-k)} = \frac{\left(RSS_R - RSS_{UR}\right)/m}{RSS_{UR}/\left(n-k\right)}$$

$$H_0 : \sum_i^n \alpha_i = 0$$

No causality

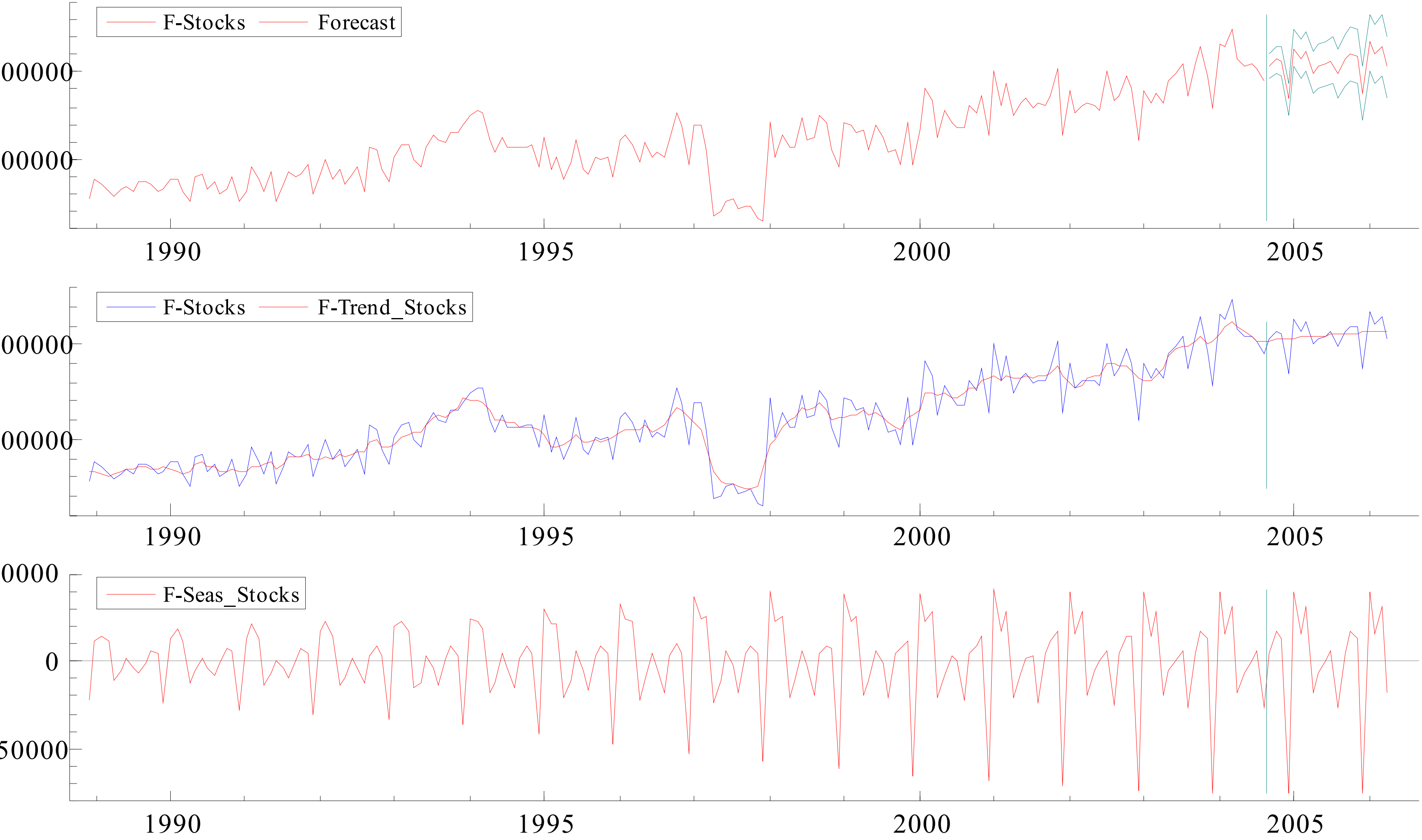
Analysis of Volatility using ARCH(p)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-2}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2$$

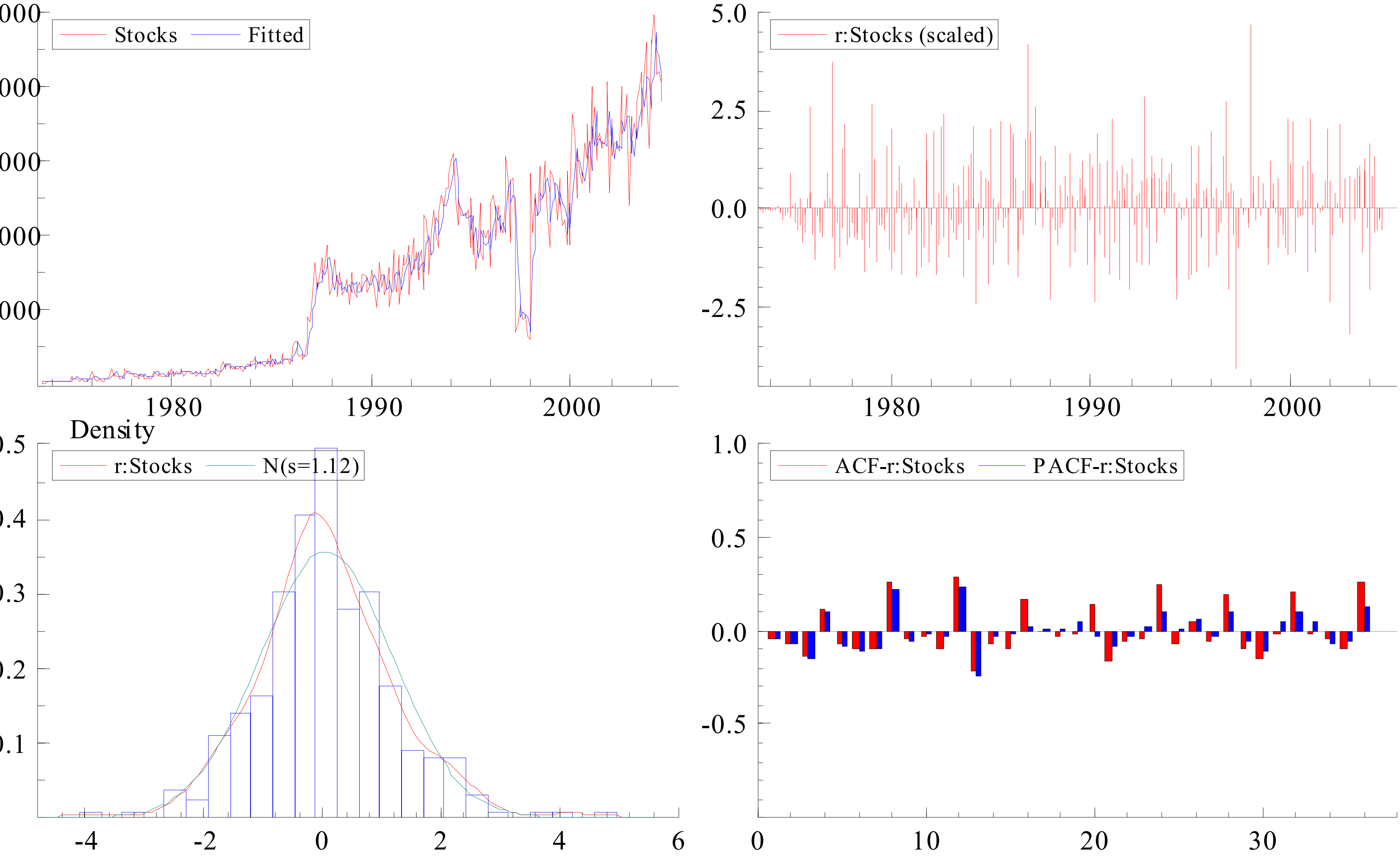
Analysis of Volatility using GARCH(p,q)

$${}_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2 + \phi_1 \sigma_{t-1}^2 + \phi_2 \sigma_{t-2}^2 + \phi_3 \sigma_{t-3}^2 + \ldots + \phi_q \sigma_{t-q}^2$$

Trend, Seasonality and Random Components of Stock Price and Forecasts

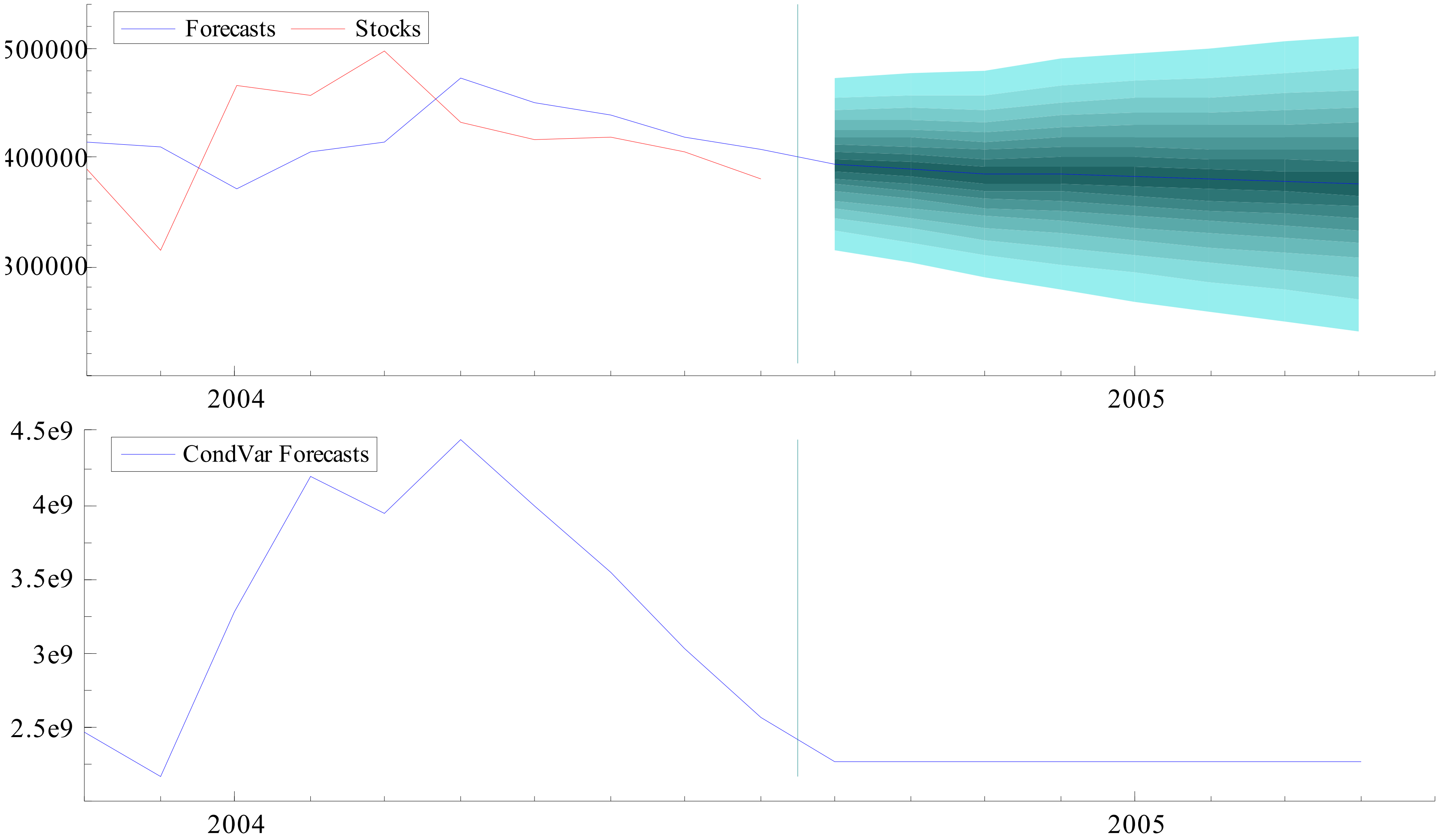


Prediction from a GARCH Model



Data source: Navidata

Confidence Interval for Forecast of Stock Prices from a GARCH Model



State-Space Model and Kalman Filter

$Y_t = T_t + C_t$ Income series is decomposed in trend and cycles.

$T_t = T_{t-1} + g_{t-1} + u_t$

$C_t = \psi_1 C_{t-1} + \psi_2 C_{t-2} + v_t$

$g_t = gc + \lambda g_{t-1} + w_t$

Here the measurement equation can be put as

$$Y_t = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_t \\ C_t \\ C_{t-1} \\ g_t \end{bmatrix}$$

$$\begin{bmatrix} T_{t+1} \\ C_{t+1} \\ C_t \\ g_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \psi_1 & \psi_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} T_t \\ C_t \\ C_{t-1} \\ g_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ gc \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ v_{t+1} \\ 0 \\ w_{t+1} \end{bmatrix}$$

In terms of the state space model

Measurement equation: $Y_t = H\xi_t + AX_t + \mu_t$

Transition equation: $\xi_{t+1} = F\xi_t + BX_{t+1} + v_{t+1}$

State-Space Model and Kalman Filter

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \psi_1 & \psi_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \text{ A=0 B =1 } \mu_t = 0 \; u'_{1_{t=}} = \begin{pmatrix} u_t & v_t & 0 & w_t \end{pmatrix}$$

$$Q = \begin{bmatrix} \sigma_u^2 & 0 & 0 & 1 \\ 0 & \sigma_v^2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sigma_w^2 \end{bmatrix}$$

(see Harvey’s Time Series Models and also Chatper 7 of Wang (2003) Financial Econometrics, Routledge)

This model can be estimated using the STAMP.

```
Variable      Coefficient      R.m.s.e.       t-value
Lvl          4.0928e+005        13705.        29.864  [ 0.0000]
Slp              1074.5       739.27        1.4535  [ 0.1469]
Cy3_ 1          -2056.5        12528.
Cy3_ 2           36.425        14759.
Sea_ 1          -6624.9        4908.1       -1.3498  [ 0.1779]
Sea_ 2          -1928.7        4947.0      -0.38988  [ 0.6969]
Sea_ 3           10023.        4023.8        2.4909  [ 0.0132]
Sea_ 4           9006.1        4056.2        2.2203  [ 0.0270]
Sea_ 5          -25443.        3760.5       -6.7659  [ 0.0000]
Sea_ 6          -1753.8        3788.0      -0.46299  [ 0.6436]
Sea_ 7          -1212.4        3645.3      -0.33259  [ 0.7396]
Sea_ 8          -12152.        3670.5       -3.3106  [ 0.0010]
Sea_ 9           10984.        3591.4        3.0584  [ 0.0024]
Sea_10           9769.1        3617.2        2.7007  [ 0.0072]
Sea_11          -14643.        3013.8       -4.8589  [ 0.0000]
```

Markov Chain and Its Use in Economic Modelling

Markov process
Transition matrix
Convergence
Likelihood function
Expected values and Policy Decision

A stochastic process $\{x_t\}$ has the Markov process if for all $k \geq 2$ and all t

$$\Pr ob\left(x_{t+1} / x_t, x_{t-1}, \ldots., x_{t-k}\right) = prob\left(x_{t+1} / x_t\right)$$

A Markov process is characterised by three elements:

1) an N dimensional vector $\bar{x} \in R^n$ of all possible values of the state of the system
2) a t $N \times N$ transition matrix P, that shows possibility of moving from one state. to another
3) $\pi_{0,i}$ the probability of being in each state i at time 0.

A typical Transition matrix

$$P_{i,j} = \begin{bmatrix} \pi_{01} & \pi_{02} & \pi_{03} & . & \pi_{0N} \\ \pi_{11} & \pi_{12} & \pi_{13} & . & \pi_{1N} \\ \pi_{21} & \pi_{22} & \pi_{23} & . & \pi_{2N} \\ . & . & . & . & . \\ \pi_{N1} & \pi_{N2} & \pi_{N3} & . & \pi_{N.N} \end{bmatrix}$$

$$\sum_{j}^{N} \pi_{i,j} = 1 \qquad \sum_{i}^{N} \pi_{0,i} = 1$$

Chapman-Kolmogorov Equations

$$\pi_{0,i} = \Pr ob\left(x_0 = \bar{x}_i\right)$$

$$\pi_{i,j} = \Pr ob\left(x_{t+1} = \bar{x}_j \big/ x_t = x_i\right)$$

$$\sum_{h=1}^{n} \Pr ob\left(x_{t+2} = \bar{x}_j \big/ x_{t+1} = \bar{x}_h\right) \Pr ob\left(x_{t+1} = \bar{x}_h / x_t = x_i\right) = \sum_{h}^{n} P_{i,h} P_{h,j} = P_{i,j}^2$$

$$\Pr ob\left(x_{t+k} = \bar{x}_j \big/ x_t = x_i\right) = P_{i,j}^k$$

$$\pi_1^{'} = prob\left(x_1\right) = \pi_0^{'} P$$

$$\pi_2^{'} = prob\left(x_2\right) = \pi_0^{'} P^2$$

$$\pi_k^{'} = prob\left(x_k\right) = \pi_0^{'} P^k$$

Likelihood Function for a Markov Chain

$$L \equiv \Pr ob\left(x_{i,T}, x_{i,T-1}, x_{i,T-2}, \ldots . x_{i,1}, x_{i,0}\right)$$

$$= P_{i_{T-1},i_T} P_{i_{T-2},i_{T-1}} P_{i_{T-3},i_{-2T}} \ldots P_{i_0,i_1} \pi_{0,i}$$

$$L = \pi_{0,i} \, {}_{\theta} \prod_{i} \prod_{j} P_{i,j}^{n_{i,j}}$$

Two uses of likelihood function

to study alternative histories of a Markov Chain

to estimate the parameter θ

Convergence of Markov Process with Finite States

$$\Pi = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \qquad \Pi^2 = \begin{bmatrix} 5/8 & 3/8 \\ 3/8 & 5/8 \end{bmatrix} \qquad \Pi^3 = \begin{bmatrix} 9/16 & 7/16 \\ 7/16 & 9/16 \end{bmatrix}$$

$$\Pi^4 = \begin{bmatrix} 17/32 & 15/32 \\ 15/32 & 17/32 \end{bmatrix} \qquad \lim_{n\to\infty} \Pi^n = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

A Markov Process Converges when each element of the of the transition matrix
pproaches to a limit like this.

$\{x_t\}$ Process is stationary in this example.

Reference: Stokey and Lucas (page 321)

Recurrent or absorbing State or Transient State in a Markov Chai

S1 is the recurrent state whenever the process

$$\{x_t\}$$

leaves, re-enters in it and stays there forever.

It is transient when it does not return to S1 when it leaves it.

$$\Pi = \begin{bmatrix} 1-\gamma & \gamma/2 & \gamma/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Here S1 is the recurrent state whenever the process leaves, re-enters in it.
S2 and S3 are transient.

Converging and Non-converging Sequences

$$\lim_{n\to\infty}\Pi^{n} = \begin{bmatrix} (1-\gamma)^{n} & \delta_n/2 & \delta_n/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\delta_n = 1-(1-\gamma)^{n}$$

$$\gamma \in (0,1)$$

$$\lim_{n\to\infty}\Pi^{n} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix}$$

$$\Pi\Pi = \begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix} = \begin{bmatrix} \Pi_1\Pi_2 & 0 \\ 0 & \Pi_1\Pi_2 \end{bmatrix}$$

$$\Pi_1 = \Pi_2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

Even

$$\Pi^{2n} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Odd

$$\Pi^{2n+1} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

One Example of Markov Chain Stochastic life cycle optimisation model (preliminary version of Bhattarai and Perroni)

$$U_{t,z} = \left\{ C_{t,z}^{\frac{1}{1+\sigma_t}} + \left\{ \frac{w_{t+1,Z}}{w_{t+1,Z} + w_{t+1,Z+d_t}} U_{t+1,z}^{1-\rho} \frac{w_{t+1,Z+d_{t+1}}}{w_{t+1,Z} + w_{t+1,Z+d_t}} U_{t+1,z+d_{t+1}}^{1-\rho} \right\}^{\frac{1}{1+\sigma_t}} \right\}^{\frac{\sigma_t}{1+\sigma_t}}$$

$$E_{t,Z} + r \cdot W_{t,Z} = C_{t,z} + V_{t,z}$$

$$W_{t+1,Z+d_t} = W_{t,z} + V_{t,z}$$

$$W_{T,Z} = -V_{T,z}$$

Probability of recurrent state π Prob of Transient state $\left(1-\pi\right)$

If transient

High income $\frac{\pi}{2}$ Low income $\frac{\pi}{2}$

Probability of being in Ambiguous state μ

Impact of Risk Aversion and Ambiguity in Expected Wealth with Markov Process

Expected nonhuman wealth with increasing risk aversion (1-3)

	SC1	SC2	SC3	SC4	SC5
T2	0.872	0.914	0.956	0.996	1.032
T3	1.580	1.654	1.727	1.796	1.862
T4	1.960	2.053	2.144	2.230	2.312
T5	1.659	1.740	1.819	1.894	1.965

Expected nonhuman wealth with increasing ambiguity (0.2-0.8)

	SC1	SC2	SC3	SC4	SC5
T2	0.872	0.906	0.938	0.968	0.995
T3	1.580	1.646	1.709	1.768	1.825
T4	1.960	2.050	2.135	2.216	2.293
T5	1.659	1.742	1.820	1.895	1.967

Markov Decision problem (refer Ross (187)).

Let there be a sequence of action a_0 , a_1 ,…., a_n corresponding to states $i = 1,2,....,n$ and the reward for this be given be $R(i,a)$. Policy makers problem with the Markov process is:

$$Max \quad \sum\sum \pi_{i,a} R(i,a)$$

Subject to

1. $\pi_{i,a} \geq 0$ for all i and a.

2. $\sum_{j}^{n} \pi_{i,a} = 1$

3. $\sum_{a}^{n} \pi_{j,a} = \sum_{i}\sum_{a} \pi_{j,a} P_{i,j}(a)$

Optimal policy is $\beta_i^*(a) = \dfrac{\pi^*_{i,a}}{\sum_t \pi^*_{i,t}}$

Use of Markov Chain in analysis of Duopoly

Sargent and Ljungqvist (133)

$$R_{i,t} = p_t y_{i,t} - 0.5d\left(y_{i,t+1} - y_{i,t}\right)^2$$

$$p_t = A_0 - A_1\left(y_{1,t+1} + y_{2t}\right)$$

$$R_{i,t} = A_0 y_{i,t} - A_1 y_{i,t}^2 - A_1 y_{i,t+1} y_{j,t} - 0.5d\left(y_{i,t+1} - y_{i,t}\right)^2$$

$$v_i\left(y_{i,t}, y_{j,t}\right) = \max_{y_{i,t+1}} \left\{R_{i,t} + \beta v_i\left(y_{i,t+1}, y_{j,t+1}\right)\right\}$$

$$y_{j,t+1} = f_j\left(y_{i,t}, y_{j,t}\right)$$

Markov perfect equilibrium is the pair of
value functions and a pair of policy functions for i=1,2
that satisfies the above Bellman equation.
Equilibrium is computed by backward induction and
he optimising behaviours of firms by iterating forward for all
conceivable future states.

Other Application of Markov Process

- Regime -Switch analysis in economic time series (Hamilton pp. 677-699; Harvey (285))
- Industry investment under uncertainty (SL chap 10)
- Stochastic dynamic programming (SL chapter 8,9)
- Weak and strong convergence analysis (SLChap 11-13)
- Arrow Securities (Ljungqvist and Sargent Chapter 7).
- Life cycle consumption and saving: An example
- Precautionary saving

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Markov Chain Example in GAMS

```
*retire1.gms
$title model with Knightian uncertainty
scalar  pi     transition probability                        /0.33/
        mu      cond probability of ambiguous state            /0/
        beta   pure rate of time preference                  /0.02/
        r      interest rate                              /0.05/
        rho     relative risk aversion                       /4.0/
        eh     high earnings                               /2.0/
        el     low earnings                                /0.5/;

option iterlim   = 1000000000;
option reslim    = 1000000000;

set t /t1*t5/
   z /s1*s16/;
alias(t,tt);
alias(z,zz);

* card(z) = 2**(card(t)-1)

beta     = (1+beta)**(50/card(t))-1;
r       = (1+r)**(50/card(t))-1;
parameter
        act(t,z) a tree generator
        d(t)                remaining states
        l(z)                odd number generator
        nlst(t)   non-last period
        prob(t,z)           probability of occurence
        weight(t,z)         weight with ambiguity
        e(t,z)              earnings
        trans(t,z)          transition index
        sex(t)              discount factor
;
act(t,z)  = round(ord(z) - trunc(ord(z)/(card(z)/(2**(ord(t)-1))))*card(z)/(2**(ord(t)-1))
act(t,z)  = 1$((act(t,z) eq 1) or (ord(t) eq card (t)));
d(t)      = round(2**(card(t)-ord(t))) ;
l(z)      = round(ord(z)- trunc(ord(z)/2)*2);
nlst(t)   = 1$(ord(t) ne card(t));
```

Markov Chain Example in GAMS

```
e("t1","s1") = eh;
loop((t,z)$(act(t,z) and nlst(t)),
  e(t+1,z)        = eh;
  e(t+1,z+d(t+1)) = el;
);
trans(t,z) = 0;
trans("t1","s1") = 0;
loop((t,z)$(act(t,z) and nlst(t)),
  trans(t+1,z)        = 1$(e(t+1,z)        ne e(t,z));
  trans(t+1,z+d(t+1)) = 1$(e(t+1,z+d(t+1)) ne e(t,z));
);
prob("t1","s1") = 1;
loop((t,z)$(act(t,z) and nlst(t)),
  prob(t+1,z)        = prob(t,z)*( (1-pi+pi/2)$(trans(t+1,z) eq 0)
                                  +(pi/2)     $(trans(t+1,z) eq 1)
                                 );
  prob(t+1,z+d(t+1)) = prob(t,z)*( (1-pi+pi/2)$(trans(t+1,z+d(t+1)) eq 0)
                                  +(pi/2)     $(trans(t+1,z+d(t+1)) eq 1)
                                 );
);
weight("t1","s1") = 1;
loop((t,z)$(act(t,z) and nlst(t)),
  weight(t+1,z)        =  weight(t,z)*( (    (1-pi+(1-mu)*pi/2) $(trans(t+1,z) eq 0)
                                         +(1-(1-pi+(1-mu)*pi/2))$(trans(t+1,z) eq 1)
                                        )$(e(t,z) eq eh)
                                      +( (1-(1-mu)*pi/2)       $(trans(t+1,z) eq 0)
                                         +(  (1-mu)*pi/2)       $(trans(t+1,z) eq 1)
                                        )$(e(t,z) eq el)
                                      );
  weight(t+1,z+d(t+1)) =  weight(t,z)*( (    (1-pi+(1-mu)*pi/2) $(trans(t+1,z+d(t+1)) eq 0)
                                         +(1-(1-pi+(1-mu)*pi/2))$(trans(t+1,z+d(t+1)) eq 1)
                                        )$(e(t,z) eq eh)
                                      +( (1-(1-mu)*pi/2)       $(trans(t+1,z+d(t+1)) eq 0)
                                         +(  (1-mu)*pi/2)       $(trans(t+1,z+d(t+1)) eq 1)
                                        )$(e(t,z) eq el)
                                      );
);
```

Markov Chain Example in GAMS: Model Equations

```
defu(t,z)$act(t,z)..
 u(t,z)    =e= ((  c(t,z)**(1/(1+sex(t)))
                * ( ((  weight(t+1,z)       /(weight(t+1,z)+weight(t+1,z+d(t+1)))
                                               *u(t+1,z)         ** (1-rho)
                       +weight(t+1,z+d(t+1))/(weight(t+1,z)+weight(t+1,z+d(t+1)))
                                               *u(t+1,z+d(t+1))  ** (1-rho)
                   ) ** (1/(1-rho))
                  )$(rho ne 1)
                  +
                  (   u(t+1,z)       **(weight(t+1,z)       /(weight(t+1,z)+weight(t+1,z+d(t+1))))
                   *
u(t+1,z+d(t+1))**(weight(t+1,z+d(t+1))/(weight(t+1,z)+weight(t+1,z+d(t+1))))
                  )$(rho eq 1)
                 )    **(sex(t)/(1+sex(t)))
               )$nlst(t)
            +c(t,z)$(nlst(t) eq 0)
            )$(insure eq 0)
          +  ((  c(t,z)**(1/(1+sex(t)))
                * ( weight(t+1,z)       /(weight(t+1,z)+weight(t+1,z+d(t+1)))
                                               *u(t+1,z)
                  +weight(t+1,z+d(t+1))/(weight(t+1,z)+weight(t+1,z+d(t+1)))
                                               *u(t+1,z+d(t+1))
                 )**(sex(t)/(1+sex(t)))
               )$nlst(t)
            +c(t,z)$(nlst(t) eq 0)
            )$(insure eq 1);

defc(t,z)$act(t,z)..
 c(t,z) + v(t,z) =e= e(t,z)+ w(t,z)*r;

defwl(t,z)$(act(t,z) and nlst(t))..
 w(t+1,z)         =e= w(t,z) + v(t,z);

defwn(t,z)$(act(t,z) and nlst(t))..
 w(t+1,z+d(t+1))  =e= w(t,z) + v(t,z);

defwt(t,z)$(act(t,z) and (nlst(t) eq 0))..
 w(t,z)    =e= -v(t,z);

dobj..
 obj    =e= u("t1","s1");

model lc /all/;
```

```
variables
 u(t,z)
 c(t,z)
 v(t,z)
 w(t,z)
 obj;
```

```
equations
 defu(t,z)
 defc(t,z)
 defwl(t,z)
 defwn(t,z)
 defwt(t,z)
 dobj;
```

Calculation of Weight Among Various States

```
loop(cases,
mu = 0;
weight("t1","s1") = 1;
loop((t,z)$(act(t,z) and nlst(t)),
  weight(t+1,z)        =  weight(t,z)*( (    (1-pi+(1-mu)*pi/2) $(trans(t+1,z) eq 0)
                                              +(1-(1-pi+(1-mu)*pi/2))$(trans(t+1,z) eq 1)
                                              )$(e(t,z) eq eh)
                                          +( (1-(1-mu)*pi/2)       $(trans(t+1,z) eq 0)
                                              +(  (1-mu)*pi/2)       $(trans(t+1,z) eq 1)
                                              )$(e(t,z) eq el)
                                          );
  weight(t+1,z+d(t+1)) =  weight(t,z)*( (    (1-pi+(1-mu)*pi/2) $(trans(t+1,z+d(t+1)) eq 0)
                                              +(1-(1-pi+(1-mu)*pi/2))$(trans(t+1,z+d(t+1)) eq 1)
                                              )$(e(t,z) eq eh)
                                          +( (1-(1-mu)*pi/2)       $(trans(t+1,z+d(t+1)) eq 0)
                                              +(  (1-mu)*pi/2)       $(trans(t+1,z+d(t+1)) eq 1)
                                              )$(e(t,z) eq el)
                                          );
);
```

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An Empirical Study of Interest Rate Determination Rules

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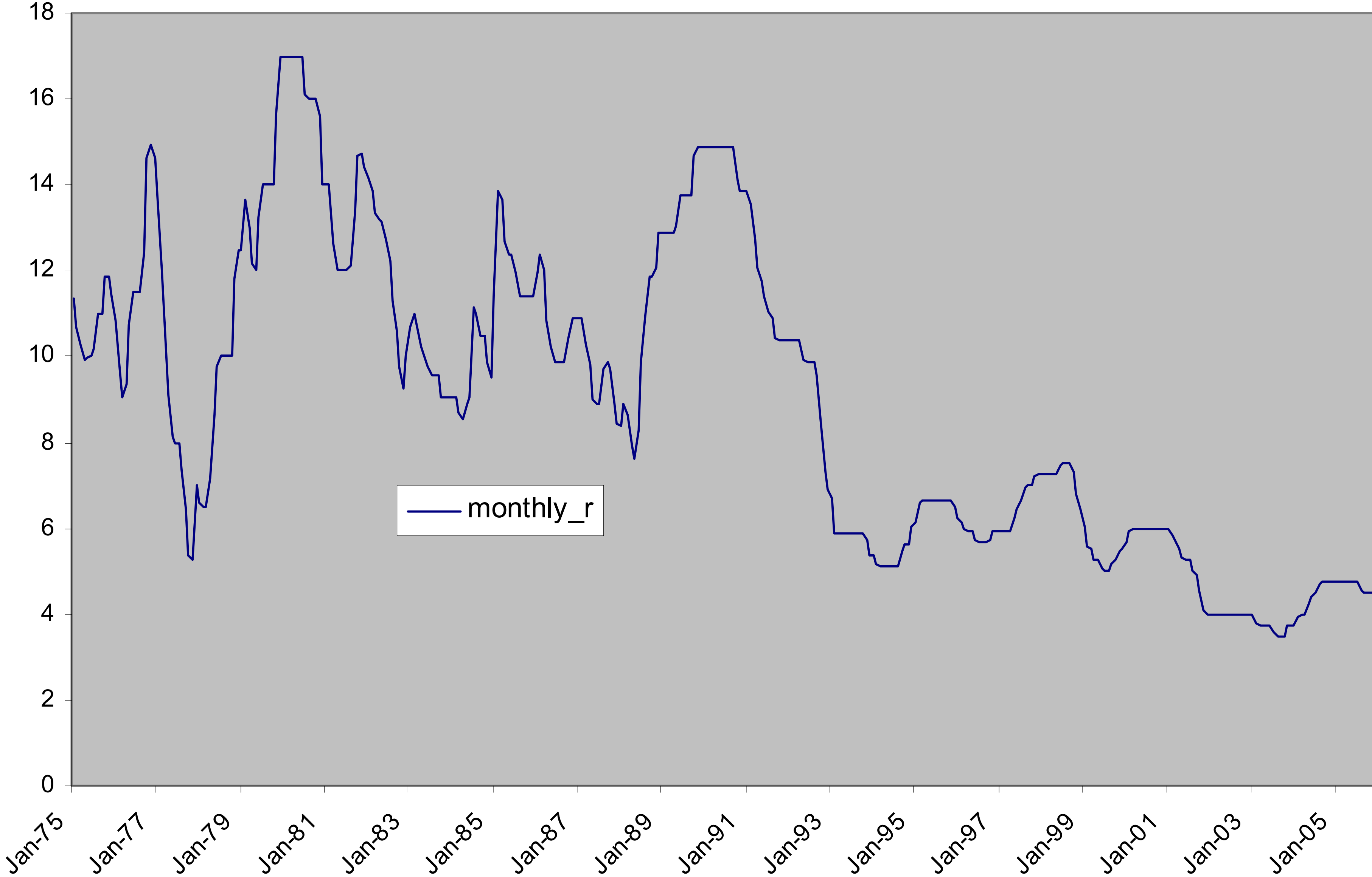
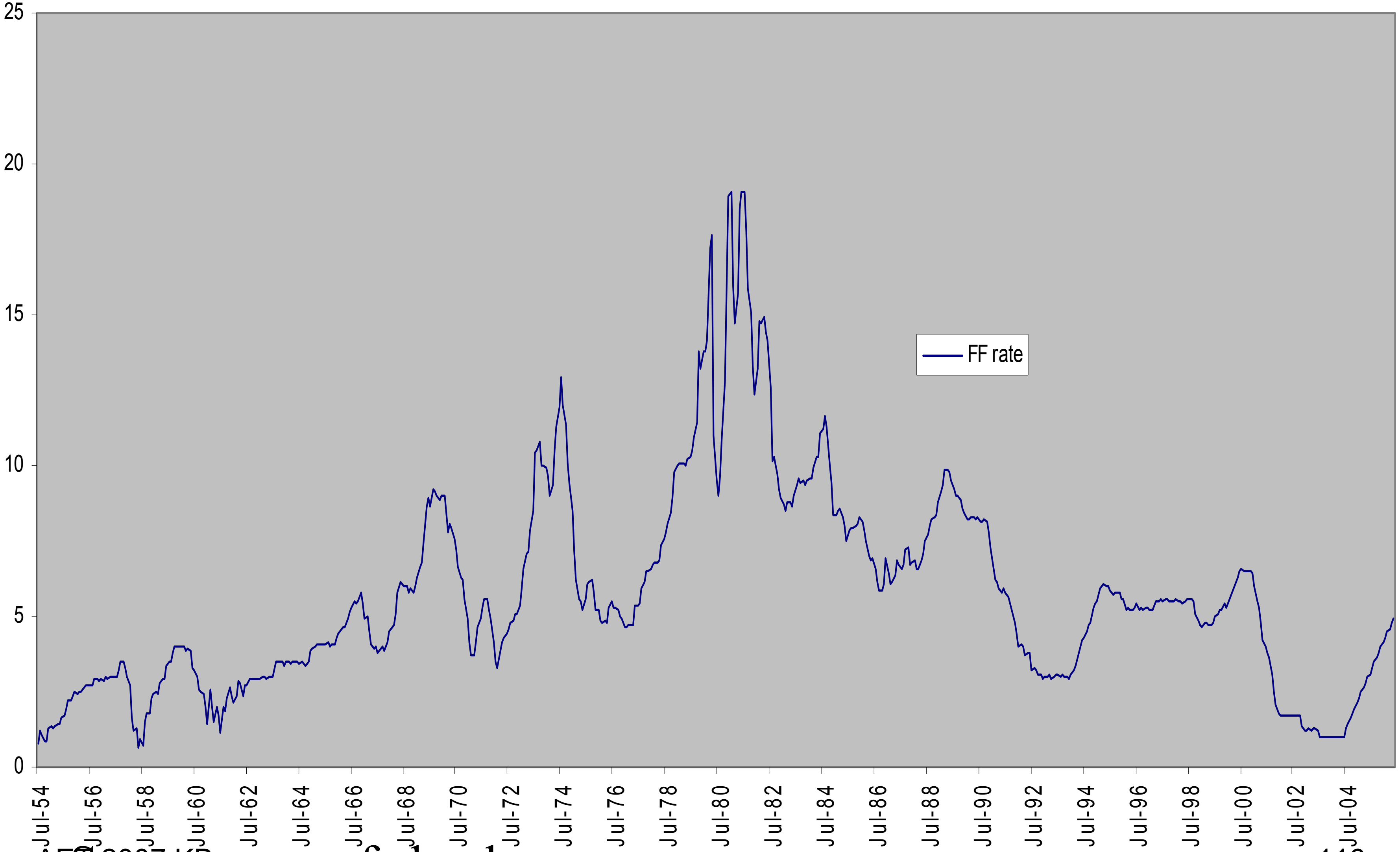
University of Hull Business School

International Economic Policy Conference, Hong Kong
28-30 June 2006

Bank of England Rate, Monthly Average Series, 1975-2006

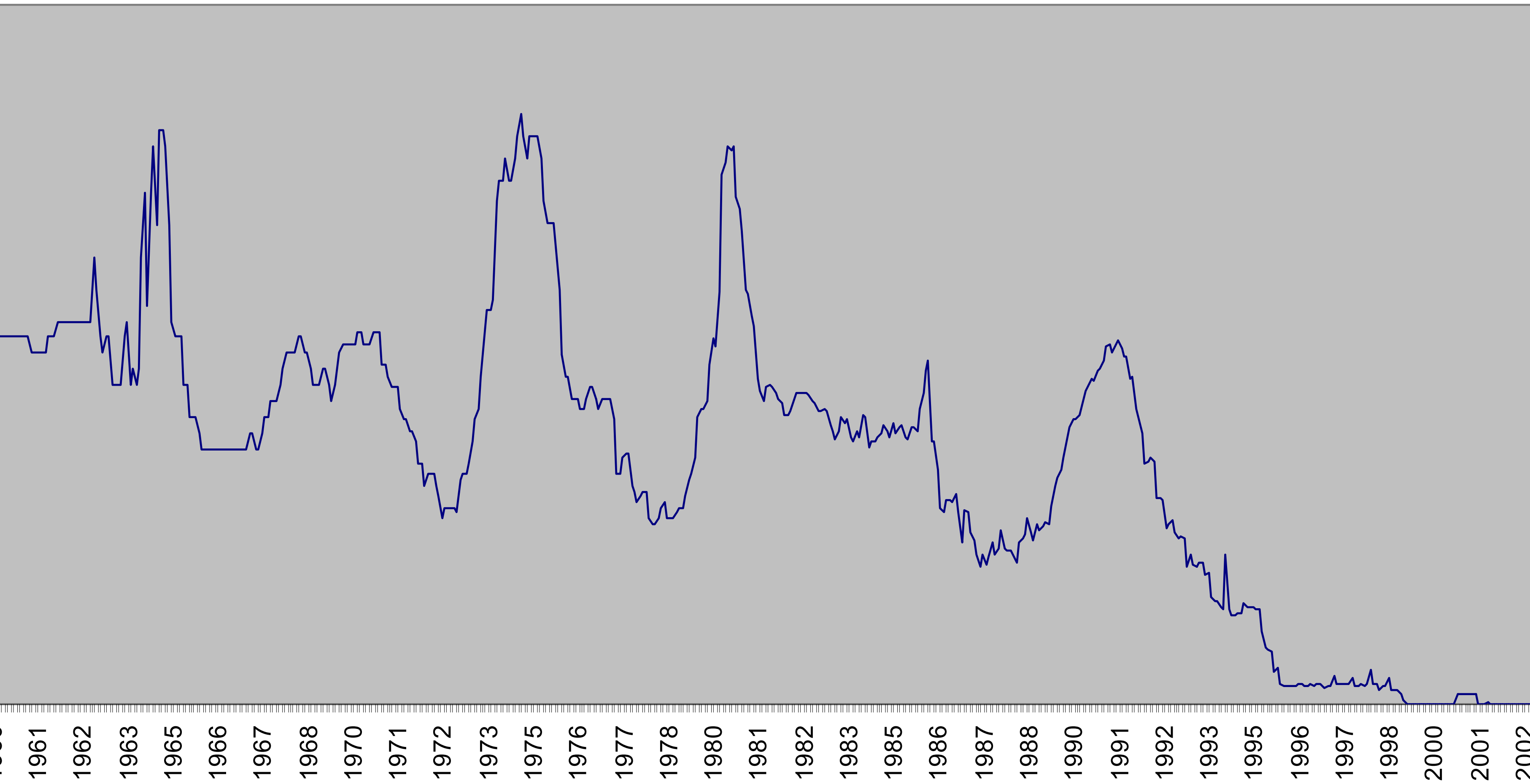


Federal Fund Rate, 1954-2006

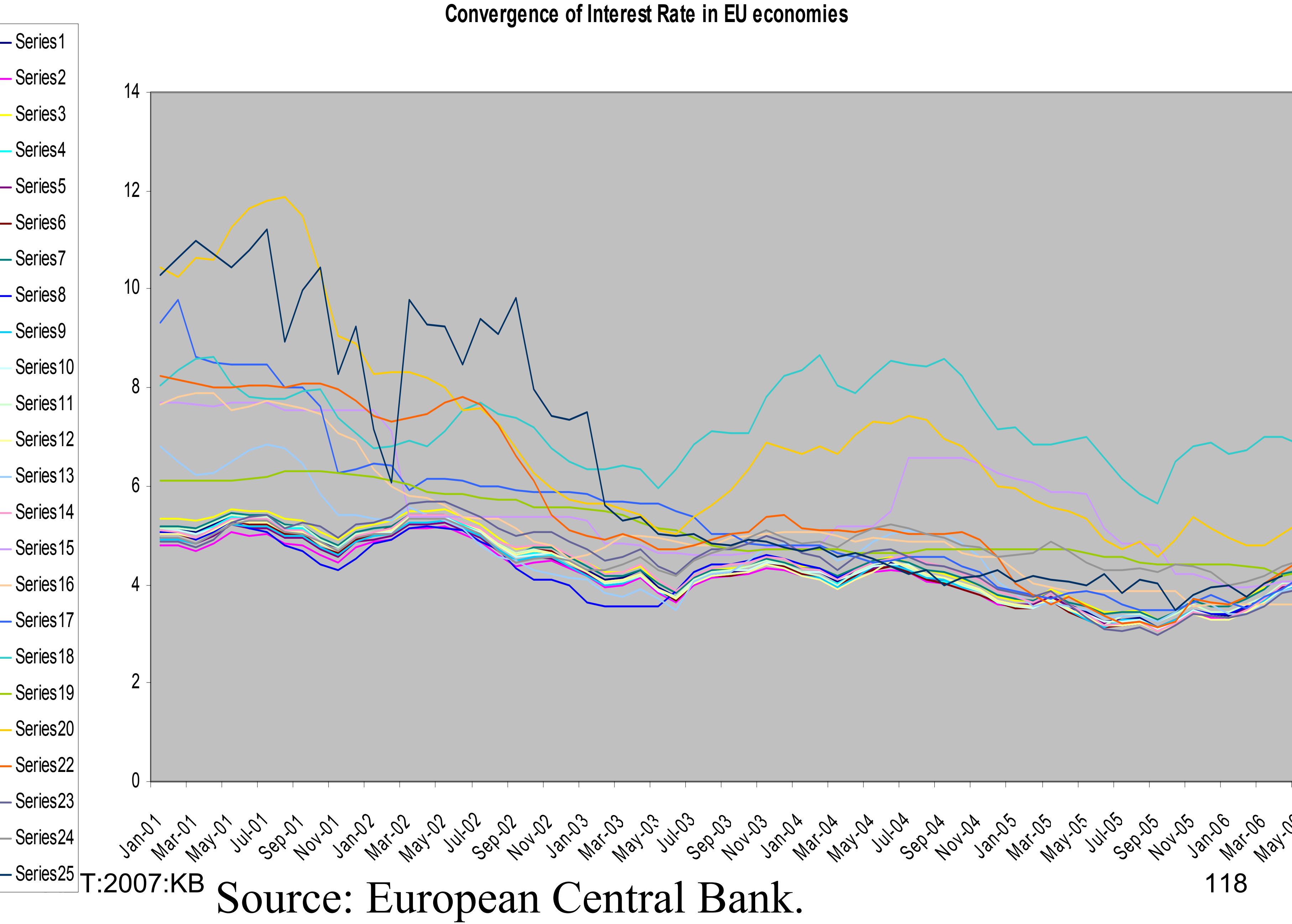


Source: www.federalreserve.gov

Monthly Interest Rate Series in Japan

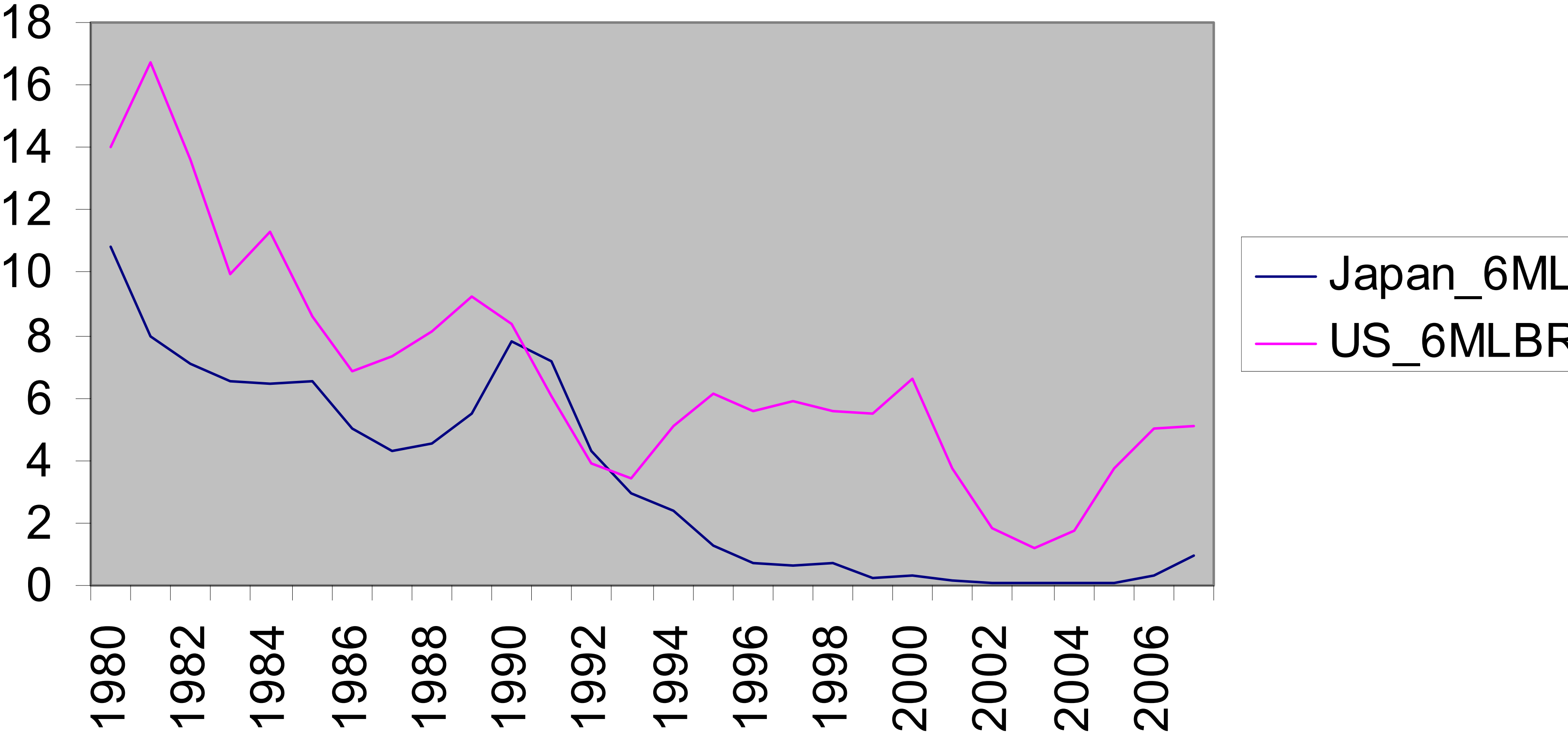


Source: Bank of Japan



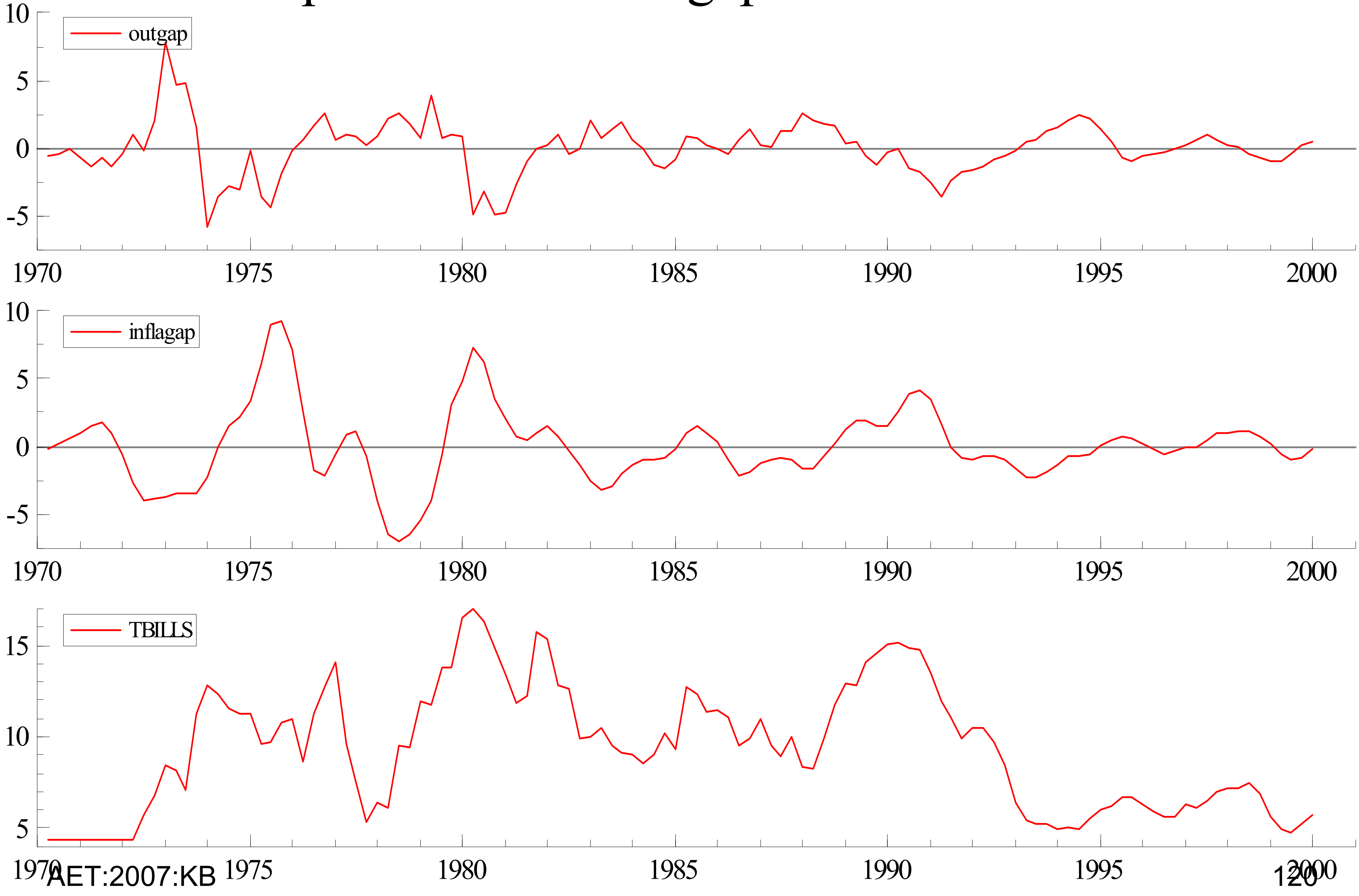
Source: European Central Bank.

LIBOR in Janan and the US

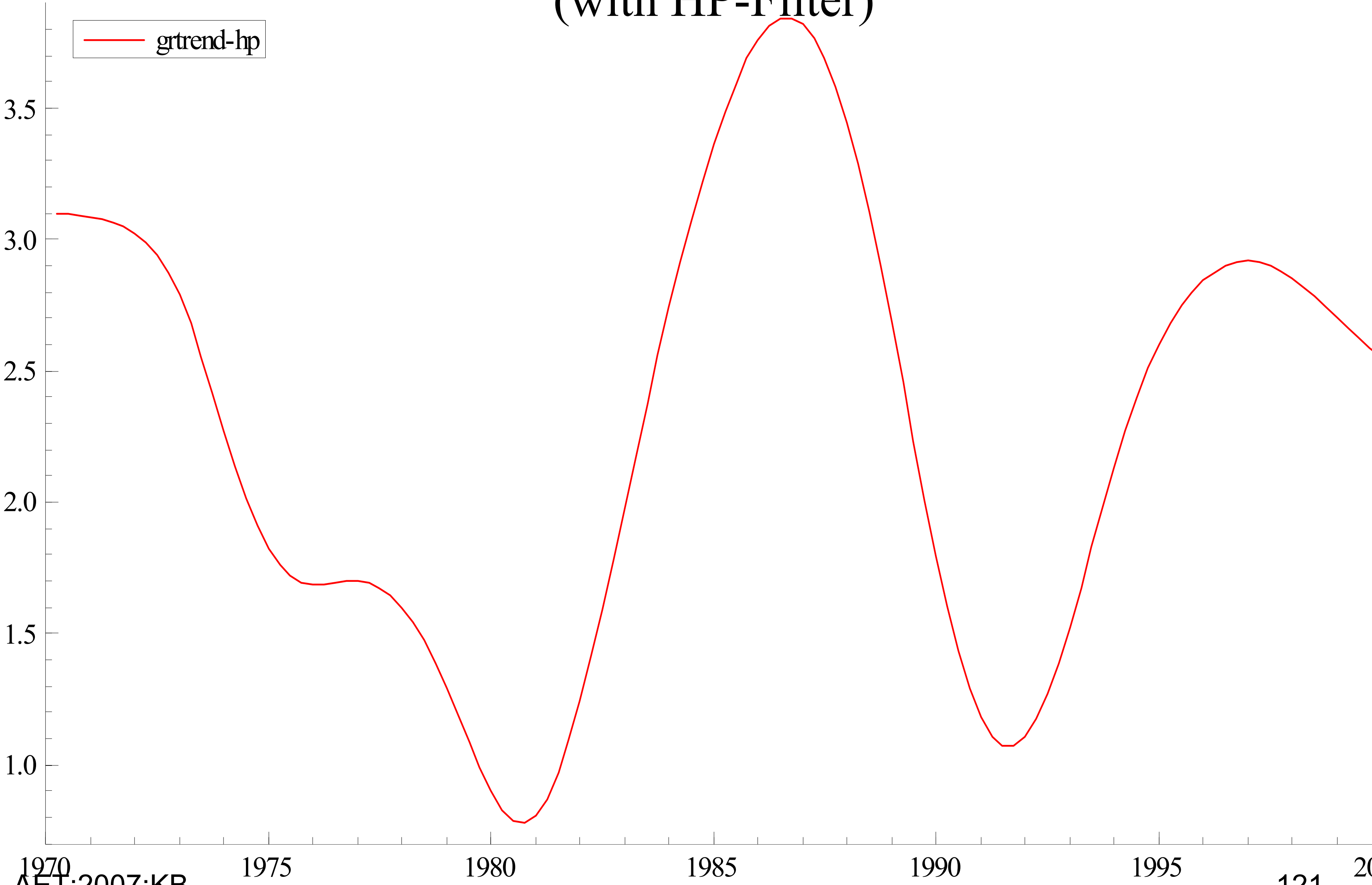


Source: World Economic Outlook, IMF.

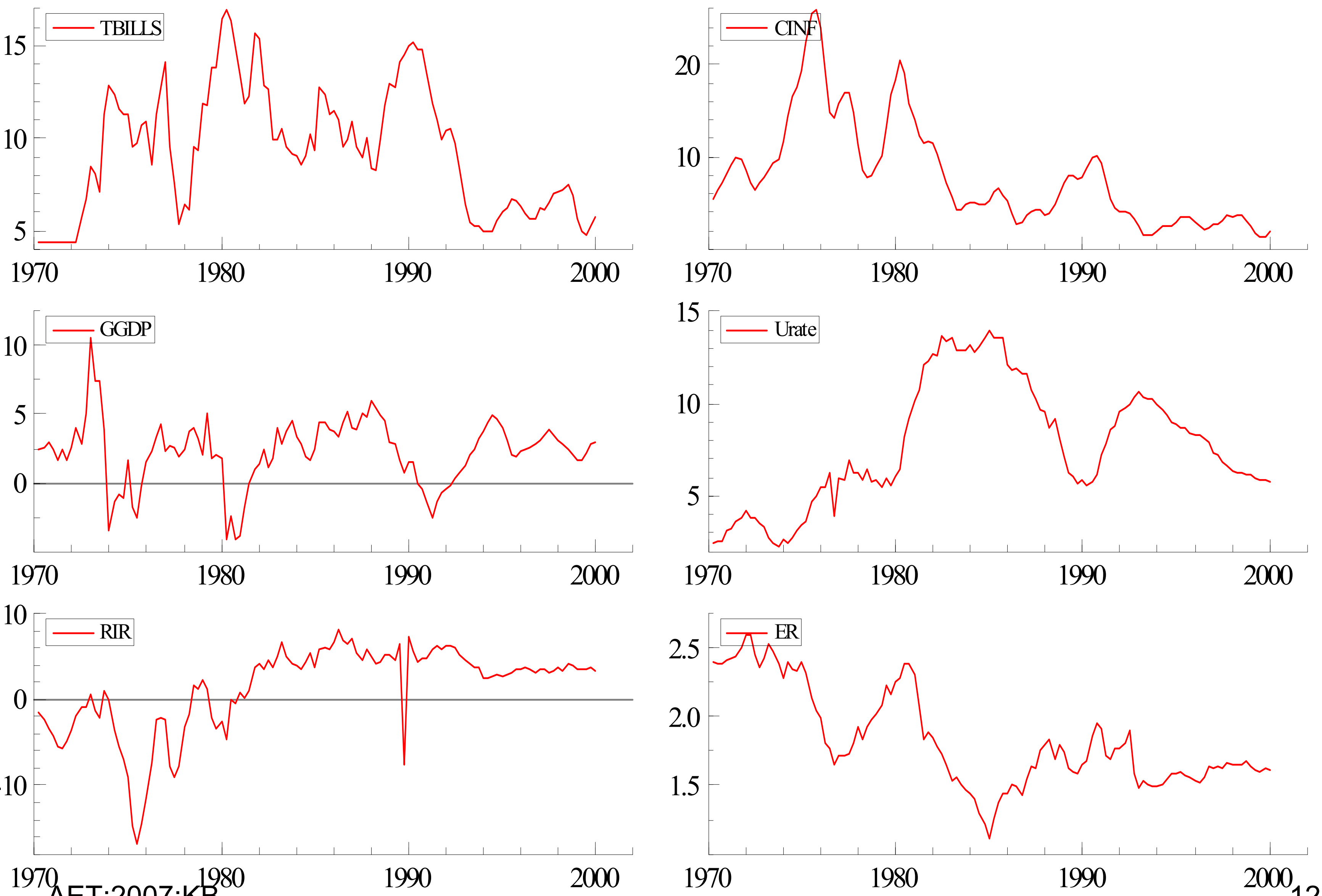
Question: Is there any systematic link between output and inflation gaps and the interest rate?



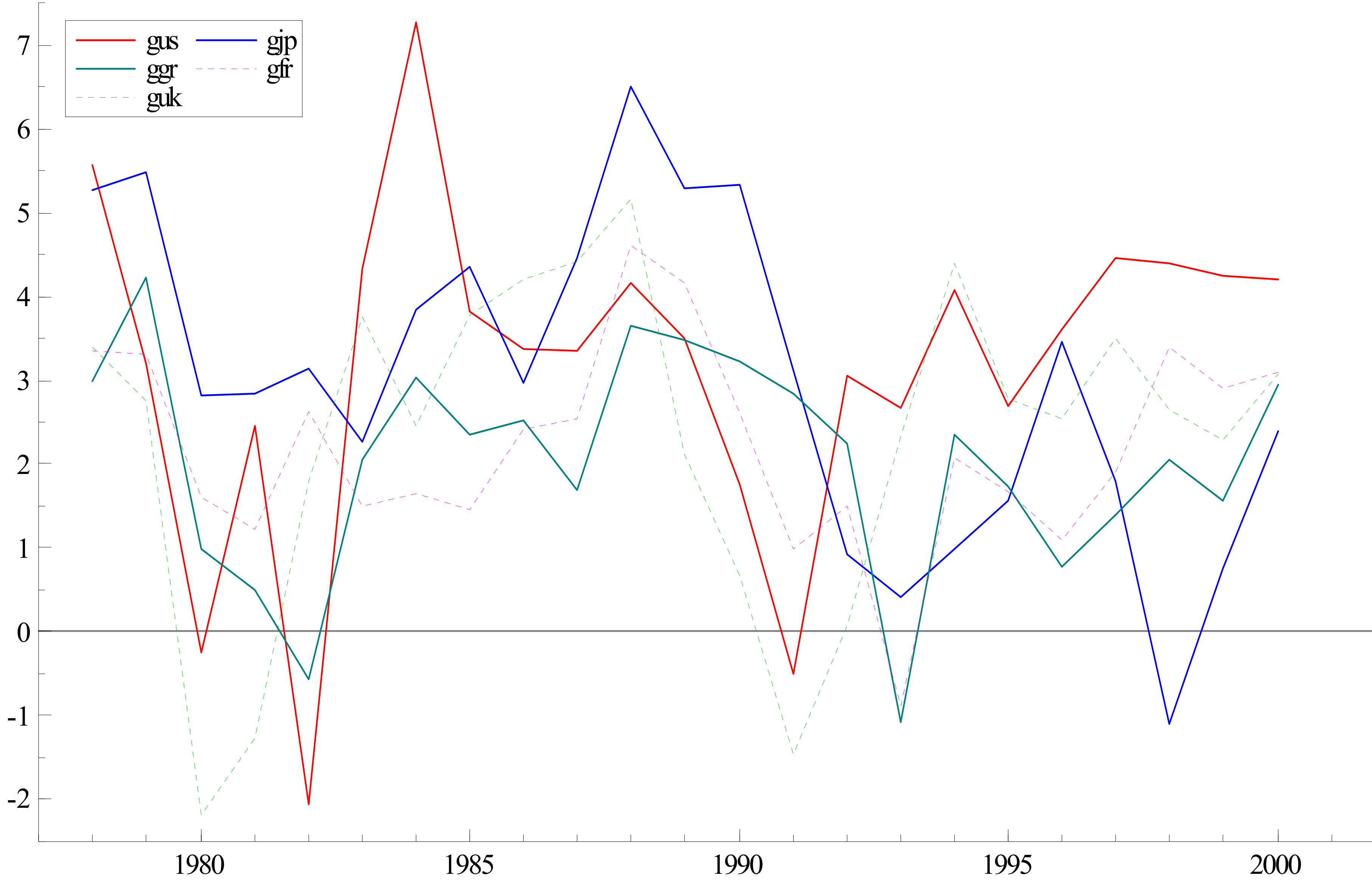
Does Changes in the interest remove fluctuations in output?
Quarterly Trend Growth Rate in the UK:1970:2-2000:1
(with HP-Filter)



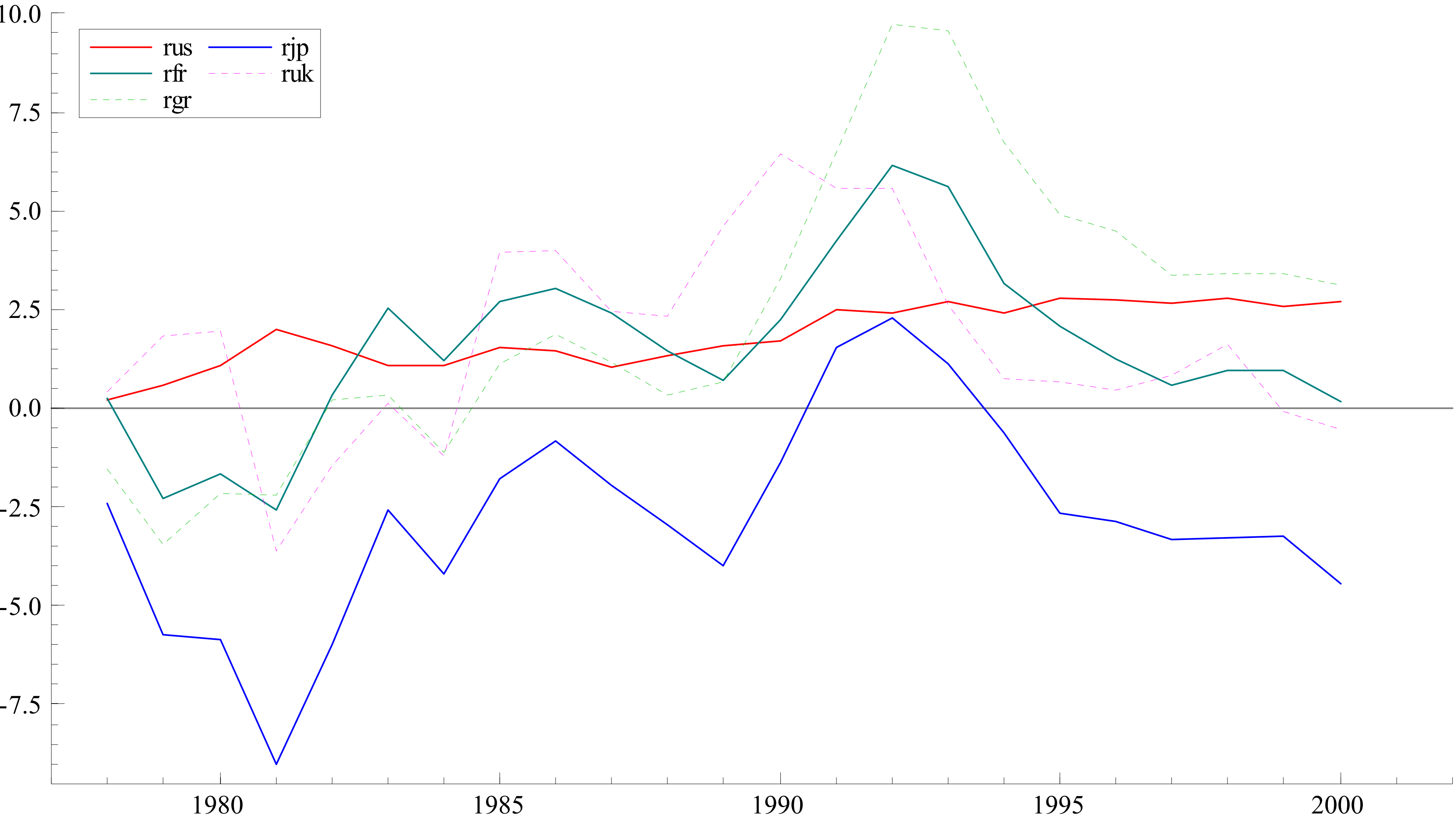
Interest Rate, Inflation, Growth Rate, Unemployment and Exchange Rate in the UK 1970:2-2000:1



Growth Rate of Output in Germany, France, Japan, UK and the U

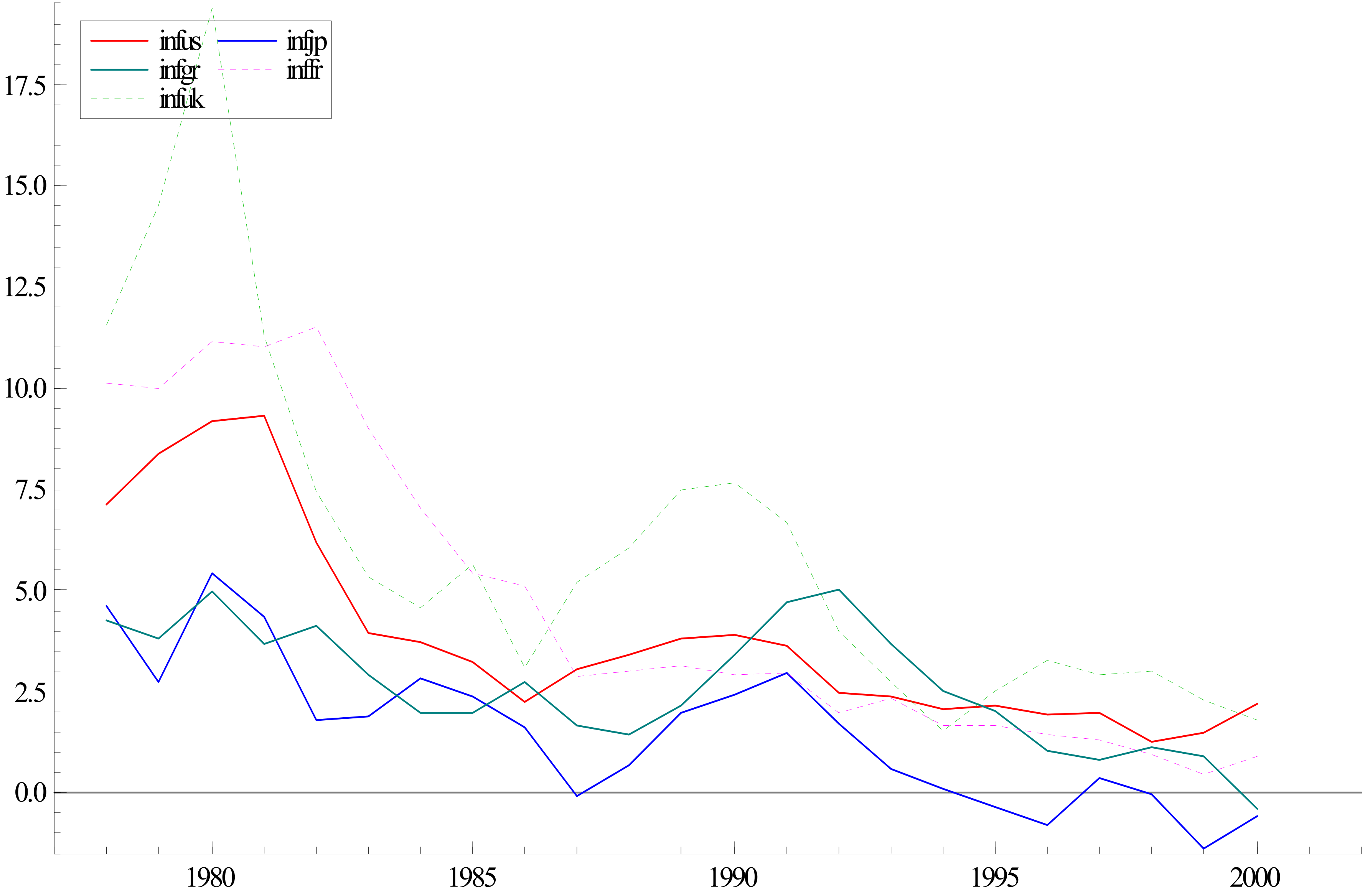


Real Interest Rate in Germany, France, Japan, UK and the US



Source: World Bank CD Database

Inflation Rate in Germany, France, Japan, UK and the US



Background Literature

General Literature on Role of Money in the Economy

Keynes (1936), Hicks (1937), Bailey(1956) Phillips (1958), Friedman (1968), Phelps (1968),Tobin (1969)) Taylor (1972). Taylor (1993), Laidler and Parkin (1975) Kydland and Prescott (1977), Phelps and Taylor (1977) Aghevli (1977), Gordon (1983), Barro and Gordon (1983), Sargent (1986) Goodhart (1989), Nickell (1990), Buiter and Patel (1992), Ball and Romer (1990) Dornbusch (1992), MPC (1999), Lockwood Miller and Zhang (1998), MPC (1999).

Natural Rate of Unemployment Hypothesis

Friedman (1968), Phelps (1968)

Research in Time Inconsistency, Policy Co-ordination and

Time inconsistency

Kydland and Prescott (1977), Phelps and Taylor (1977), Gordon (1983), Barro and Gordon (1983)
Lockwood Miller and Zhang (1998), Rogoff (1985), Miller and Salmon (1985)),

Policy Co-ordination at National and International Level

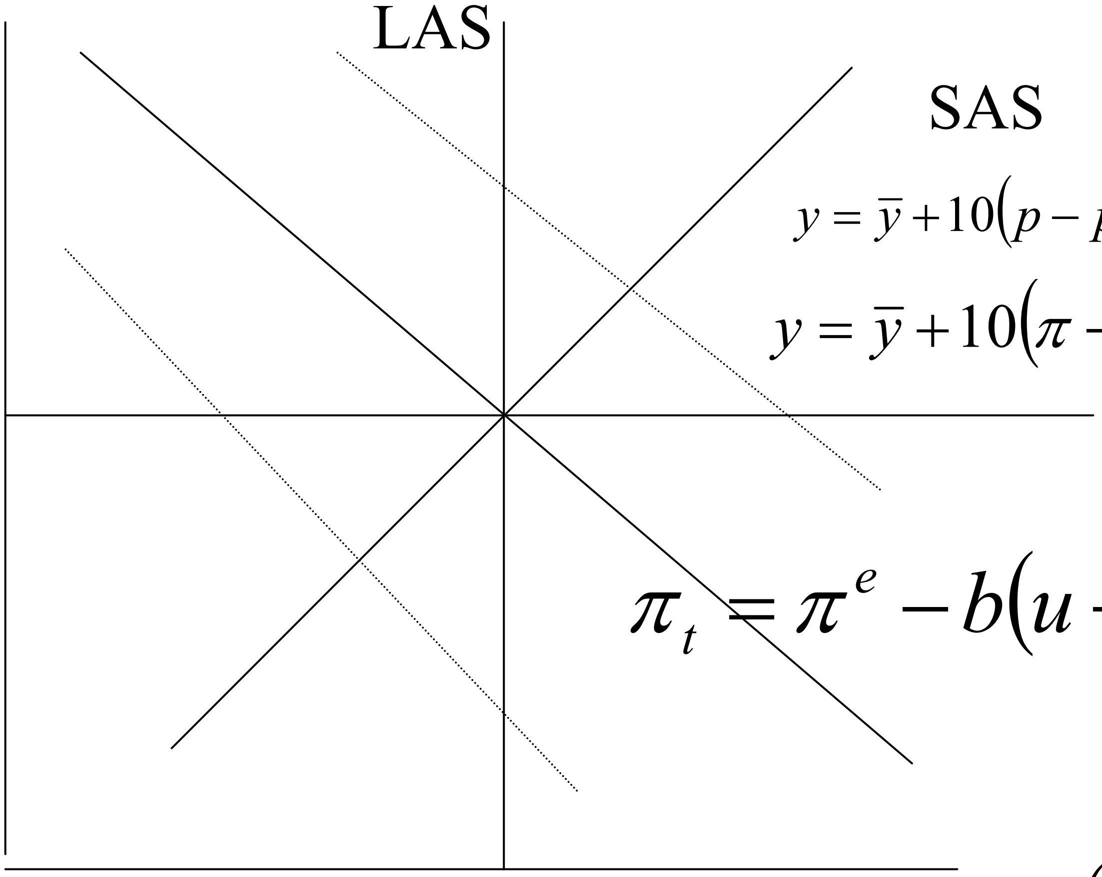
Krugman (1979), Barro and Gordon (1983), Canzoneri M. B. and J A Gray (1985), Cukierman (1994), Goodhart (1994), Nardhaus (1994) Eijffinger SCW and J.D. Haan (2000)

Money in General Equilibrium

Tobin (1969) Altig D E, C.T. Carlstrom and K.L. Lansing (1995), Lockwood Miller and Zhang (1998), Holly and Weale (2000), Corsetti and Pesenti (2001), Benigno(2002)

Authors	Economic issues	Method	Summary	Journal
Asimakopoulos, Goddard and Siriopoulos(2000)	US and major European equity markets	spectral analysis	Lead-lag relation in stock returns in EU and US	AE
Bacchetta and Ballabriga(2000)	The impact of monetary policy and banks' balance sheets: some international evidence.	VAR	Strong relation between the Interest and output in the US and 13 EU economies	AFE
Brooks and Skinner(2000)	What will be the risk-free rate and benchmark yield curve following European monetary union?	Linear factor model	UK 3-month yield curve best approximates others in EU	AFE
Berument and Jelashi (2002)	Fisher hypothesis a multi-country analysis	ADF, OLS, ARCH-LM	Support for Fisher hypothesis for 13 of 26 countries	AE
Mills and Wood (2002)	Wages and prices in the UK	VECM	Wage growth does not predict inflation	AE
Silvapulle and Hewarathna (2002)	Robust estimation and inflation forecasting	ECM	Support Fisher effect on inflation for Australia	AE
Camarero, Ordonez and Tamarit (2002)	Monetary transmission in Spain	S-CVAR	Support for endogenous policy reaction of monetary policy	AE
Cheung and Westerman (2002)	Output dynamics in G7 countries: stochastic trends and cyclical movements	VAR-Co integration	Existence of common business cycles among G7 countries	AE
Lee (2002)	Real interest rate in regional economic blocks,	VEC, ARIMA	Long run relation in real interest rates of APEC, EU and the US	AE
Castelnouvo (2003)	Taylor rules, omitted variables, and the interest rate smoothing in the US	OLS in first differences	Test of forward looking Taylor rule in the US	EL
Yamada (2002)	Real interest rate equalisation: some evidence from three major world financial markets	VAR cointegration	Departure from long-run real interest rate equalisation is not very large	AE
Ferris and Galbraith(2003)	Indirect convertibility as a money rule for inflation targeting	Relative price concept	How indirect convertibility brings price stability(fixing a basket/ unit of money)	AFE
Valente (2003)	Monetary policy rules and regime shifts	MS-VAR	Time varying parameter and Markov Switching VAR model for policy rule	AFE
Ghazali and Ramlee (2003)	A long memory test of the long-run Fisher effect in the G7 countries	ARIMA, ARFIMA	Long run relation between interest rate and inflation in G7 countries.	.AFE
Wetherilt (2003)	Money market operations and short-term interest rate volatility in the United Kingdom	GARCH and VECM	Reduction in the volatility of market rates along with that in repo rates in uk	AFE
Buch (2004)	Cross-border banking and transmission mechanisms in Europe: evidence from German data.	Credit data analysis	Activities of commercial banks cause transmission of shocks across countries	AFE
Staikouras(2004)	The information content of interest rate futures and time-varying risk premia	VAR cointegration	Tests speculative efficiency hypothesis and supports price discovery hypothesis	AFE
Butter and Jenson (2004)	An empirical analysis of German long term interest rate	ARIMA, ECM	Four theories of interest rate explain German short term rate	AFE

Aggregate Supply, Inflation and natural rate of unemployment hypothesis



$$y = \bar{y} + 10\left(p - p^{e}\right)$$

$$y = \bar{y} + 10\left(\pi - \pi^{e}\right)$$

$$\pi > \pi^{e}$$

$$\pi = \pi^{e}$$

$$\pi < \pi^{e}$$

$$\pi_t = \pi^{e} - b\left(u - u_n\right) +$$

$$y<\bar{y} \quad y=\bar{y} \quad y>\bar{y}$$

$$u>u_n \quad u=u_n \quad u<u_n$$

$$\left(u-u_n\right)$$

Summary:

$> \pi_t^e \Rightarrow u_t < u_N \Rightarrow y_t > \bar{y}$

$< \pi_t^e \Rightarrow u_t > u_N \Rightarrow y_t < \bar{y}$

$= \pi_t^e \Rightarrow u_t = u_N \Rightarrow y_t = \bar{y}$

Main Points on Interest Determination Rule

- Higher interest rate is contractionary. Effect of interest rate in output is felt after some lag.
- Higher level of output puts pressure in the price level. Increase in the output at the current period may raise the rate of inflation in the next period (Aggregate supply curve)
- Interest should be raised when the economy is overheating output is above the trend to reduce the inflationary pressure.
- It should be raised also when the rate of inflation is above the target inflation to reduce aggregate demand.
- Interest rate should be lowered in recession.
- Interest rates should be determined based on economic facts but not according to whims of the policy makers.
- An independent central can take such an independent

Three Equations of the Interest Determination Rule: Taylor Rule

$$y_t - y_t{*} = -d\left(i_{t-1} - i_{t-1}^{*}\right) \qquad d > 0 \qquad (1)$$

where i_t and i_t^{*} are actual and natural level of output, i_t is the actual rate of interest in period *t*, i is the interest target of the monetary authority.

One period lag is assumed between the interest rate decision and the change in the output.

$$\pi_t = \pi_t^{*} + c\left(y_{t-1} - y_{t-1}^{*}\right) \quad c > 0 \qquad (2)$$

where π_t and π_t^{*} are actual and target inflation rates.

$$i_t = i_t^{*} + a\left(y_t - y_t^{*}\right) + b\left(\pi_t - \pi_t^{*}\right) \quad a > 0; b > 0 \qquad (3)$$

Reduced Form Equation of the Interest Determination Model

$$i_t = i_t^* - ad\left(i_{t-1} - i_{t-1}^*\right) - bcd\left(i_{t-2} - i_{t-2}^*\right)$$

$$i_t + adi_{t-1} + bcdi_{t-2} = i_t^* + adi_{t-1}^* + bcdi_{t-2}^* \qquad (4)$$

$_0 = \left(i_t^* + adi_{t-1}^* + bcdi_{t-2}^*\right)$, and $\beta_1 = ad$ and $\beta_2 = bcd$.

Reduced form: $i_t + \beta_1 i_{t-1} + \beta_2 i_{t-2} = \beta_0$ (5)

teady State: $i_t = i_{t+1} = i_{t+2} = \ldots = i_{t+n}$:

Natural rate of Interest: Steady State

Natural Rate of Interest: $\bar{i} = \dfrac{i_t^* + adi_{t-1}^* + bcdi_{t-2}^*}{1 + \beta_1 + \beta_2}$;

$\bar{i} = \dfrac{i_t^* + adi_{t-1}^* + bcdi_{t-2}^*}{1 + ad + bcd}$ with flexible targets and

$\bar{i} = \dfrac{i_t^* + adi_t^* + bcdi_t^*}{1 + ad + bcd}$ with fixed targets (6)

General Solution of the Interest Rule Model

$$\lambda_1^t = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2} \text{ and } \lambda_2^t = \frac{-\beta_1 - \sqrt{\beta_1^2 - 4\beta_2}}{2}.$$

$$i_t = A_1\lambda_1^t + A_2\lambda_2^t + \bar{i} \qquad (7)$$

$$i_t = A_1\left(\frac{-\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}\right)^t + A_2\left(\frac{-\beta_1 - \sqrt{\beta_1^2 - 4\beta_2}}{2}\right)^t + \bar{i} \qquad (8)$$

$$i_t = A_1\left(\frac{ad + \sqrt{(ad)^2 + 4bcd}}{2}\right)^t + A_2\left(\frac{ad - \sqrt{(ad)^2 + 4bcd}}{2}\right)^t + \bar{i} \qquad (9)$$

Convergence or Divergence from the Steady State

Reduced form:

$$i_t + \beta_1 i_{t-1} + \beta_2 i_{t-2} = 0 \qquad (10)$$

(a) real and distinct root if $\beta_1^2 - 4\beta_2 > 0$

(b) real and equal roots case if $\beta_1^2 - 4\beta_2 = 0$

(c) complex roots case if $\beta_1^2 - 4\beta_2 < 0$. The general solutions of the model in these three different cases are :

$$i_t = A_1\lambda_1^t + A_2\lambda_2^t + \bar{i} \qquad (11)$$

Specification of Vector Autoregression Model

$$Y_t = A_1 Y_{t-1} + \varepsilon_t$$

Y_t **vector of variables**
interest rate,
output gap
inflation gap

ε_t **normally and identically distributed random error te**

$$\Delta Y_t = \left(A_1 - I\right) Y_{t-1} + \varepsilon_t$$

$$\Pi = \left(A_1 - I\right)$$

Application of VAR Analysis

$$y_t - y_t^* = d\left(i_{t-1} - i_{t-1}^*\right) + \varepsilon_{1,t}$$

$$\pi_t = \pi_t^* + c\left(y_{t-1} - y_{t-1}^*\right) + \varepsilon_{2,t}$$

$$i_t = i_t^* + a\left(y_t - y_t^*\right) + b\left(\pi_t - \pi_t^*\right) + \varepsilon_{3,t}$$

Stationarity of variables in the model
ADF tests (T=116, Constant; 5%=-2.89 1%=-3.49)

	Interest rate	Difference of Interest rate	Output gap	Inflation gap
Coefficient	-2.723	-6.463**	-6.160**	-7.428**
Lags	2	2	3	1

Decomposition of Long and Short Run Responses

Π **Long run response**

$$\Pi = \alpha\beta'$$

α **Dynamic process of adjustment**

β **Long run Steady State Relation**

$$\Pi = \alpha\beta'$$

$$\Pi = \begin{bmatrix} -0.03654 & -0.12011 & 0.18569 \\ 0.02982 & -0.24185 & -0.08453 \\ -0.09433 & -0.07224 & -0.53655 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 0.01810 & 0.09100 & 0.01089 \\ 0.01667 & -0.00346 & -0.00208 \\ -0.00793 & -0.20723 & 0.00692 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1.0000 & 0.19498 & -6.6460 \\ -13.850 & 1.0000 & 3.6355 \\ -4.3680 & 2.7897 & 1.0000 \end{bmatrix}$$

Trace and Max Test for Cointegration Rank

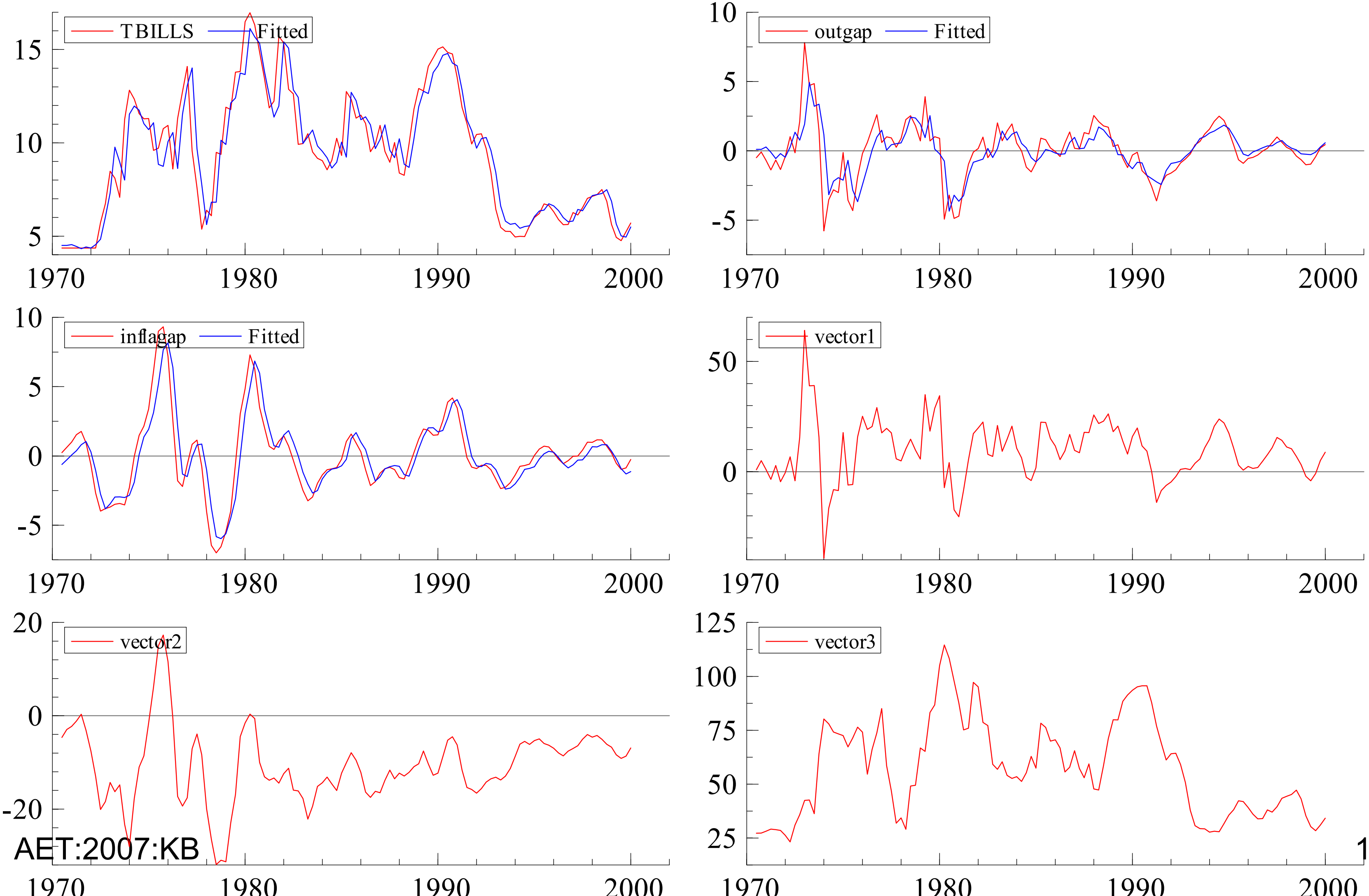
$$\lambda_{trace(r)} = -T\sum_{i=r+1}^{n}\ln\left(1-\hat{\lambda}_i\right)$$

$$\lambda_{\max(r,r+1)} = -T\ln\left(1-\hat{\lambda}_{r+1}\right)$$

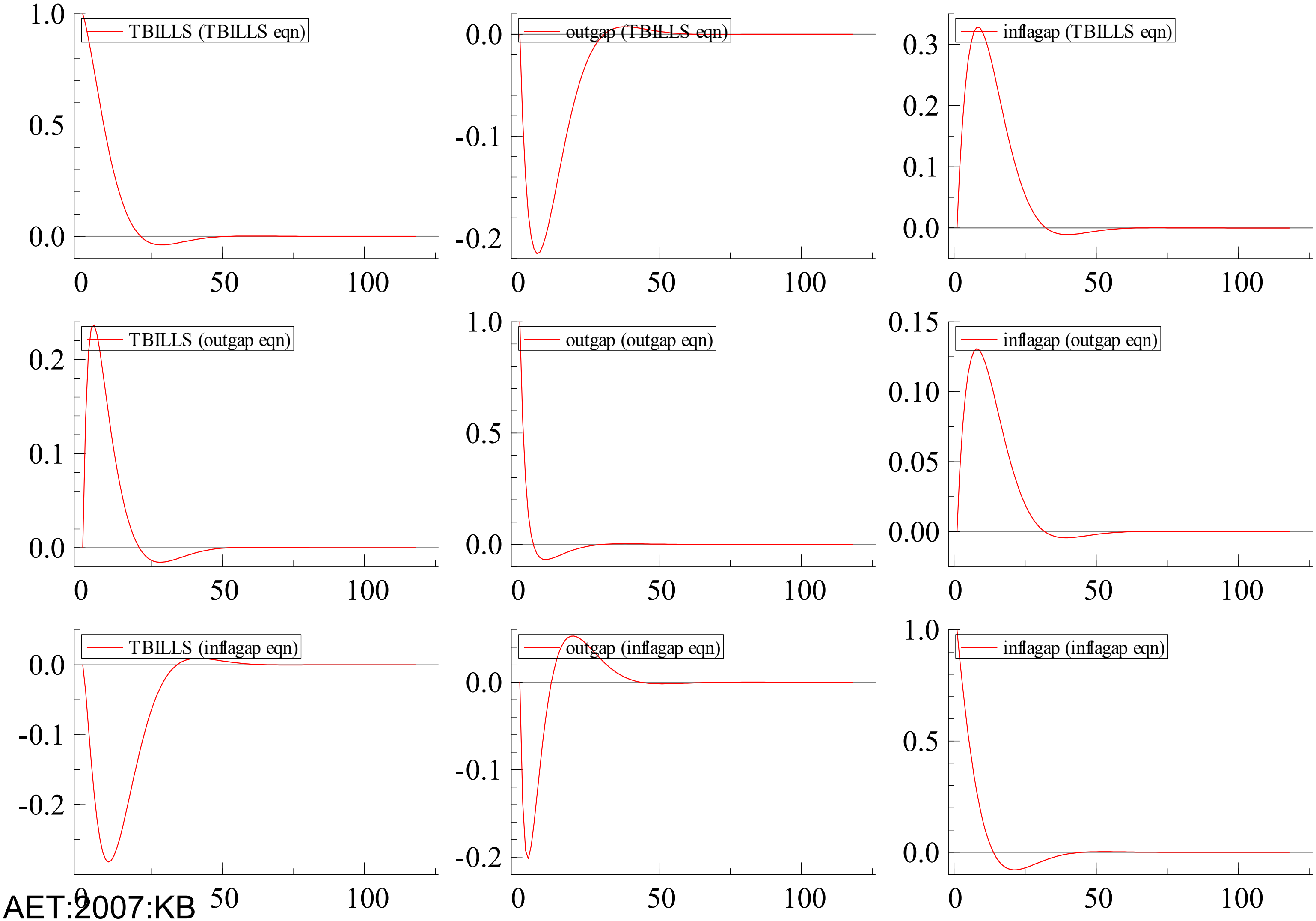
λ_i **Eigenvalues of the characteristic matrix**

rank	Trace test [Prob]	Max test [Prob]	Trace test (T-nm)	Max test (T-nm)
0	56.86 [0.000]**	34.38 [0.000]**	55.43 [0.000]**	33.52 [0.000]**
1	22.48 [0.003]**	12.68 [0.087]	21.91 [0.004]**	12.36 [0.097]
2	9.80 [0.002]**	9.80 [0.002]**	9.55 [0.002]**	9.55 [0.002]**

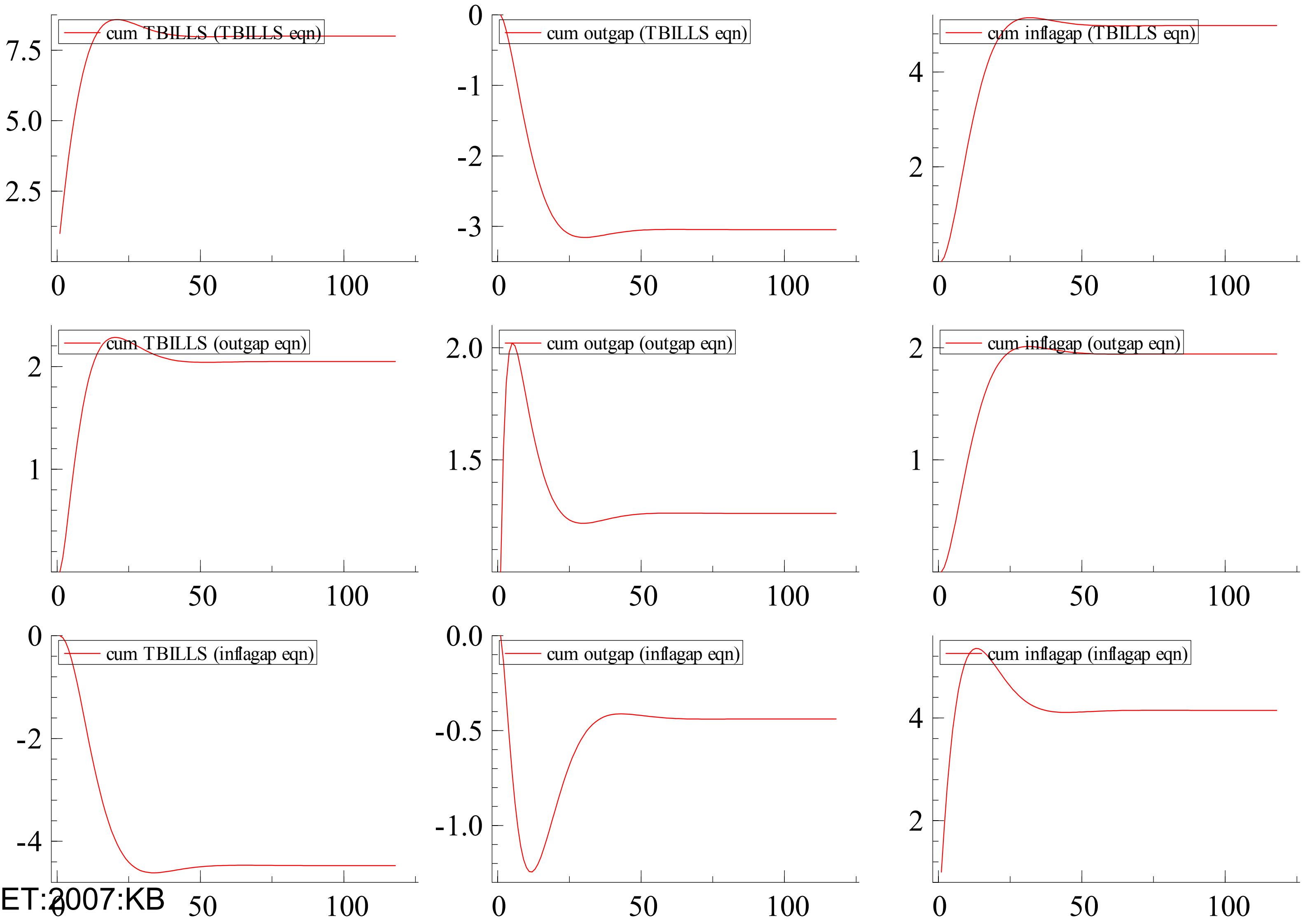
Prediction from Cointegrating VAR Model



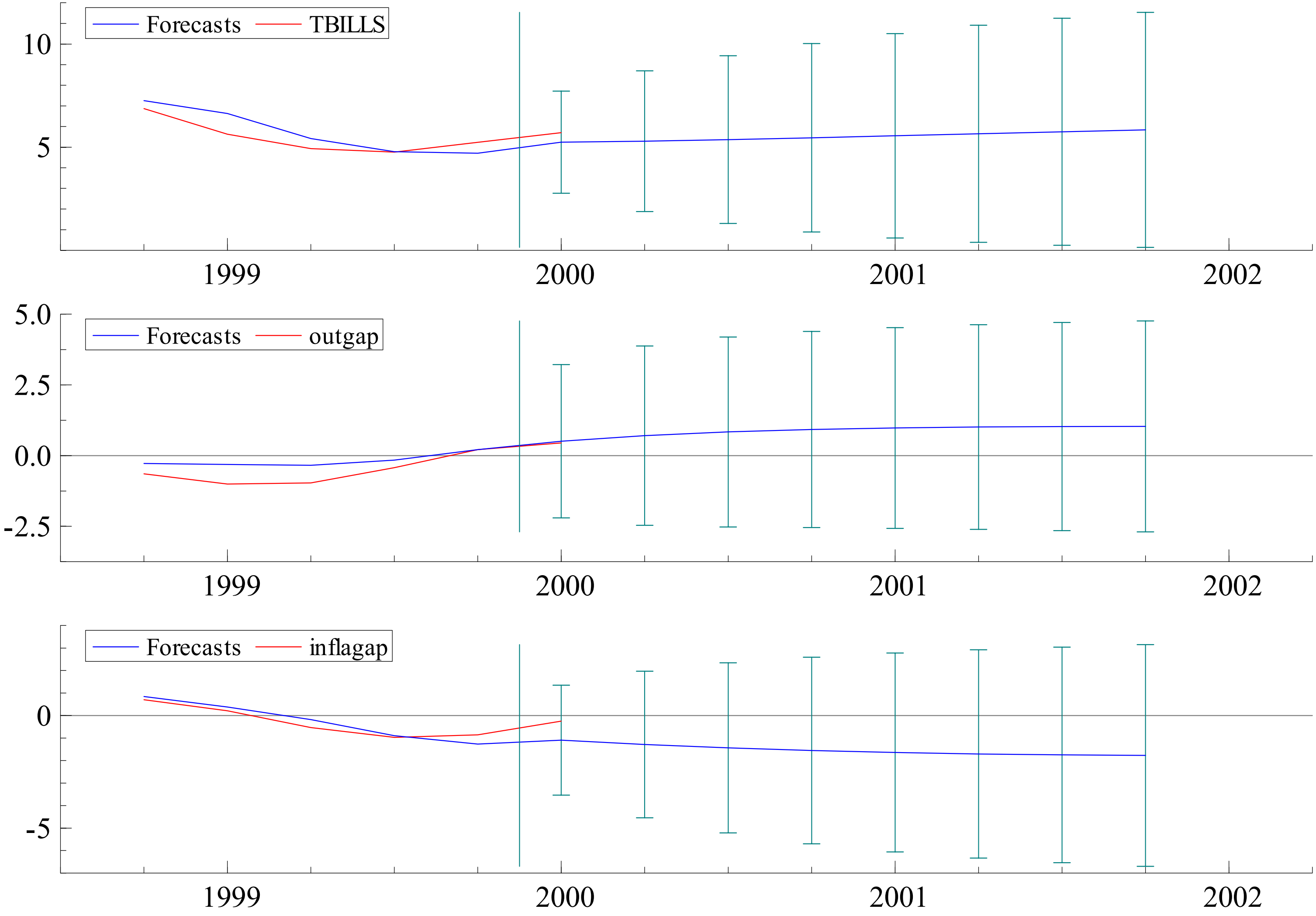
Impulse Response Analysis



Cumulative Impulse Response Analysis



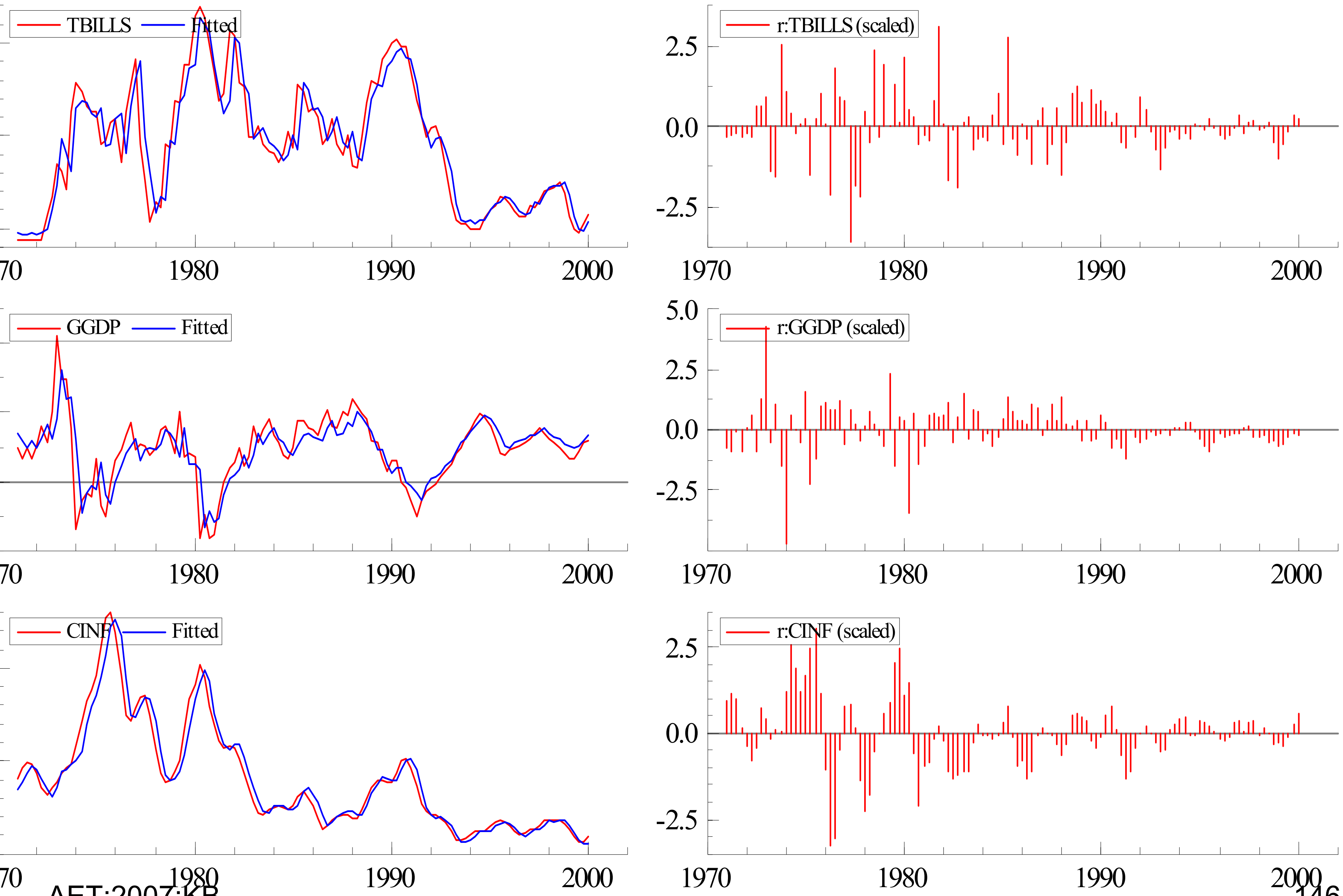
Dynamic Forecasts of Interest rate, output-gap and inflation gap



Estimates of the Interest Rule model by FIML (using uk_1.xls by GiveWin PcGive): The estimation sample is: 1971 (1) to 2000 (1)

```
                     Equation for: TBILLS
                 Coefficient  Std.Error  t-value  t-prob
TBILLS_1            0.938558    0.04090     22.9   0.000
GGDP_1              0.155536    0.05605     2.77   0.006
CINF_1             0.0321303    0.02456     1.31   0.193
Constant      U   -0.0178904     0.4473  -0.0400   0.968
                       sigma = 1.2475
                     Equation for: GGDP
                 Coefficient  Std.Error  t-value  t-prob
TBILLS_1           -0.124014    0.04638    -2.67   0.009
GGDP_1              0.689260    0.06356     10.8   0.000
CINF_1            -0.0269151    0.02785   -0.966   0.336
Constant      U      2.09479     0.5073     4.13   0.000
                       sigma = 1.41472
                     Equation for: CINF
                 Coefficient  Std.Error  t-value  t-prob
TBILLS_1           0.0793641    0.04335     1.83   0.070
GGDP_1              0.133732    0.05941     2.25   0.026
CINF_1              0.979589    0.02603     37.6   0.000
Constant      U    -0.936789     0.4741    -1.98   0.051
                       sigma = 1.32217
log-likelihood    -586.063862  -T/2log|Omega|    -88.0164366
no. of observations       117  no. of parameters          12
```

Actual and Fitted Values for Interest Rate, Output and Inflation for U



Test of Interest Determination Rule for Five Major Economies

	Output gap	Inflation gap	Constant	R^2
France	-6.641 (-14.778)	0.670 (1.341)	5.900 (1.341)	0.766
Germany	-10.732 (-15.187)	4.335 (4.953)	5.339 (11.898)	0.752
Japan	-6.775 (-6.554)	-1.794 (-7.061)	-1.312 (-3.487)	0.641
UK	-2.941 (-5.885)	1.006 (2.848)	7.416 (10.203)	0.574
USA	-1.794 (-7.061)	0.360 (0.408)	5.337 (18.955)	0.696

Estimates of the Simultaneous Interest Rule model for UK and Four Major Industrial Economies

rus = + 0.2507*infus + 0.04032*infjp - 0.1736*infgr - 0.1627*inffr
(SE) (0.103) (0.102) (0.0767) (0.0489)
- 0.1207*infuk - 0.1136*gus - 0.1429*gjp - 0.004963*ggr
(0.0421) (0.0518) (0.049) (0.0831)
- 0.07644*gfr - 0.08878*guk + 3.946
(0.0798) (0.065) (0.316)

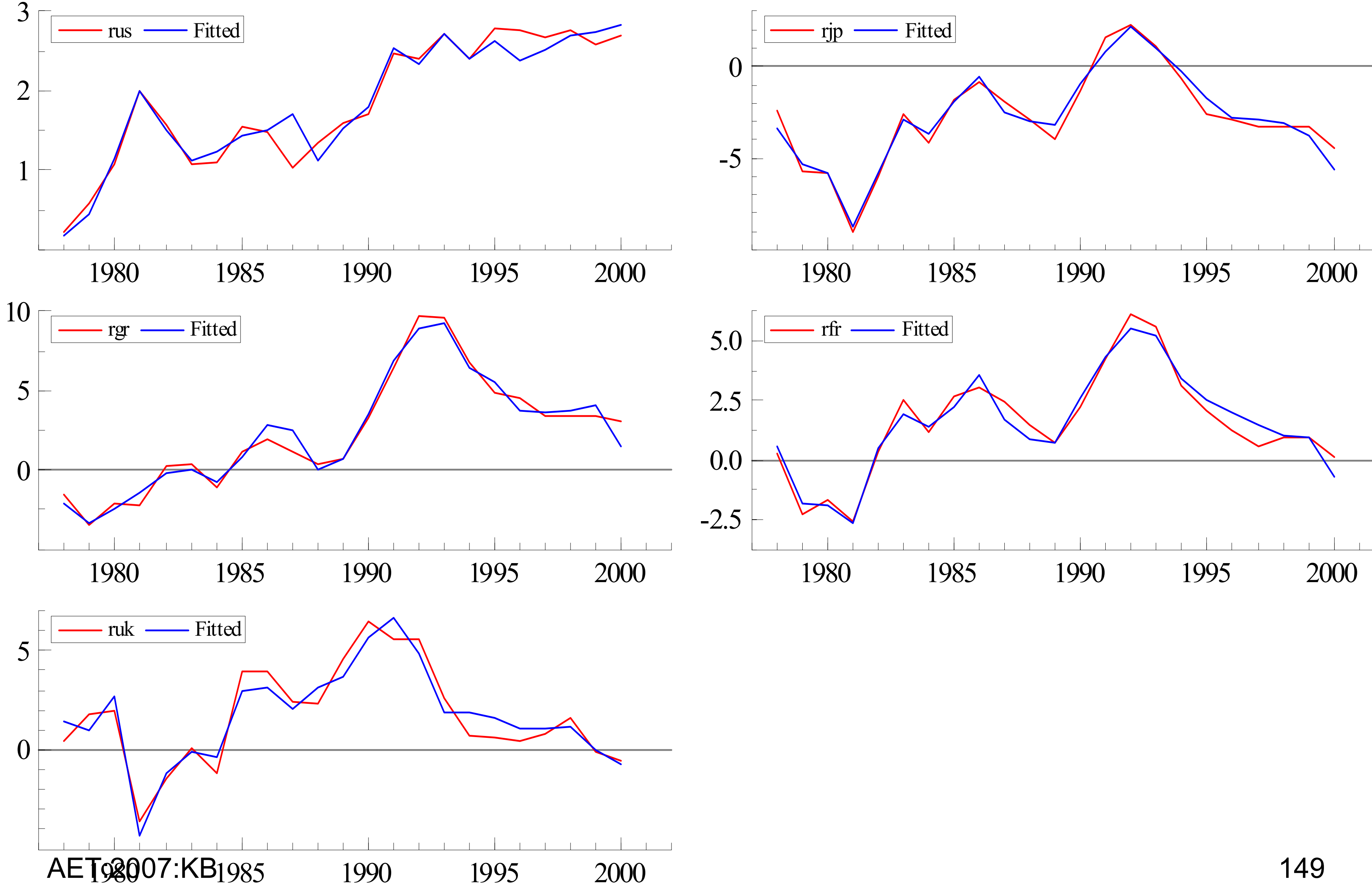
rjp = - 0.9612*infus + 0.2619*infjp + 1.541*infgr - 0.3411*inffr
(SE) (0.278) (0.276) (0.207) (0.132)
+ 0.1209*infuk - 0.01156*gus + 0.1172*gjp + 0.2549*ggr
(0.114) (0.14) (0.132) (0.224)
- 0.5457*gfr + 0.468*guk - 3.367
(0.215) (0.175) (0.85)

rgr = + 0.2813*infus - 0.05551*infjp + 1.517*infgr - 0.8322*inffr
(SE) (0.34) (0.337) (0.252) (0.161)
- 0.3356*infuk - 0.1166*gus - 0.2131*gjp - 0.1541*ggr
(0.139) (0.17) (0.161) (0.274)
- 0.4911*gfr + 0.1629*guk + 5.319
(0.263) (0.214) (1.04)

rfr = - 0.8201*infus + 0.1866*infjp + 1.228*infgr - 0.1101*inffr
(SE) (0.261) (0.258) (0.194) (0.123)
- 0.09884*infuk + 0.009484*gus + 0.2115*gjp - 0.06226*ggr
(0.106) (0.131) (0.124) (0.21)
- 0.3603*gfr + 0.2028*guk + 2.111
(0.201) (0.164) (0.796)

ruk = - 1.394*infus + 0.6449*infjp + 0.8495*infgr - 0.3876*inffr
(SE) (0.405) (0.401) (0.301) (0.192)
+ 0.501*infuk - 0.4017*gus + 0.3004*gjp + 0.4134*ggr - 0.1568*gfr
(0.165) (0.203) (0.192) (0.326) (0.313)
+ 0.2822*guk + 1.89

Actual and predicted values of interest Rate for UK and Four Major Industrial Country



Main Points of this Paper

Origin of the interest determination rule in the literature of the natural rate of unemployment hypothesis, dynamic time inconsistency and credibility and policy co-ordination at the national and international level.

The prominence of the central bank independence and rule based monetary policy in 1990s.

A simple model for interest determination and found its analytical solution using the second order difference technique.

Estimation of the model using quarterly series on treasury bills rate, growth rate of output and inflation rates for the UK and annual time series for UK and four major economies.

An evidence for such a interest rule and the interest changes to have significant impacts on output, unemployment and inflation in our estimation.

The simultaneous equation technique better than the single equation technique.

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