Basics of Game Theory

Elements of Games
Players and Strategies
Solutions:
Dominant Strategy
Nash Equilibrium
Mixed Strategy
Dynamic Game: Backward Induction
GAMES with Perfect and Imperfect Information

http://cepa.newschool.edu/het/schools/game.htm

Elements of a Game

- **Rational Players**
- **Strategic Choices**
- **Payoff matrix**

Players like to Maximise their Own pay-off.

A

B

	Strategy 1	**Strategy 2**
Strategy 1	$\left(\prod_{1,1}^{R}, \prod_{1,1}^{C}\right)$	$\left(\prod_{1,2}^{R}, \prod_{1,2}^{C}\right)$
Strategy 2	$\left(\prod_{2,1}^{R}, \prod_{2,1}^{C}\right)$	$\left(\prod_{2,2}^{R}, \prod_{2,2}^{C}\right)$

$\prod_{1,1}^{R}$ payoff to row player when both row and column play strategy 1.

Types of Games

- **Cooperative Games:**
  two or many players; oligopoly/competition
- **Non-cooperative Games: two or many players**
  between opposing political parties, countries
- **Single period of multiple period: static and dynamic**
- **Full information or incomplete information**
- **Firms and consumers; government and public;**

Among individuals, clubs, parties; nations and regions
Competition and Collusion

		A Strategy 1	**A** Strategy 2
B	**Strategy 1**	(10, −10)	(−10, 10)
B	**Strategy 2**	(−10, 10)	(10, −10)

$$B\begin{matrix}\text{Strategy 1}\\\text{Strategy 2}\end{matrix}\begin{bmatrix}(10,-10) & (-10,10)\\(-10,10) & (10,-10)\end{bmatrix}$$

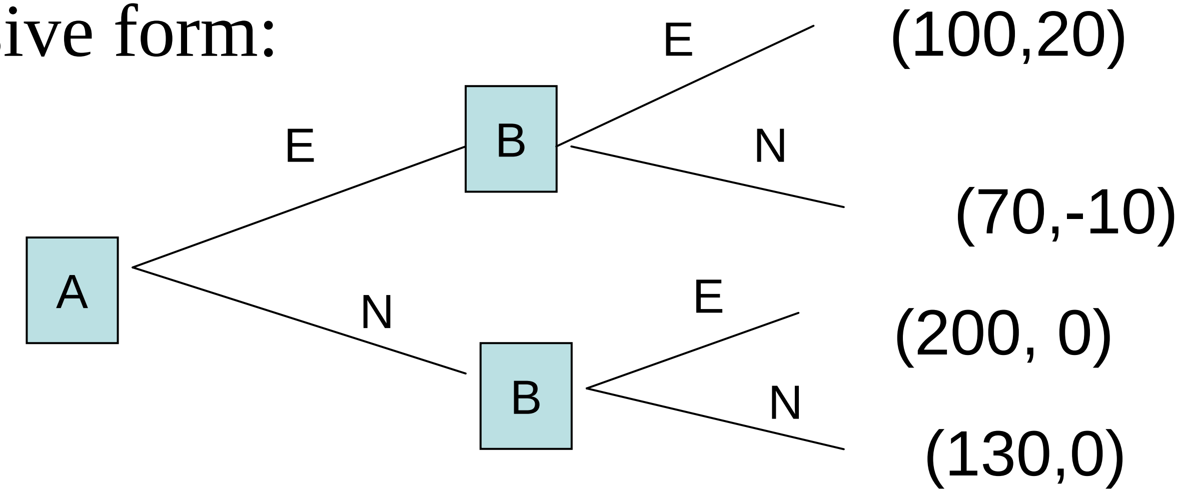
Example of a **zero sum game**: one's gain = loss of another Sports; Market shares.

Normal and Extensive Form Representation of a Game

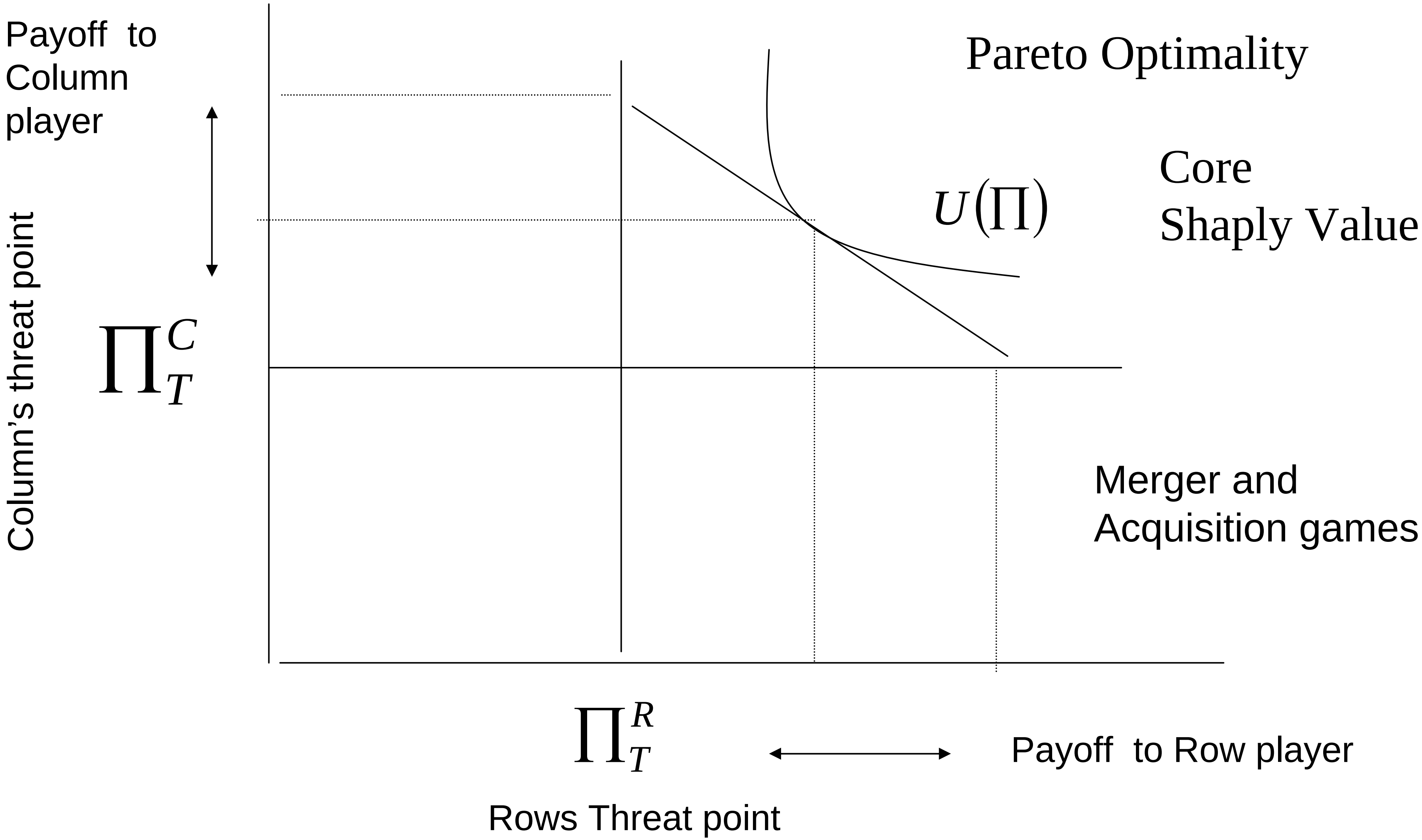
Normal form:

		A	
		Enter	Don’t Enter
B	Enter	(100,20)	(200,0)
	Don’t Enter	(70,–10)	(130,0)

Extensive form:



Gains from Co-operative Solutions and Room for a Bargain

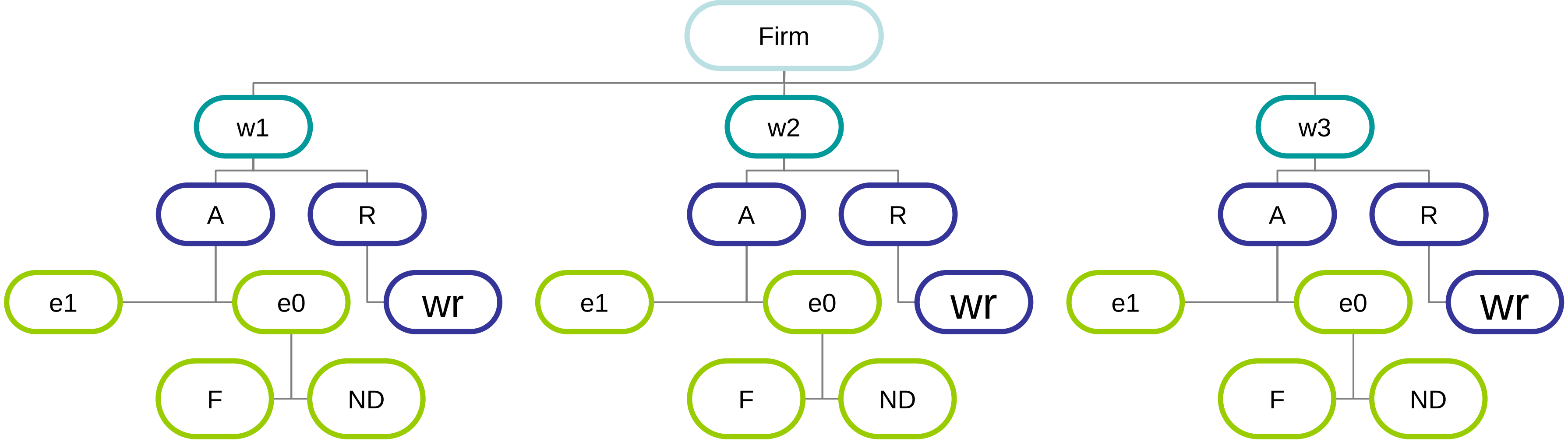


With many players there can be several coalitions.

A Dynamic GAME

Efficiency Wage Rate: Principal Agent Problem

Workers try to maximize expected Utility



First stage: Firm offers three different wage rates w1, w2 and W2

Second stage: Workers accept or reject the offer. If they reject they get wr

Third stage: either they can put hard effort (e1) or Shirk (e0), Gets efficiency wage in effort e1 but can be caught and punished in shirking.

If caught is fired (F) and gets doles (wr)

If not detected (ND) gets the efficiency wage rate as in e1

Question: How do workers and firms play this wage bargaining game?

There is an issue of Reputation and credibility over time.

Games that can be solved by a Dominant Strategy

B \ A	Adv	DntAdv
Adv	(10,5)	(15,0)
DntAdv	(6,8)	(10,2)

Adv better for B

Adv better for A

Both advertise
B get 10
A get 5

B \ A	Adv	DntAdv
Adv	(10,5)	(15,0)
DntAdv	(6,8)	(20,2)

Firm B does not have a dominant Strategy

Adv is dominant St. for Firm A

Game with A Nash Equilibrium

Prisoners dilemma

		A: Confess	A: Do not confess
B	Confess	(–5,–5)	(–1,–10)
	Do not confess	(–10,–1)	(–2,–2)

$$\begin{bmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{bmatrix}$$

$$\begin{bmatrix} (-\underline{5},-\underline{5}) & (-\underline{1},-10) \\ (-10,-\underline{1}) & (-2,-2) \end{bmatrix}$$

Nash Solution: (-5,-5)

Cooperation was better: (-2,-2)

Cooperation is better but each think that other player will cheat and therefore they Don’t cooperate, therefore stay longer in jail.

Solution by Random Mixed Strategy

All games do not have equilibrium in pure strategy.
Example: Game of Matching Pennies

		A: S1	A: S2
B	S 1	$(1,-1)$	$(-1,1)$
	S2	$(-1,1)$	$(1,-1)$

There is no solution in pure strategy.

At least on Nash Equilibrium in the Mix strategy.

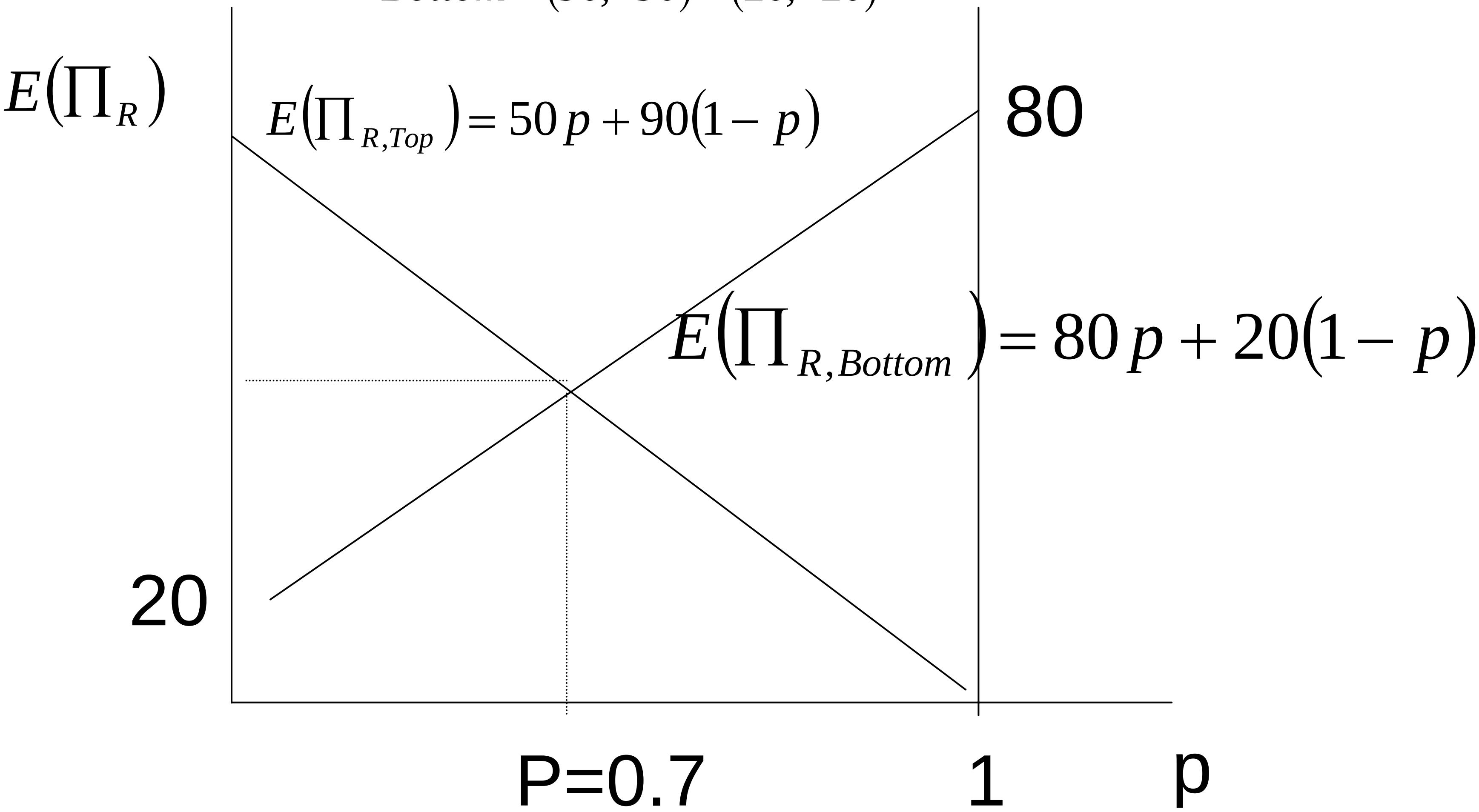
Flip the coin to randomise the chosen strategies.

If played s1 half of the time optimal payoff is zero to both players.

Probability of playing S1 or S2 is 0.5.

Finding the Mixed Strategy in a Competitive Game

	left	*right*
top	$(50,-50)$	$(80,-80)$
Bottom	$(90,-90)$	$(20,-20)$



$$50p + 90(1-p) = 80p + 20(1-p) \qquad 100p = 70$$

Does subsidy to the Airbus by EU countries deter Boeing from Producing a New Aircraft?

GAME 1

		Airbus S1	Airbus S2
Boeing	S 1	$(-10,-10)$	$(100,0)$
	S2	$(0,100)$	$(0,0)$

GAME 2

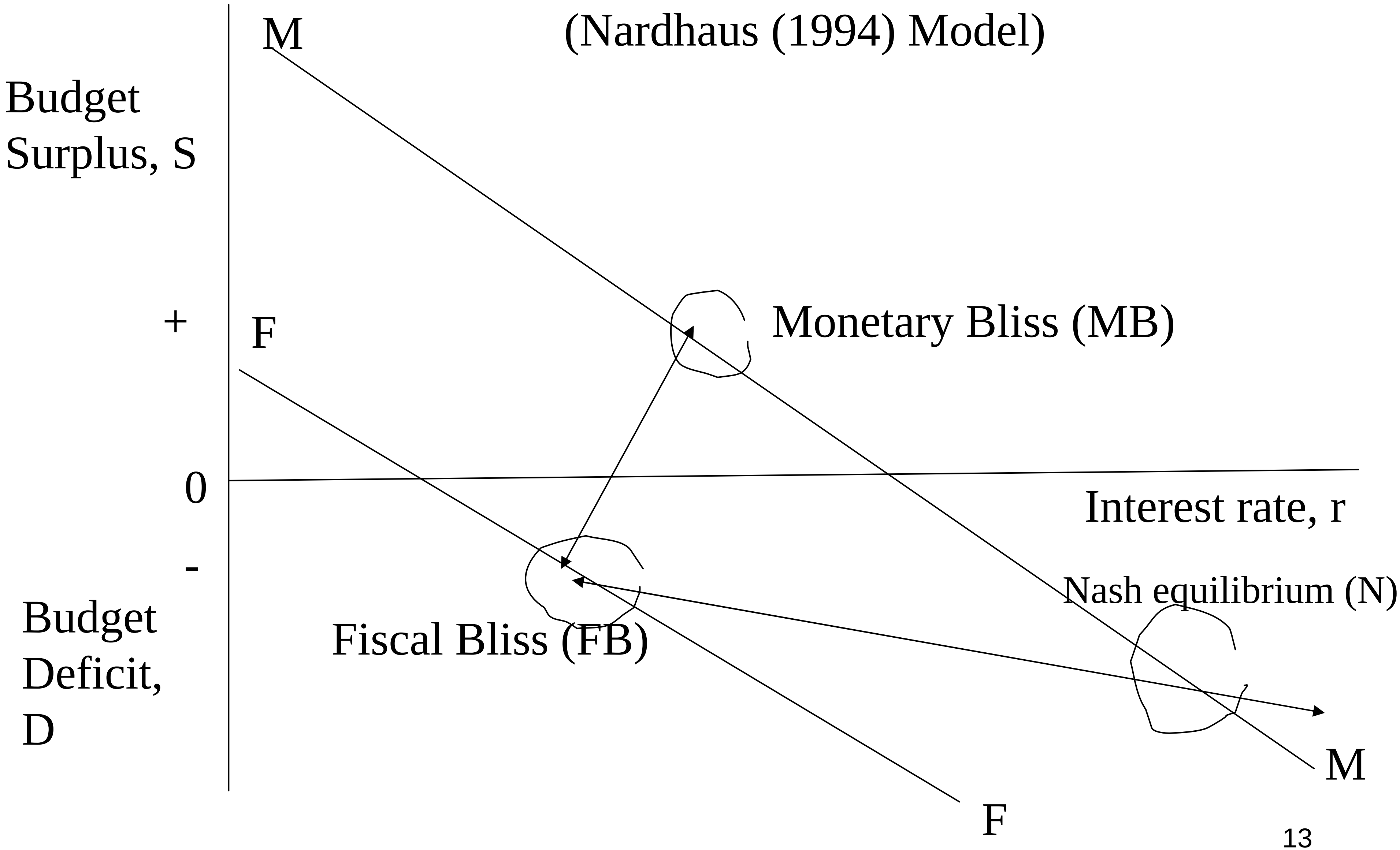
		Airbus S1	Airbus S2
Boeing	S 1	$(-10,10)$	$(100,0)$
	S2	$(0,120)$	$(0,0)$

Entry Deterrence by an Incumbent to a Potential Entrant

		Potential Entrant: Enter	Potential Entrant: Don't
Incumbent	Enter	$(-10,-10)$	$(100,0)$
	Don't	$(0,100)$	$(0,0)$

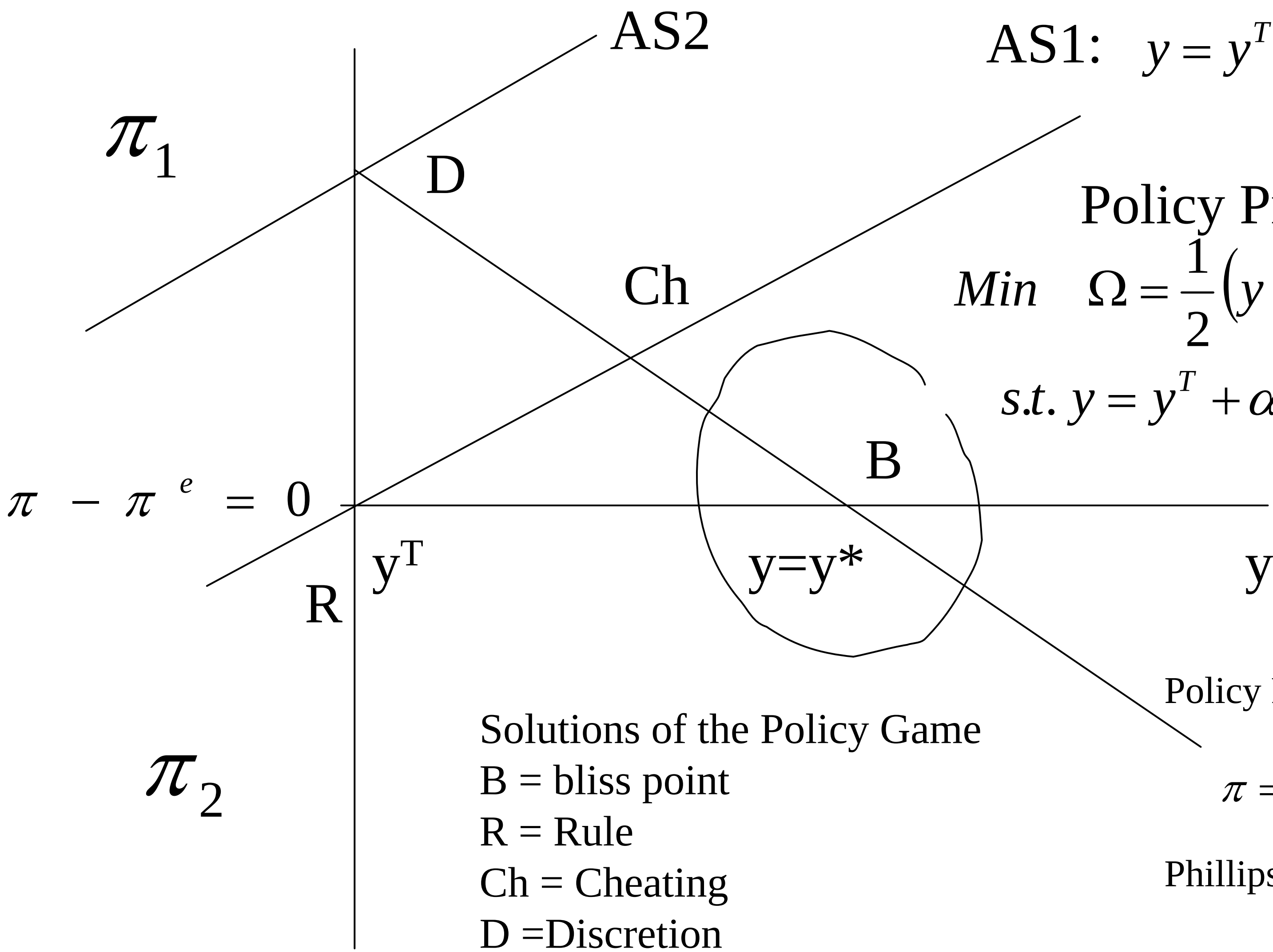
		Potential Entrant: Enter	Potential Entrant: Don't
Incumbent	Enter	$(-10,10)$	$(100,0)$
	Don't	$(0,120)$	$(0,0)$

Fiscal and Monetary Policy Game in a Diagram



Policy Reaction Function and Lucas Supply Curve

AS1: $y = y^T + \alpha(\pi - \pi^e) + \varepsilon$



Solutions of the Policy Game

B = bliss point

R = Rule

Ch = Cheating

D =Discretion

Policy Problem

$$Min \quad \Omega = \frac{1}{2}(y - y^*)^2 + \frac{\beta}{2}\pi^2$$

$$s.t.\ y = y^T + \alpha(\pi - \pi^e) + \varepsilon$$

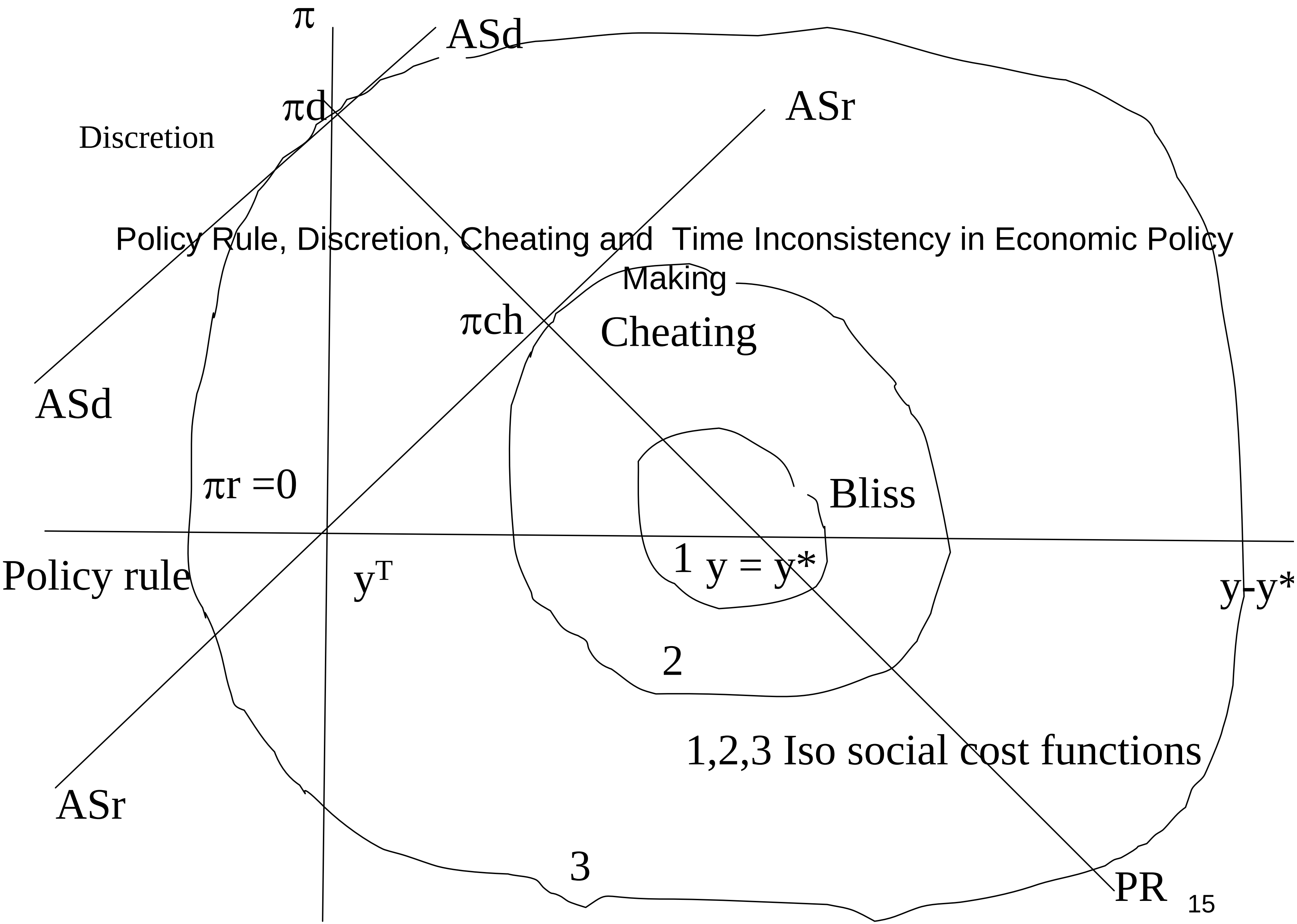
Policy Reaction Function

$$\pi = \frac{-\alpha(y - y^*)}{\beta}$$

Phillips' Curve

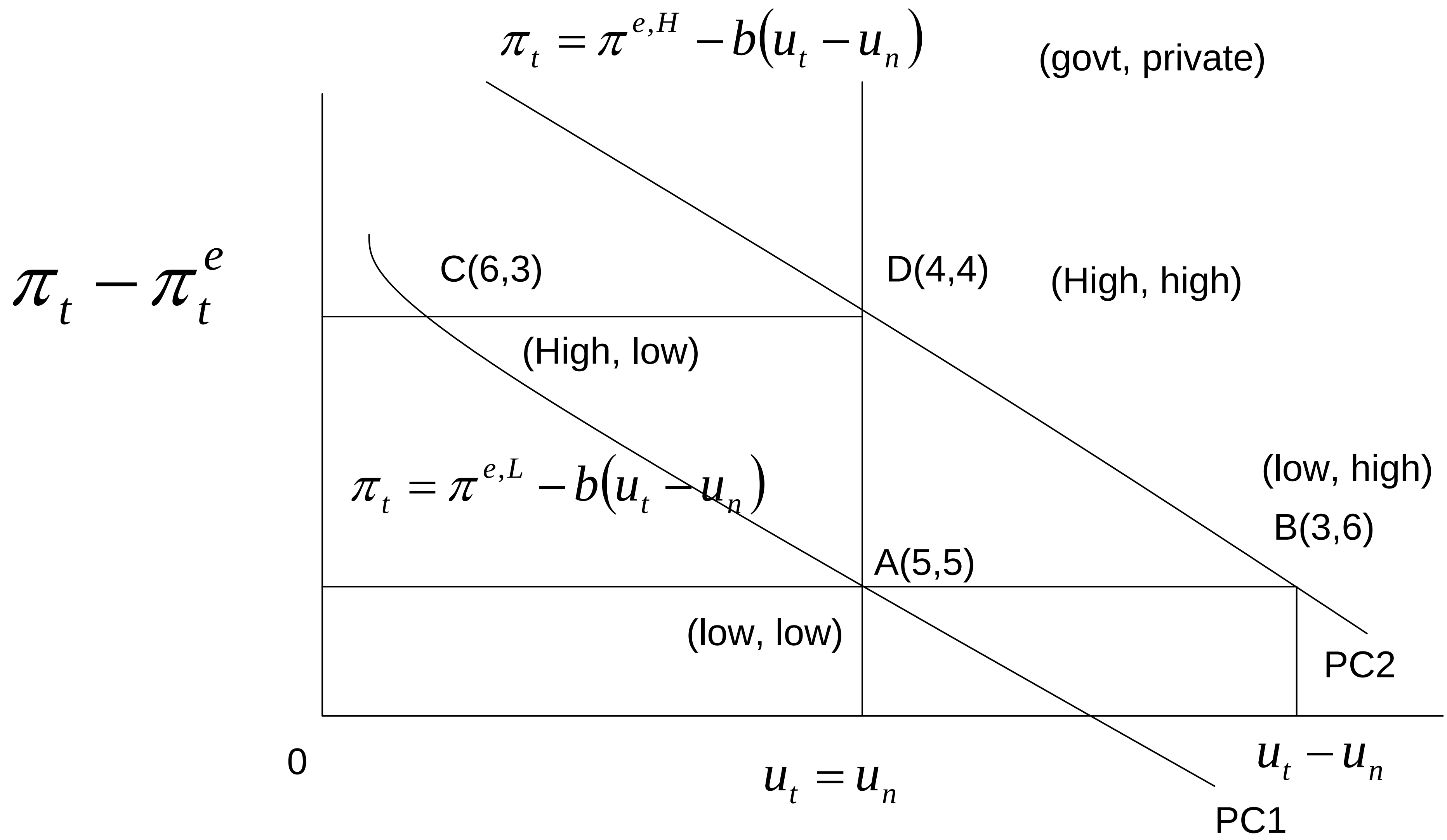
Higher rate of inflation or deflation or deviation of output from the trend are undesirable

Policy Rule, Discretion, Cheating and Time Inconsistency in Economic Policy Making



Kydland and Prescott (1977)

Inflation-Unemployment Game Between Private and Public Sectors in a diagram



First element of choice is by the government government.

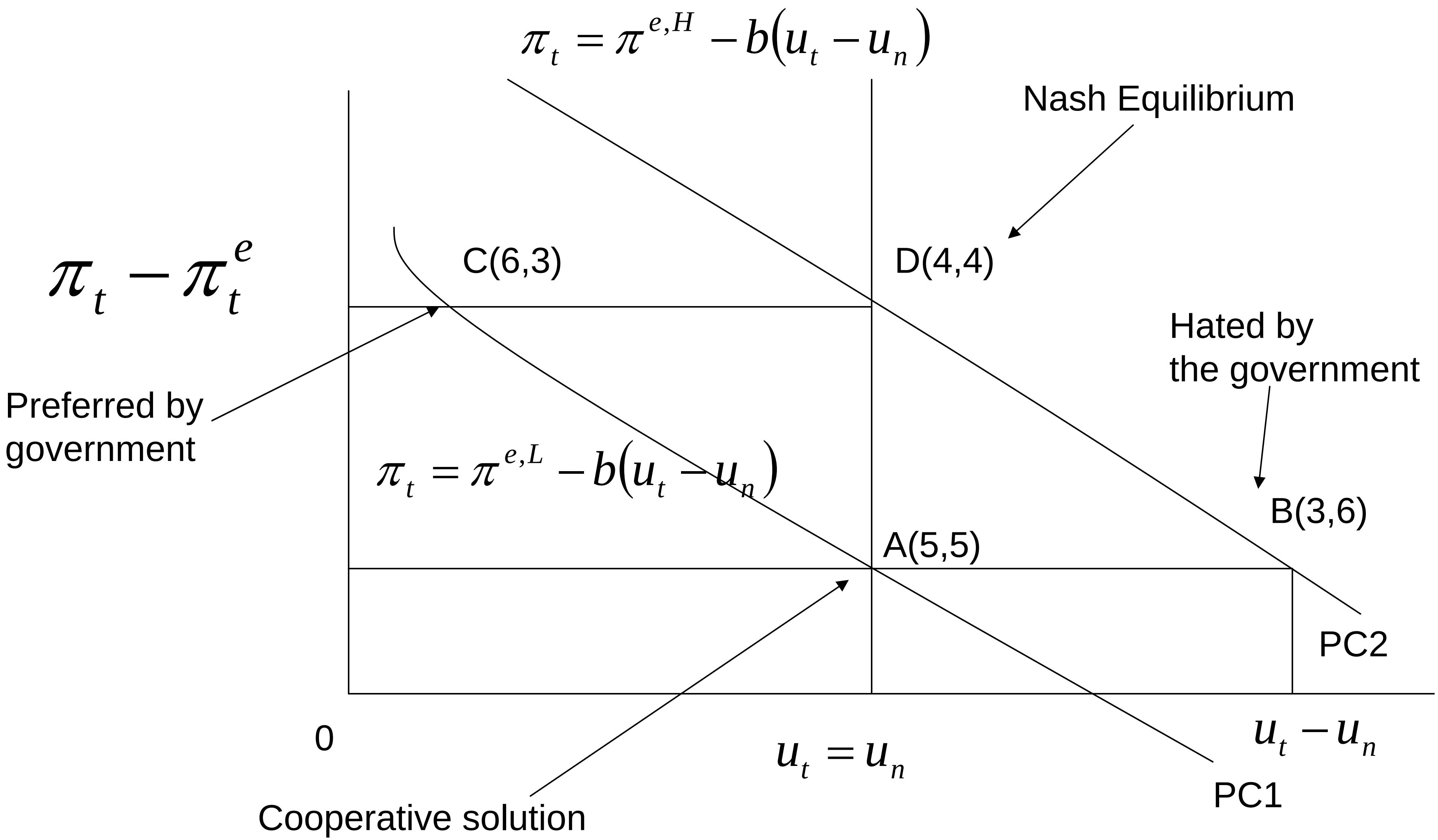
Pay-Off Matrix for Inflation-unemployment Policy Game

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \Pr ivate \quad Sector$$

$$Government \quad Sector \quad \begin{bmatrix} & H & L \\ H & 4,4 & 6,3 \\ L & 3,6 & 5,5 \end{bmatrix}$$

Tasks
Find a Nash Equilibrium.
Solve the game using Backward induction if the government moves first.
Find the discount factor if the game is played infinite number of times.

Inflation-Unemployment Game Between Private and Public Sectors



First element represents payoff to the government.

Process of Finding a Nash Equilibrium Unemployment Inflation Game

Private sector

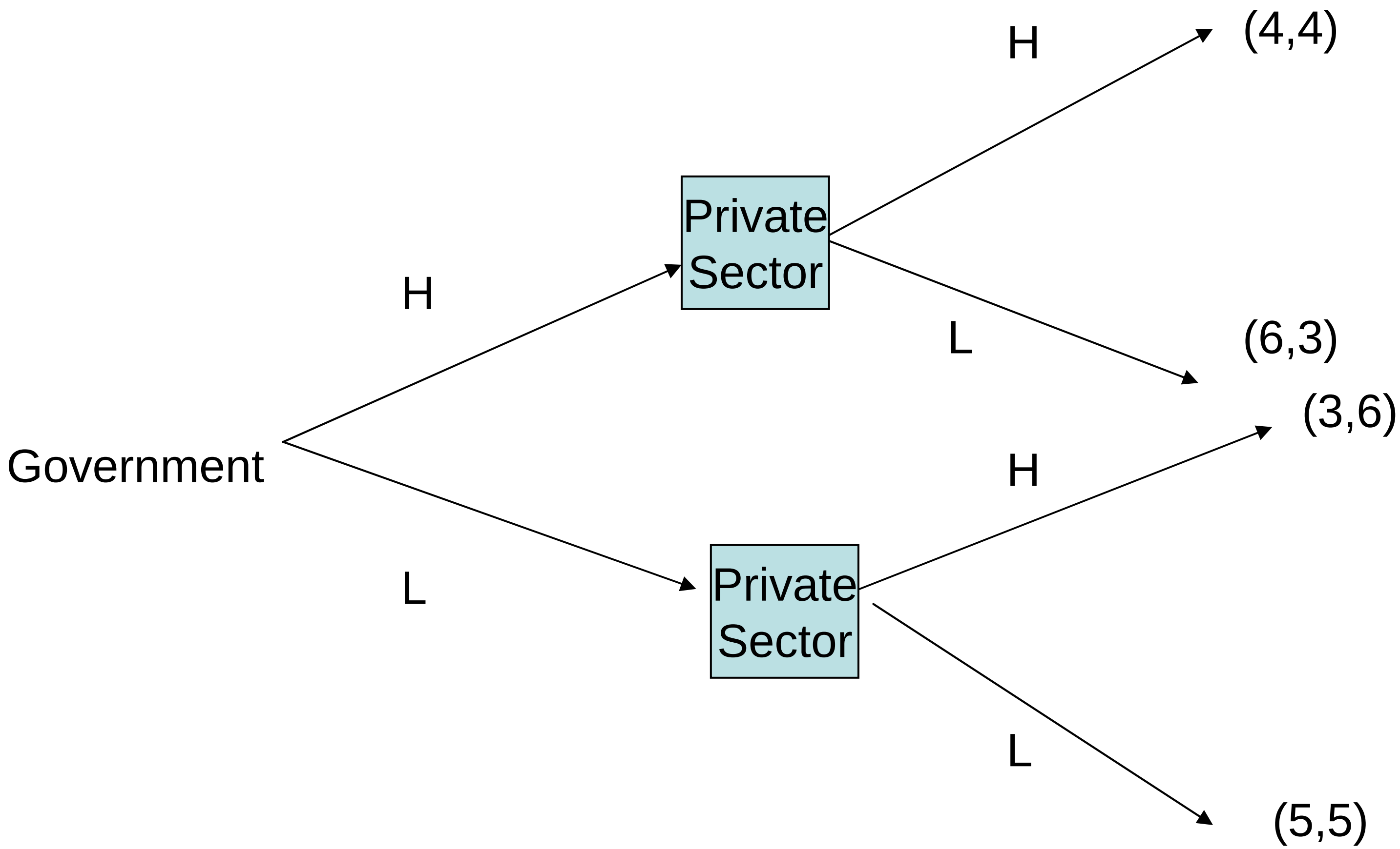
Government		H	L		
	H	4 4	6 3	H	Government Sector’ choice
	L	3 6	5 5	L	

Private sector

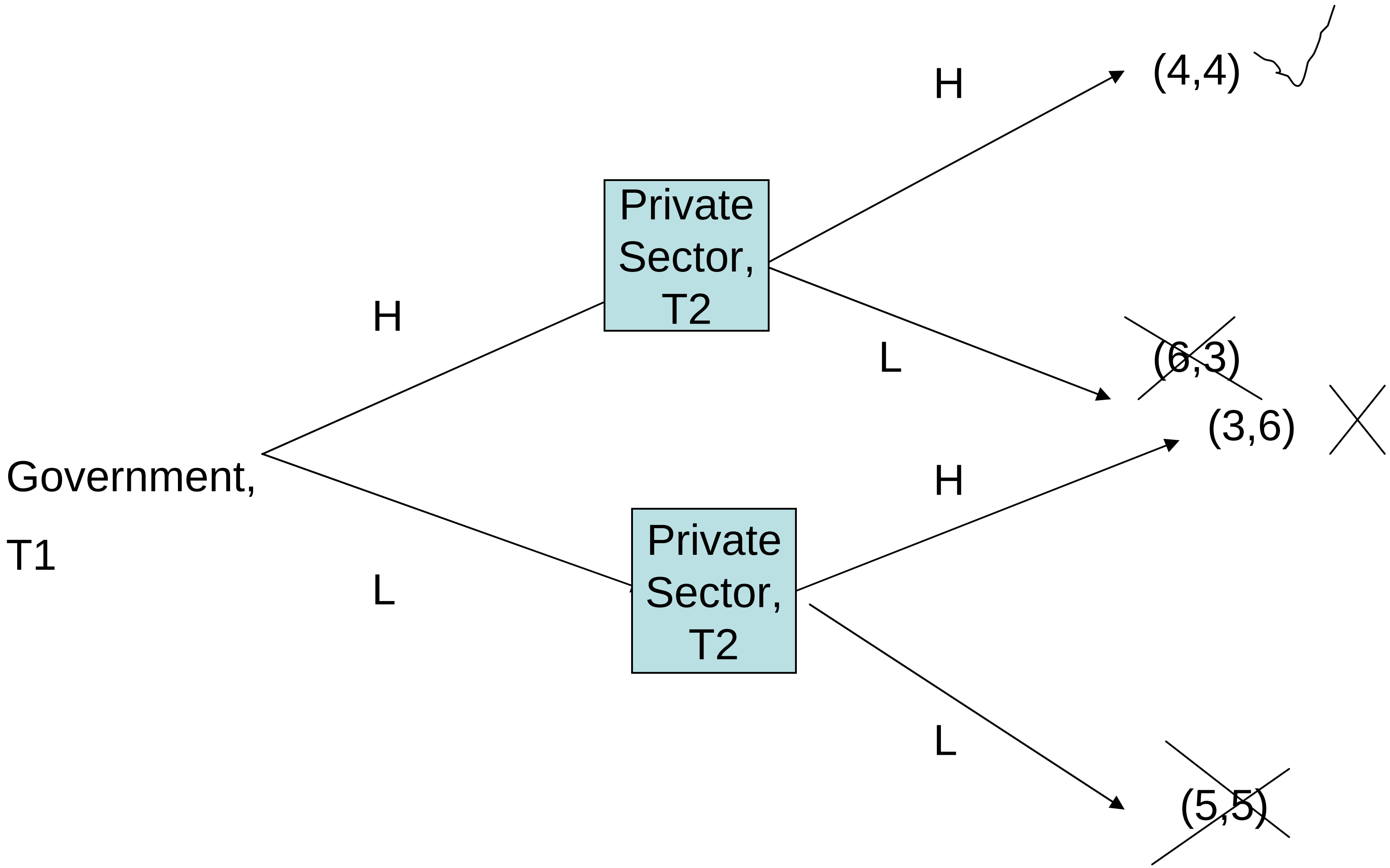
Government		H	L	
	H	4 4	6 3	Private Sector’ choice
	L	3 6	5 5	

	H	L	
H	4 4	6 3	Outcome of the Game
H	3 6	5 5	

Extensive Form of Inflation-Unemployment Game



Solution by Backward Induction
Dynamic Inflation-Unemployment Game



Credibility Problem, Cheating and Discount Factor of the Game

Both gain by playing (C,C)

But this solution is not credible.
There is incentive to deviate. Trigger Strategy

Game returns to Nash path in absence of credibility.

If the game is played infinite number of times the optimal discount value ff the game is calculated as

$$PV(C,C) = 5 + 5\delta + 5\delta^2 + 5\delta^3 + ... + 5\delta^n = \frac{5}{1-\delta}$$

$$\underset{Lim \quad n \to \infty}{PV(C,C)} = 5 + 5\delta + 5\delta^2 + 5\delta^3 + ... + 5\delta^n = \frac{5}{1-\delta}$$

$$PV(cheat) = 6 + 4\delta + 4\delta^2 + 4\delta^3 + ... + 4\delta^n$$

Solution for the Discount Factor of the Game

$$\underset{Lim\ n\to\infty}{PV(L,L)} = 5 + 5\delta + 5\delta^2 + 5\delta^3 + \ldots + 5\delta^n = \frac{5}{1-\delta}$$

$$PV(cheat) = 6 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots + 4\delta^n$$

$$\delta PV(cheat) = 6\delta + 4\delta^2 + 4\delta^3 + \ldots + 4\delta^{n+1}$$

$$(1-\delta)PV(cheat) = 6 - 6\delta + 4\delta \qquad \underset{\lim n\to\infty}{\delta^{n+1}} \approx 0$$

$$PV(cheat) = 6 + 4\frac{\delta}{(1-\delta)}$$

$$\frac{5}{1-\delta} = 6 + 4\frac{\delta}{(1-\delta)}$$

$$5 = 6(1-\delta) + 4\delta \qquad 6 - 5 = 2\delta \qquad \delta = \frac{1}{2}$$

Cooperation or non-Cooperation?

$$
\begin{matrix} .... & Advanced \quad Countries \\ Developing \quad Countries & \begin{bmatrix} & NC & C \\ NC & 4,4 & 6,3 \\ C & 3,6 & 5,5 \end{bmatrix} \end{matrix}
$$

Nash Solution is non-cooperation (NC,NC) =(4,4)

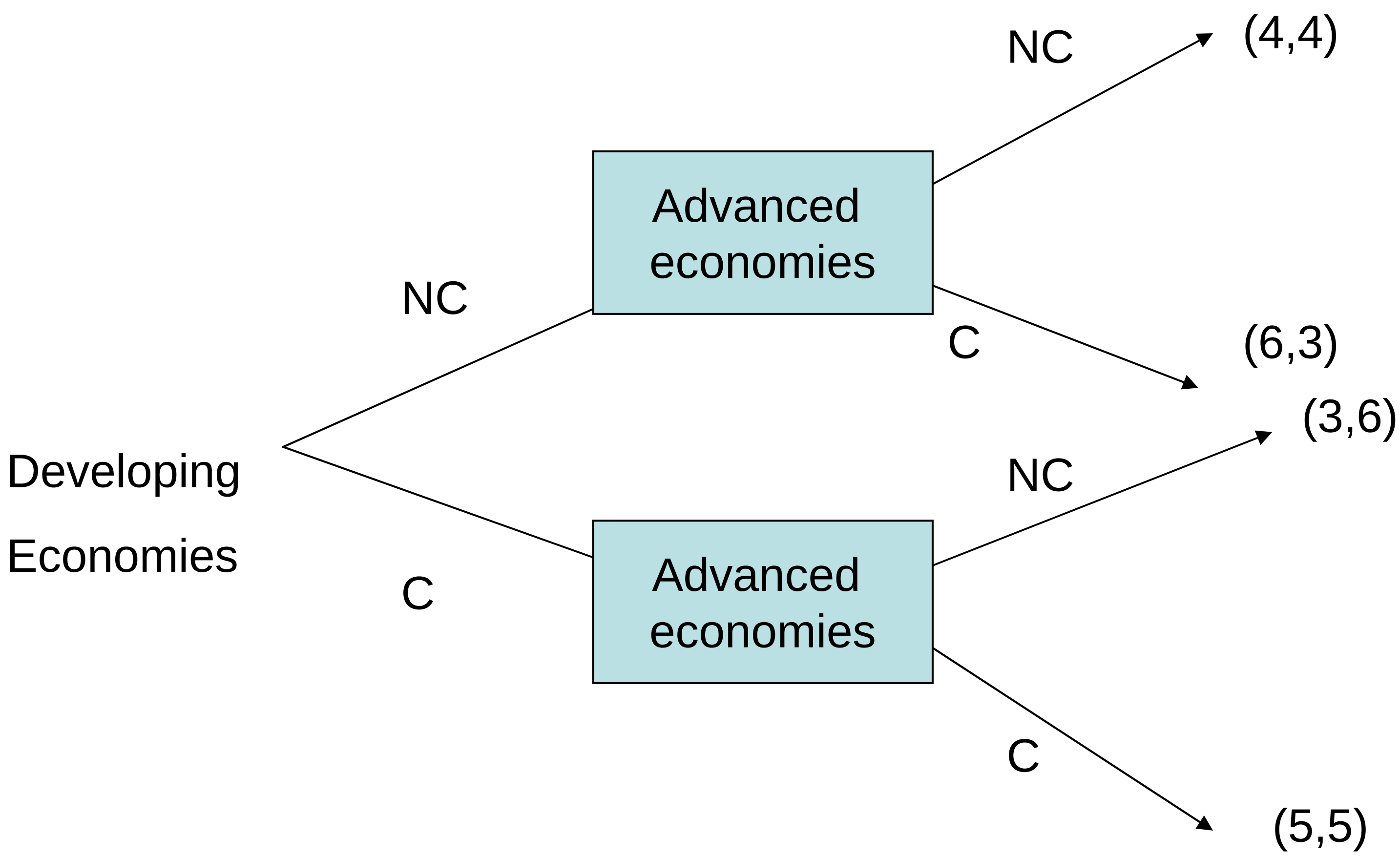
$$
\begin{matrix} .... & Advanced \quad Countries \\ Developing \quad Countries & \begin{bmatrix} & NC & C \\ NC & \underline{4}, \underline{4} & \underline{6},3 \\ C & 3, \underline{6} & 5,5 \end{bmatrix} \end{matrix}
$$

Cooperative Solution (C,C) =(5,5)

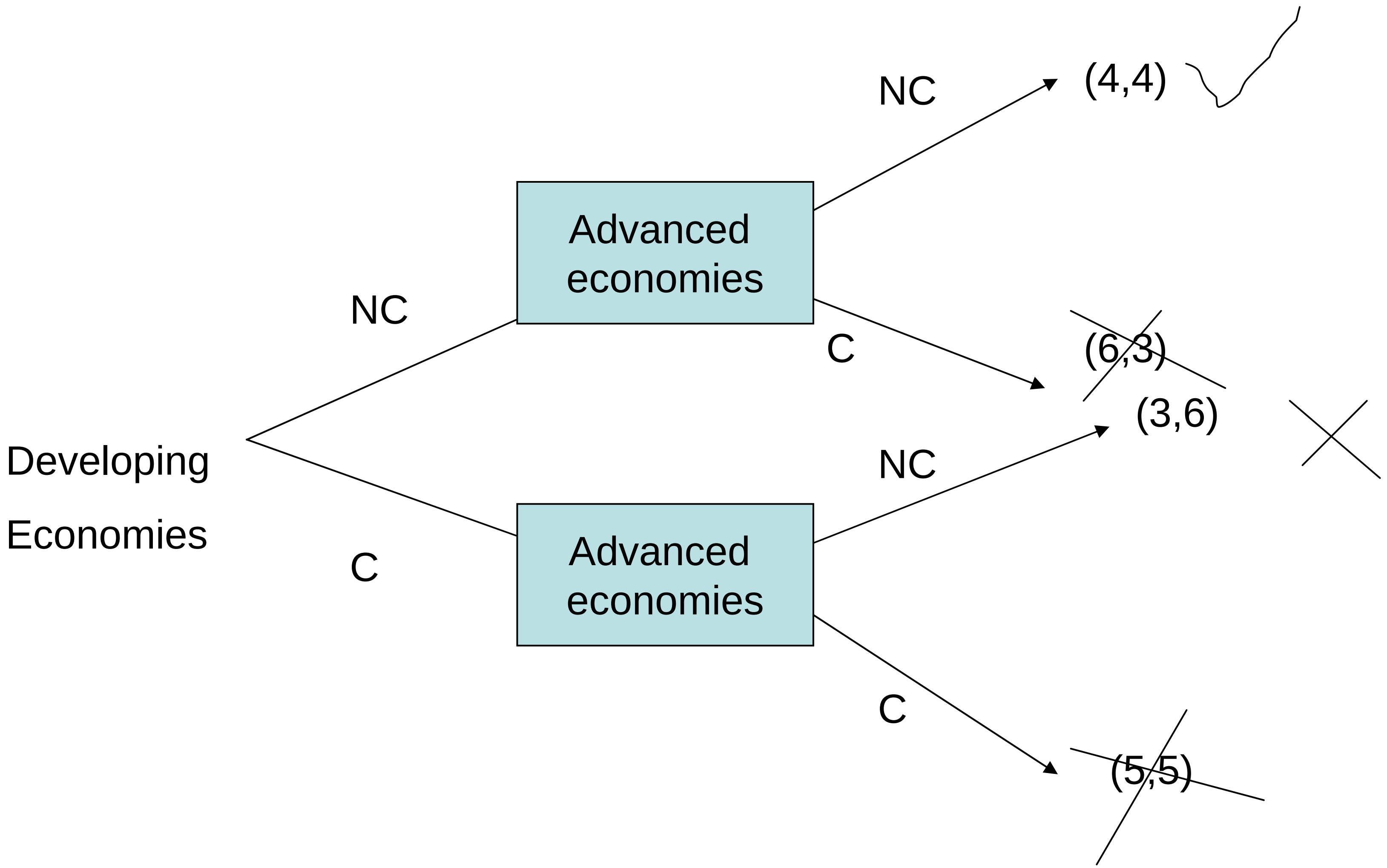
Cooperative solution Pareto dominated Non-cooperative solution.

Pareto efficiency: at least one party gains without hurting the other.

Extensive Form of International Cooperation Game



Dynamics of International Policy Cooperation Game: Solution by Backward Induction



Solution for the Discount Factor of the Game

$$\underset{Lim\ \ n\to\infty}{PV(C,C)} = 5 + 5\delta + 5\delta^2 + 5\delta^3 + ... + 5\delta^n = \frac{5}{1-\delta}$$

$$PV(cheat) = 6 + 4\delta + 4\delta^2 + 4\delta^3 + ... + 4\delta^n$$

$$\delta PV(cheat) = 6\delta + 4\delta^2 + 4\delta^3 + ... + 4\delta^{n+1}$$

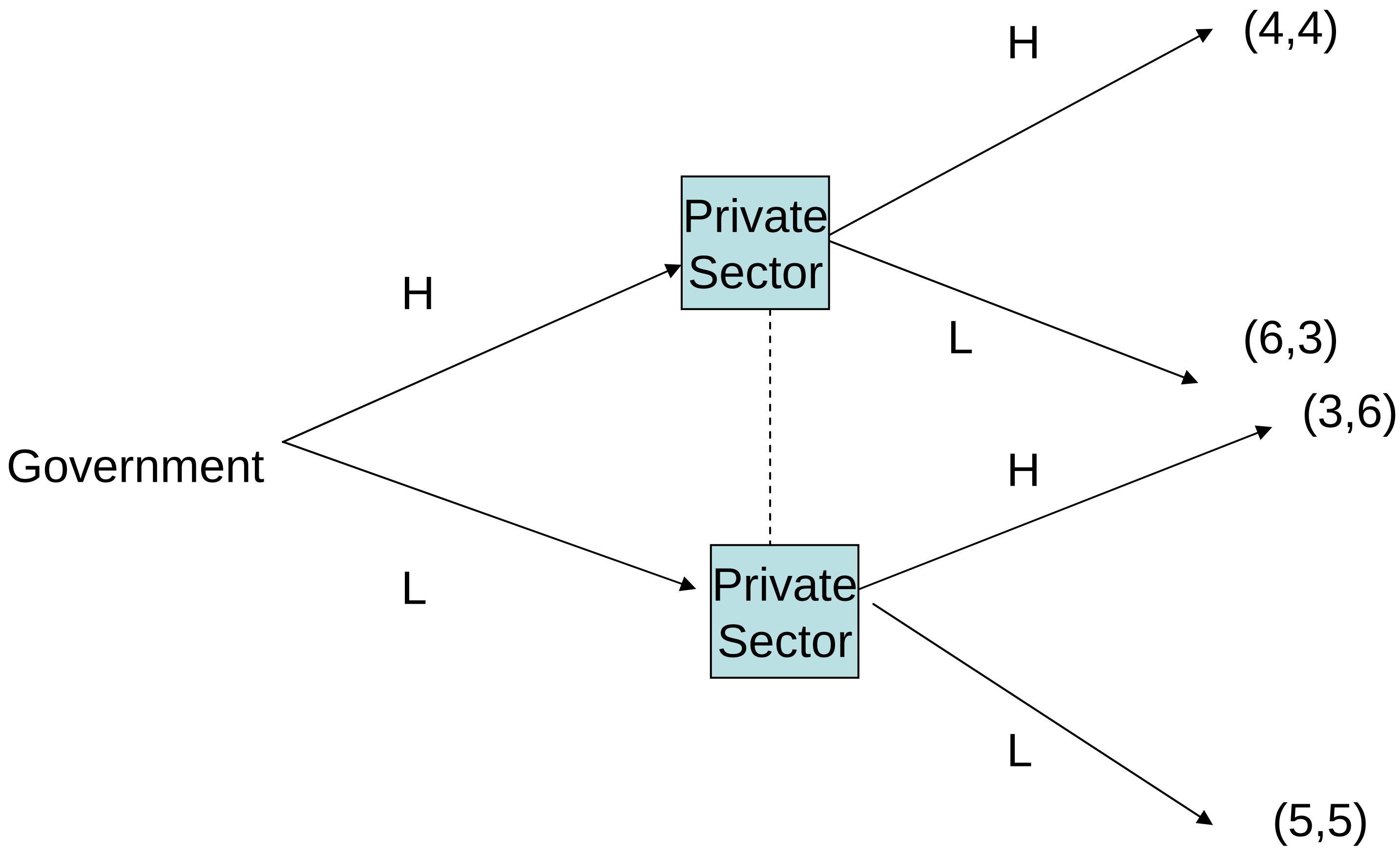
$$(1-\delta)PV(cheat) = 6 - 6\delta + 4\delta \qquad \underset{\lim n\to\infty}{\delta^{n+1}} \approx 0$$

$$PV(cheat) = 6 + 4\frac{\delta}{(1-\delta)}$$

$$\frac{5}{1-\delta} = 6 + 4\frac{\delta}{(1-\delta)}$$

$$5 = 6(1-\delta) + 4\delta \qquad 6 - 5 = 2\delta \qquad \delta = \frac{1}{2}$$

GAMES with Incomplete Information



Some Popular games

- Advertising
- Always the low price
- Bankrupty
- Blackjack
- Chicken
- Competition
- Coordination
- Devide a dollar
- Free riding
- Hawks versus Dove
- Lemongs
- Liar’s poker
- Majority rule
- Matching pennies
- Money back guarantee
- Pick the largest number
- Poker
- Principal agent
- Prisoner’s dilemma
- Roulette
- Solitaire
- Take it or leave it
- Tic-tac-toe
- This offer is good for limited time only
- Tragedy of commons

References and Readings

- Gardner Roy (2003) Games for Business and Economics, John Wilely
- Pindyck and Rubinfeld (2005) Microeconomics, Chapter 13 .
- Varian H (2003) Microeconomics, Chapter 28-29.
- Romp Graham (1997) Game Theory, Oxford University Press.

Bargaining and Cooperative Games

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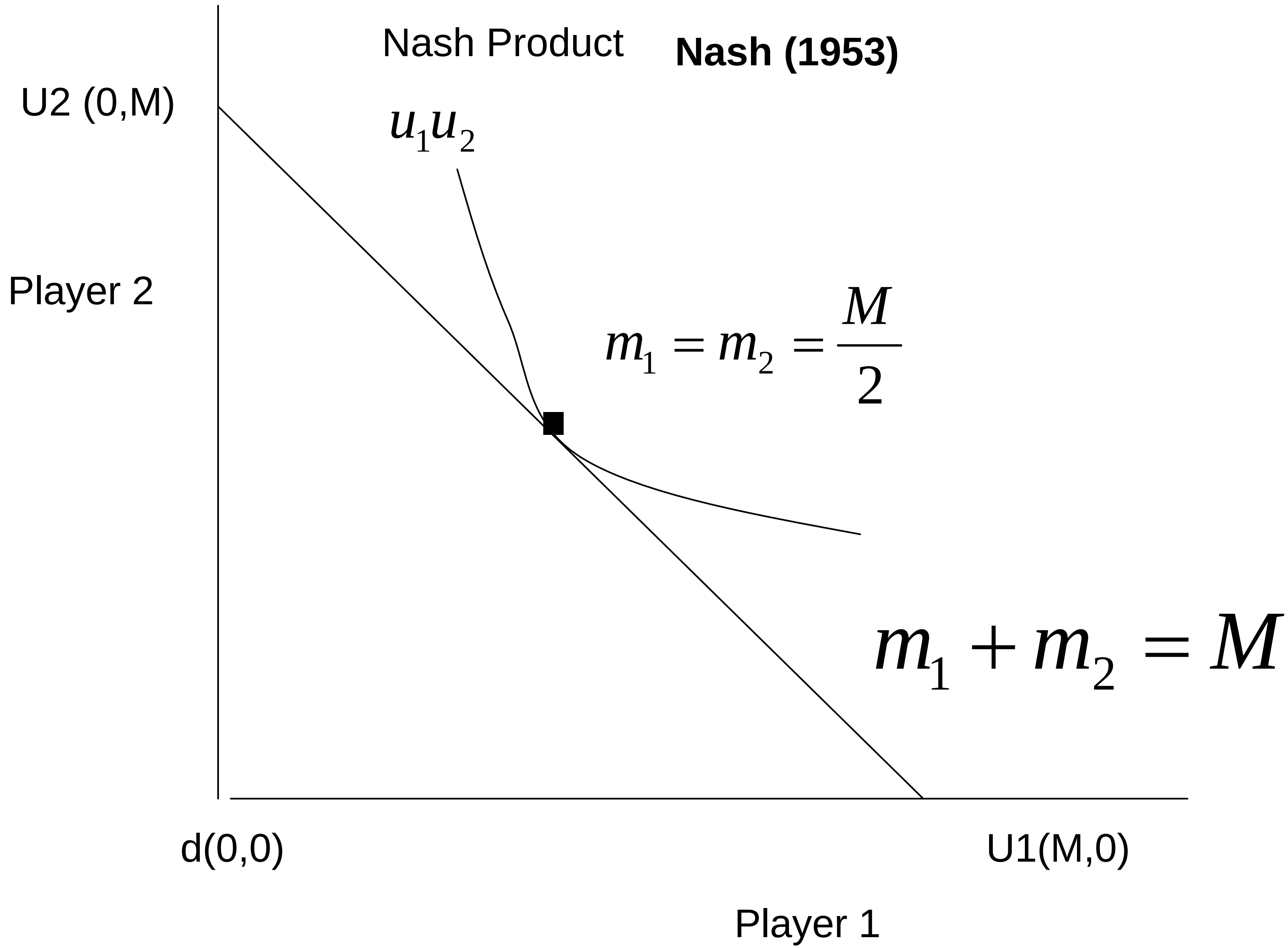
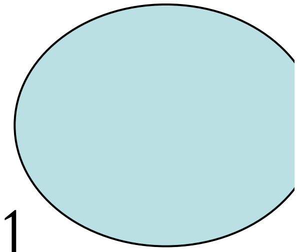
April 2007

Nash Product: Utility Possibility Frontier



Bargaining on Splitting a Pie

The very common example for bargaining game is splitting a pie between two individuals.



The sum of the shares of the pie claimed by both cannot exceed more than 1, otherwise each will get zero.

If we denote these shares by θ_i and θ_j then $\theta_i + \theta_j \leq 1$ is required for a meaningful solution of the game where each get $\pi_i \geq 0$ and $\pi_i \geq 0$ payoff. When $\theta_i + \theta_j > 1$ then $\pi_i = 0$ and $\pi_i = 0$.

Standard technique to solve this problem is to use the concept of Nash Product .

Nash Product Solution of Splitting a Pie Game

$$\max \quad U = (\theta_i - 0)(\theta_j - 0)$$

subject to

$\theta_i + \theta_j \leq 1$ or by non-satiation property $\theta_i + \theta_j = 1$

Using a Lagrangian function

$$L(\theta_i, \theta_j, \lambda) = (\theta_i - 0)(\theta_j - 0) + \lambda[1 - \theta_i - \theta_j]$$

First order conditions

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_i} = \theta_j - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_j} = \theta_i - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \lambda} = 1 - \theta_i - \theta_j = 0$$

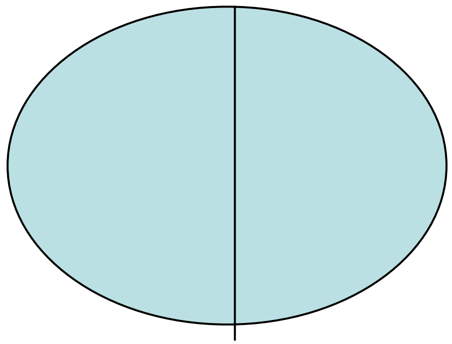
Nash Product Solution of Splitting a Pie Game

From the first two first order conditions

$\theta_j - \lambda = \theta_i - \lambda$ implies

$\theta_j = \theta_i$ and putting this into the third first order condition

$$\theta_j = \theta_i = \frac{1}{2}.$$



This is called focal point.

Thus Nash solution of this problem is to divide the pie symmetrically into two equal parts. Any other solution of this not stable.

Roy Gardner (2003) and Rasmusen (2007) have a number of interesting examples on bargaining game.

Nash Product Solution: A numerical Example

Suppose there is 1000 in the table to be split between two players.

What is the optimal solution from a symmetric bargaining game if the threat point is given by d(0,0)?

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda[1000 - u_1 - u_2]$$

Nash Product Solution A numerical Example

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 1000 - u_1 - u_2 = 0$$

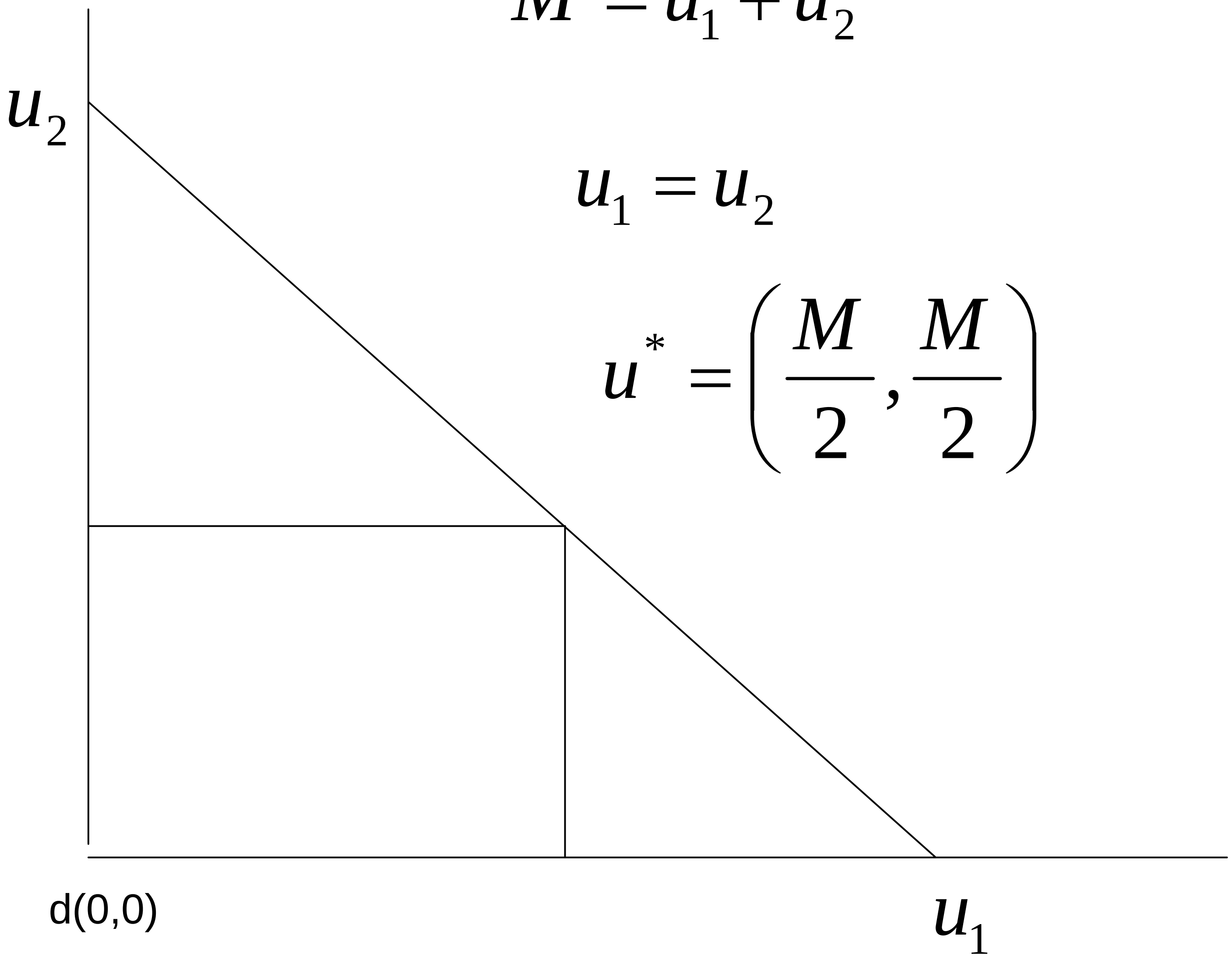
Thus $u_1 - \lambda = u_2 - \lambda$ implies $u_1 = u_2$ and putting this into the third first order condition $u_1 = u_2 = \frac{1000}{2} = 500$.

Splitting M amount between two players

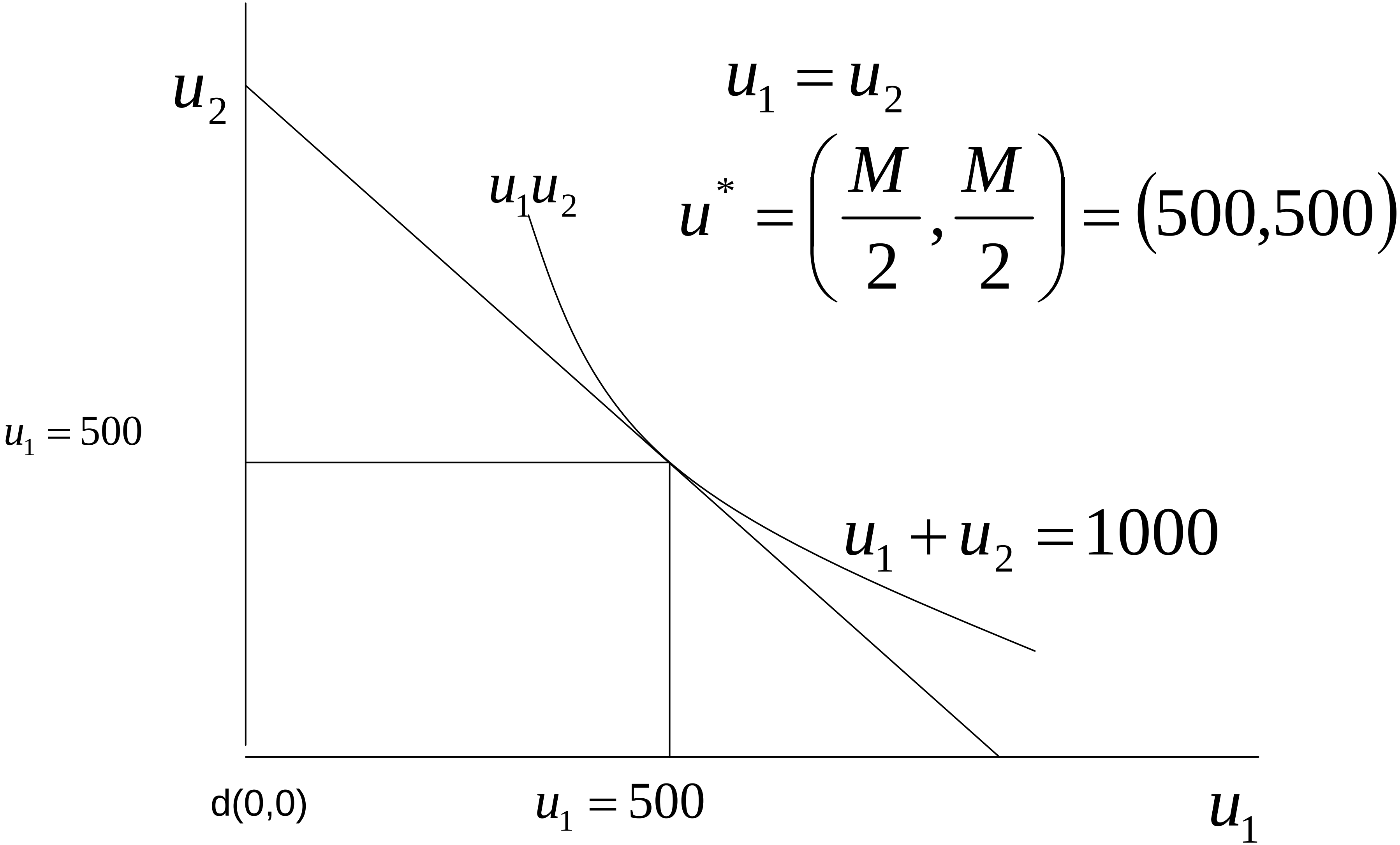
$$M = u_1 + u_2$$

$$u_1 = u_2$$

$$u^* = \left(\frac{M}{2}, \frac{M}{2}\right)$$



Symmetric Allocation of Amount 1000 Between Two Players



Linear Invariance in a Bargaining Game

$$L(u_1, u_2, \lambda) = (u_1 - d_1)(u_2 - d_2) + \lambda[1 - u_1 - u_2]$$

Suppose the player 1 has side payment $d_1 = 15000$

$$L(u_1, u_2, \lambda) = (u_1 - 15000)(u_2 - d_2) + \lambda[50000 - u_1 - u_2]$$

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 15000 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 50000 - u_1 - u_2 = 0$$

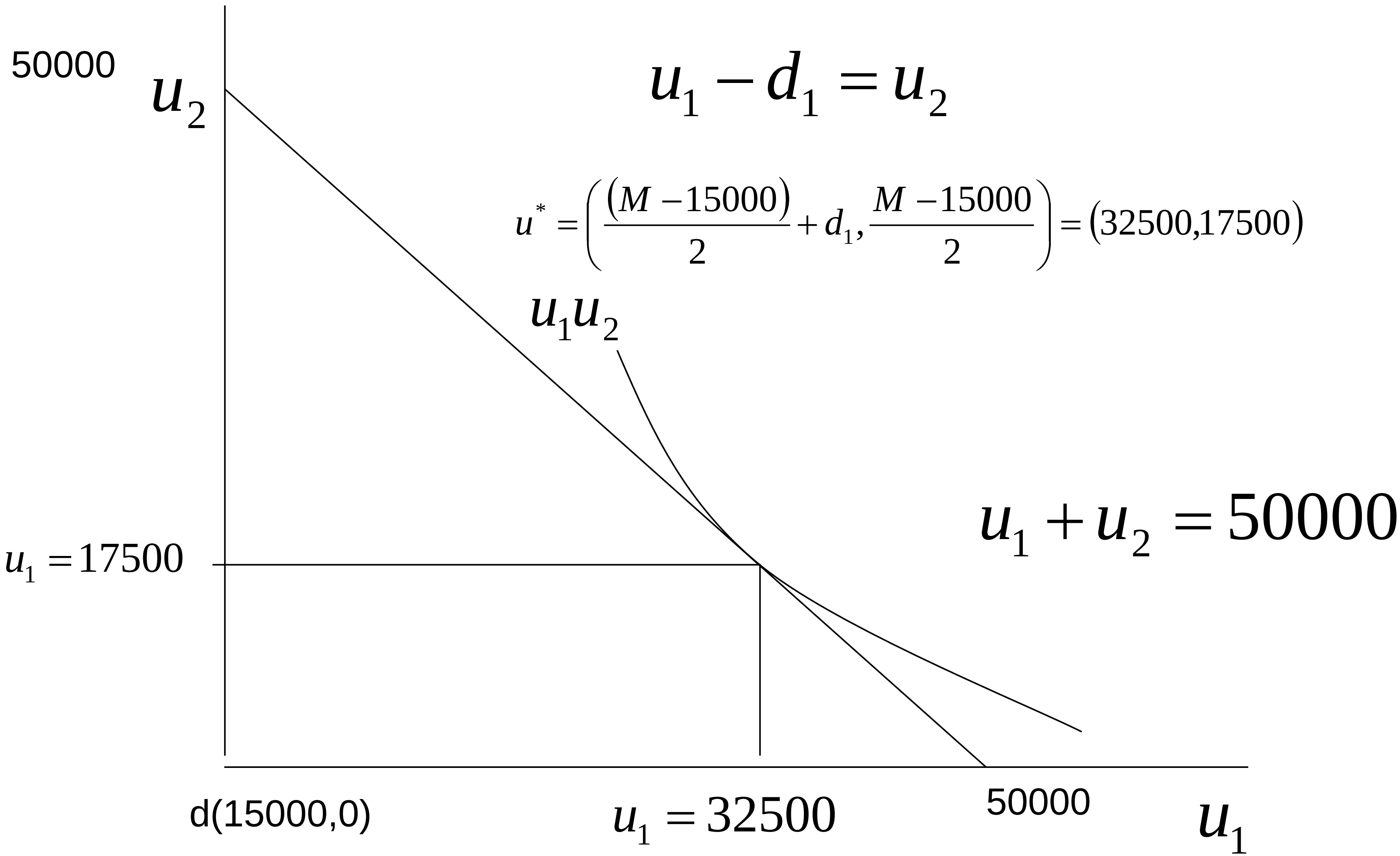
$u_1 - 15000 - \lambda = u_2 - \lambda$ implies $u_1 = 15000 + u_2$ and putting this into the third first order condition $u_2 + 15000 + u_2 = 50000$

$$2u_2 = 50000 - 15000$$

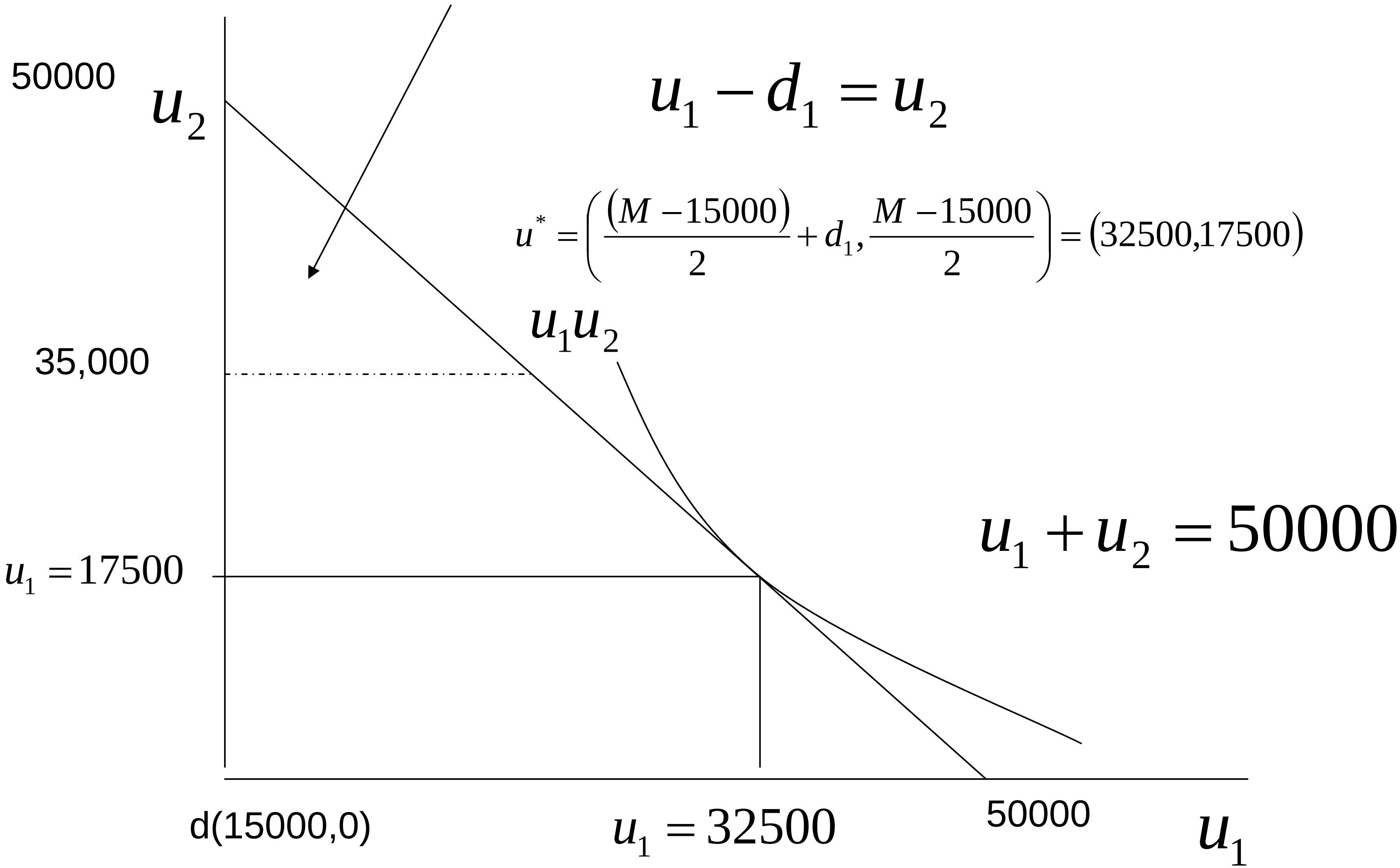
$$2u_2 = 35000 \quad u_2 = \frac{35000}{2} = 17500$$

$$u_1 = 15000 + 17500 = 32500$$

Symmetric Allocation of Amount 50,000 Between Two Players With Linear Invariance



Independence of Irrelevant Alternative (IIA)



Division of Gains between Risk Averse and Risk Neutral Players

A risk averse person looses in bargaining but the risk neutral person gains.

Suppose the utility functions of risk averse person is given by

$u_2 = (m_2)^{0.5}$ but the risk neutral person has a linear utility

$u_1 = m_1$.

$$m_1 + m_2 = M$$

$$u_1 + u_2^2 = 100$$

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda\left[100 - u_1 - u_2^2\right]$$

Division of Gains between Risk Averse and Risk Neutral Players

First order conditions

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 2\lambda u_2 = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 100 - u_1 - u_2^2 = 0$$

From the first two first order conditions $\frac{u_2}{u_1} = \frac{\lambda}{2\lambda u_2}$ implies $u_1 = 2u_2^2$

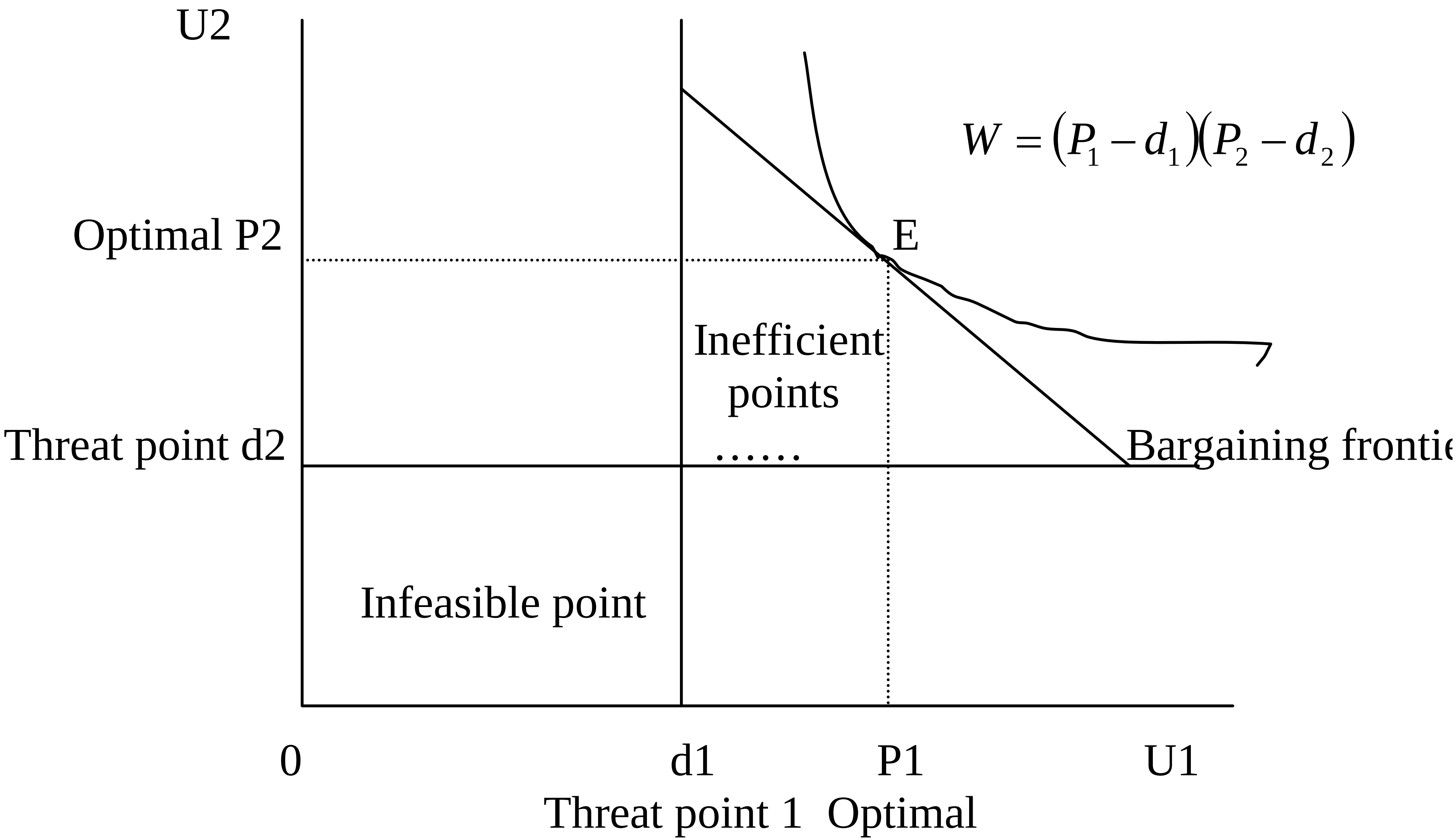
and putting this into the third first order condition $3u_2^2 = 100$.

$$u_2^2 = \frac{100}{3} = 33.33; \; u_2 = 5.77$$

$$u_1 = 2u_2^2 = 2(5.77)^2 = 66.6$$

$$u_1 + u_2^2 = 66.67 + 33.33 = 100$$

Efficient and Inefficient Bargaining Solutions



Coalition Formation and Cooperation and Core

- 2^N -1 rule for possible coalition
- Consider Four Players A,B,C,D
- A, B, C, D
- AB, AC, AD
- BC, BD, CD
- ABC, ACD, BCD
- ABCD
- $2^4 - 1 = 15$

Recommended Texts for GAME Theory

- Carmichael F.(2005) A Guide to Game Theory, ISBN: 0273684965
- Gardner R. (2003) Games for Business and Economics, Wiley, ISBN 0471451754
- Rasmusen E (2007) Games and Information, Blackwell, ISBN 1-140513666-9.
- Varian HR (2003) Intermediate Microeconomics: Modern Approach, Norton.
- Pindyck R.S. and D.L. Rubinfeld (2005) Microeconomics, 6th Edition, Pearson; ISBN 0-13-191207-0.

Game theory: Asymmetric information
Signalling and Screening and Sequential Equilibrium
Principal agent model

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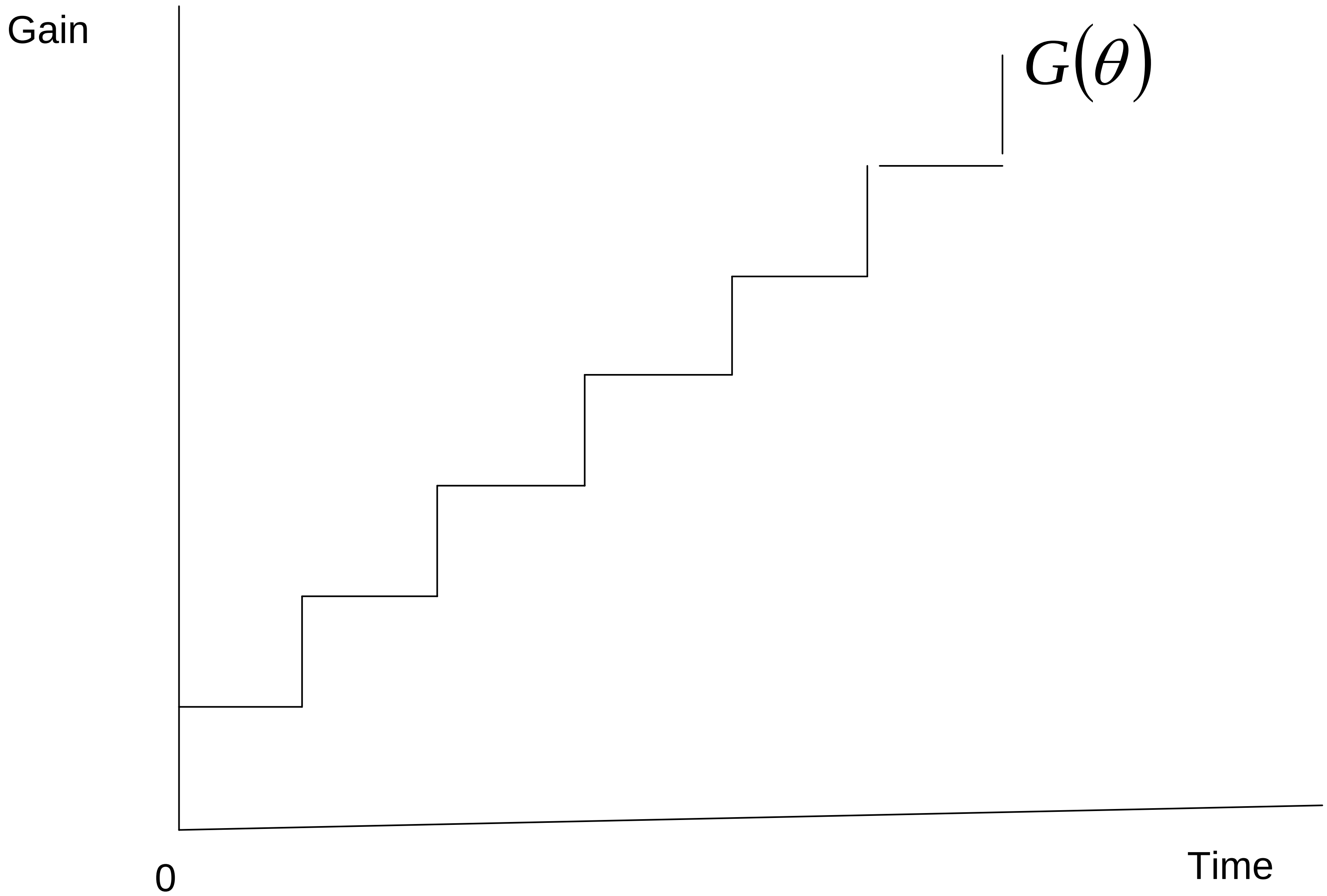
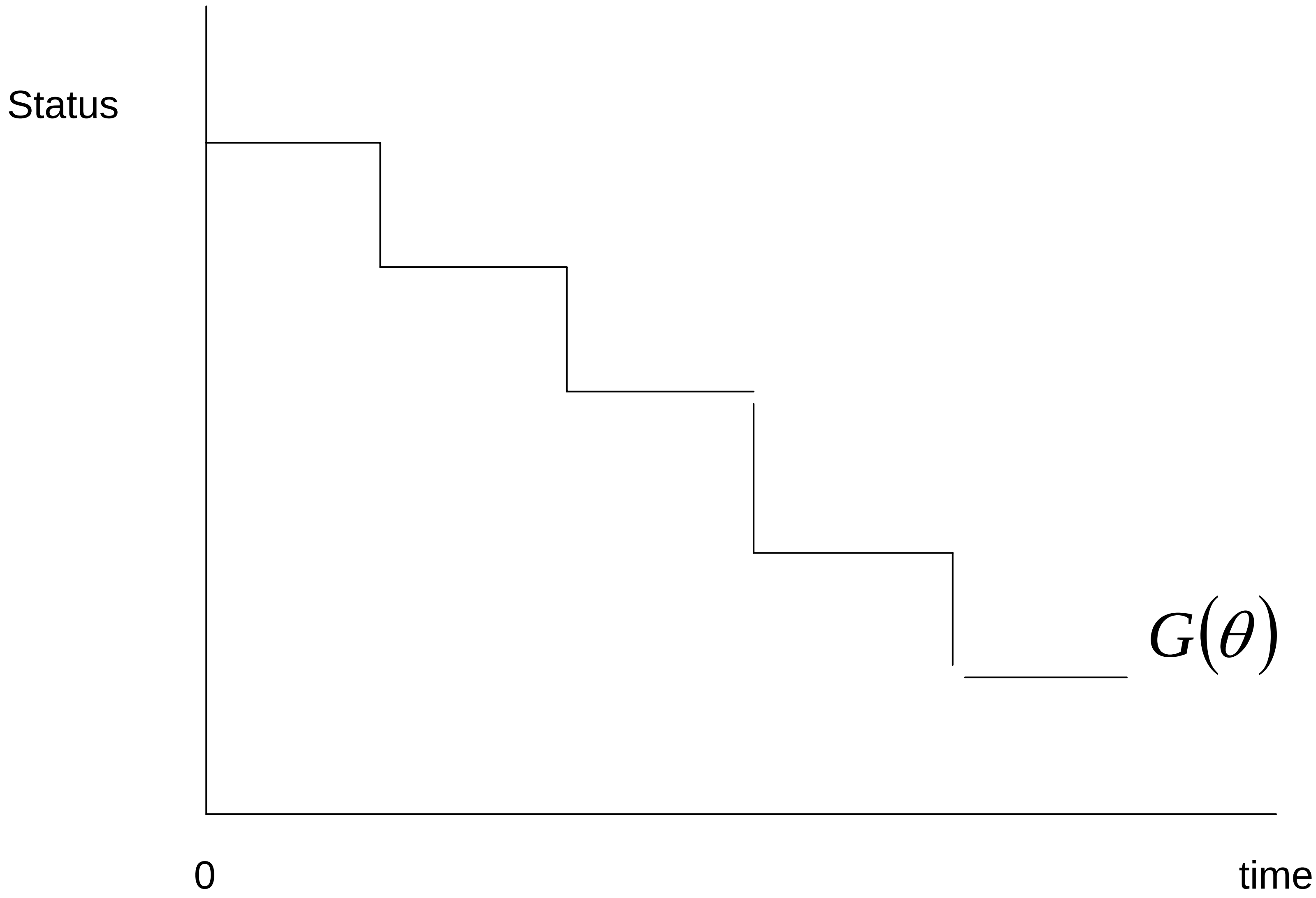
Reference texts: Gardner(2003) and Rasmusen (2007)

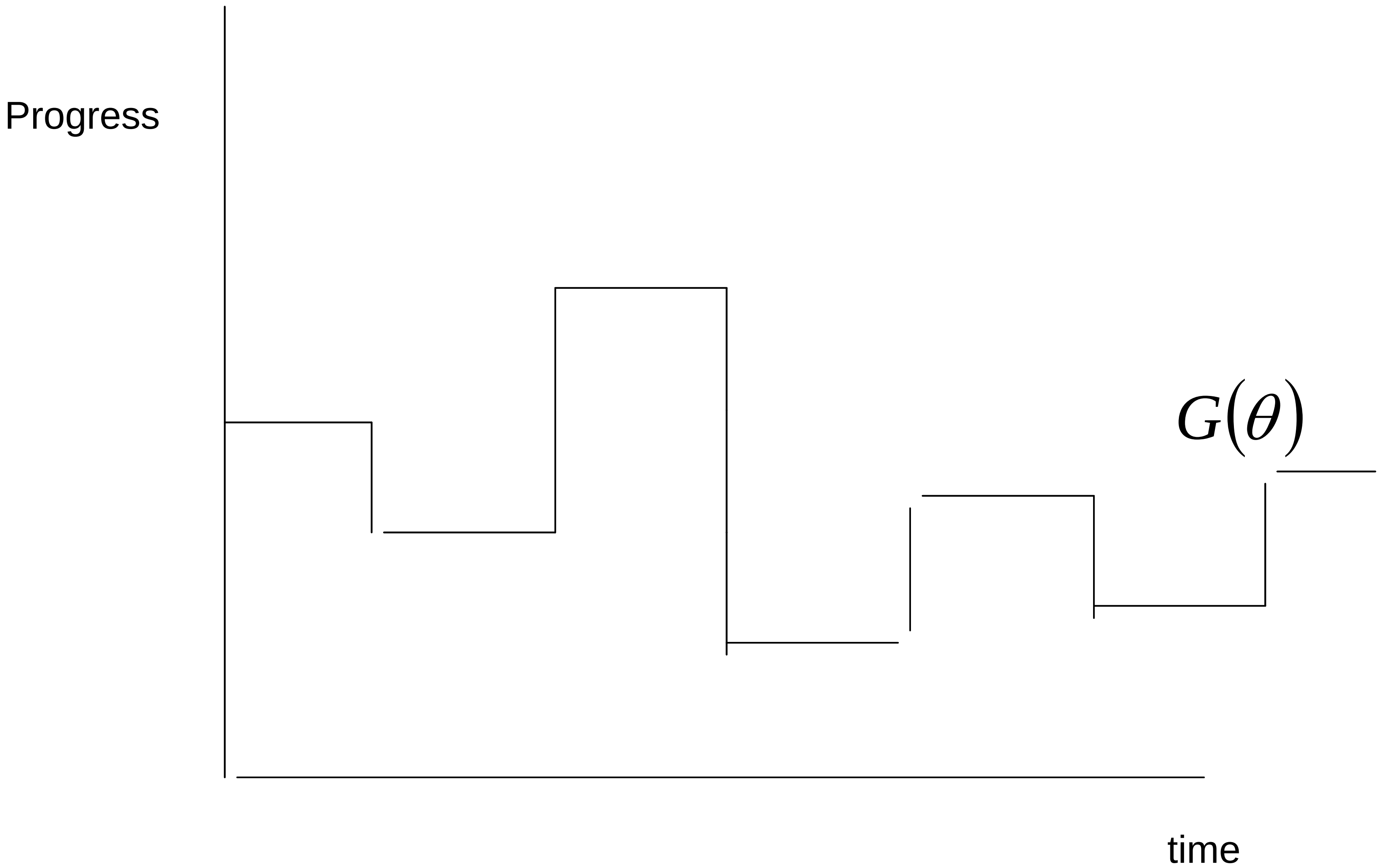
Results of a sequence of right actions following correct signals



Results of a sequence of wrong actions following reading signals incorrectly



Results of sequences of right and wrong actions reading signals correctly and incorrectly

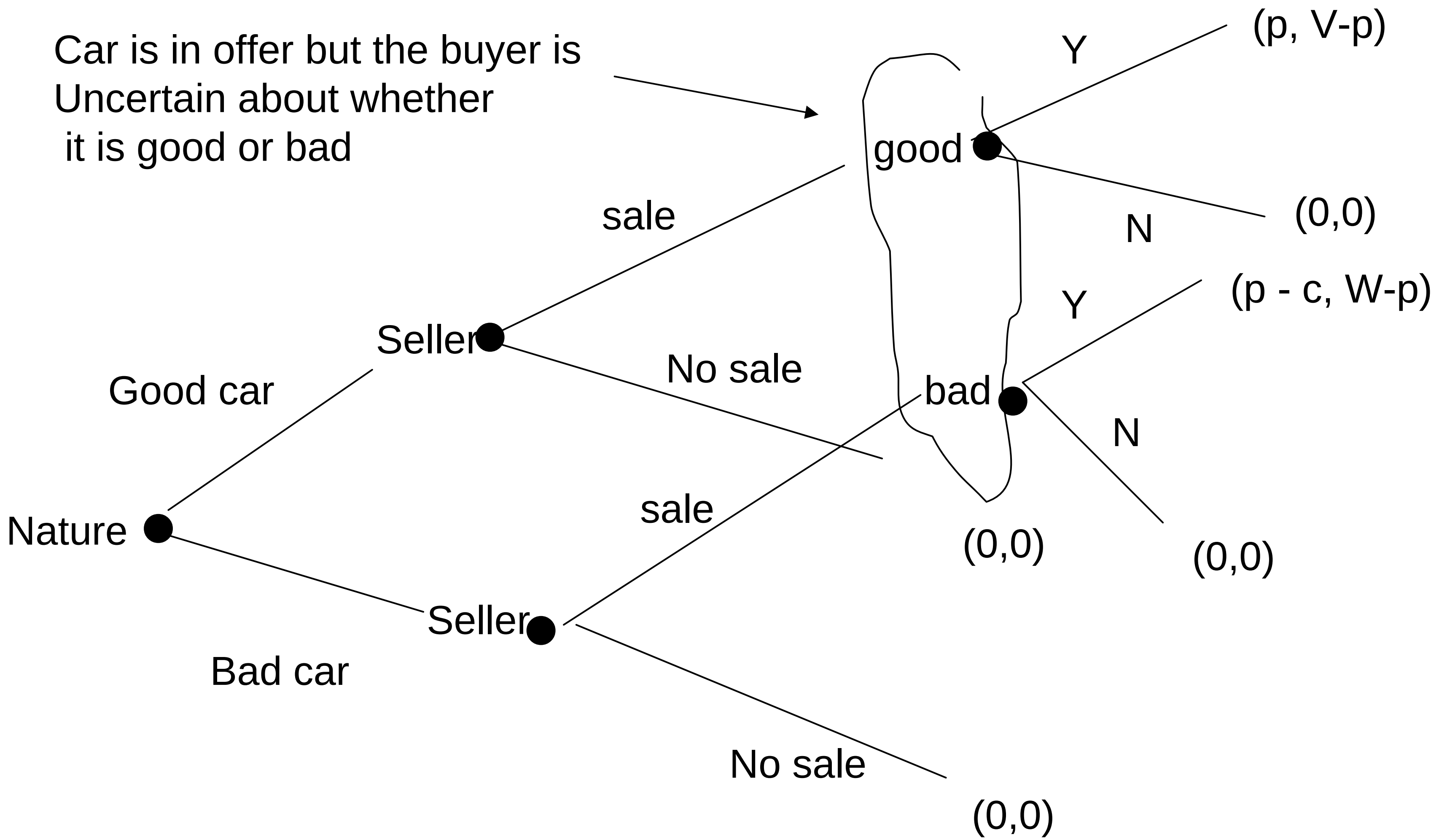


Market Situations in Signalling Game

- Complete market failure
  - pooling equilibrium (same price for good and bad cars; good cars disappear from the market)
- Complete market success
  - Separating equilibrium where players act as they should according to the signal (prices according to quality)
- Partial market success
  - (both good and bad cars are bought, some feel cheated)
- Near Market failure (mixed strategies)

Bayesian updating mechanism at work

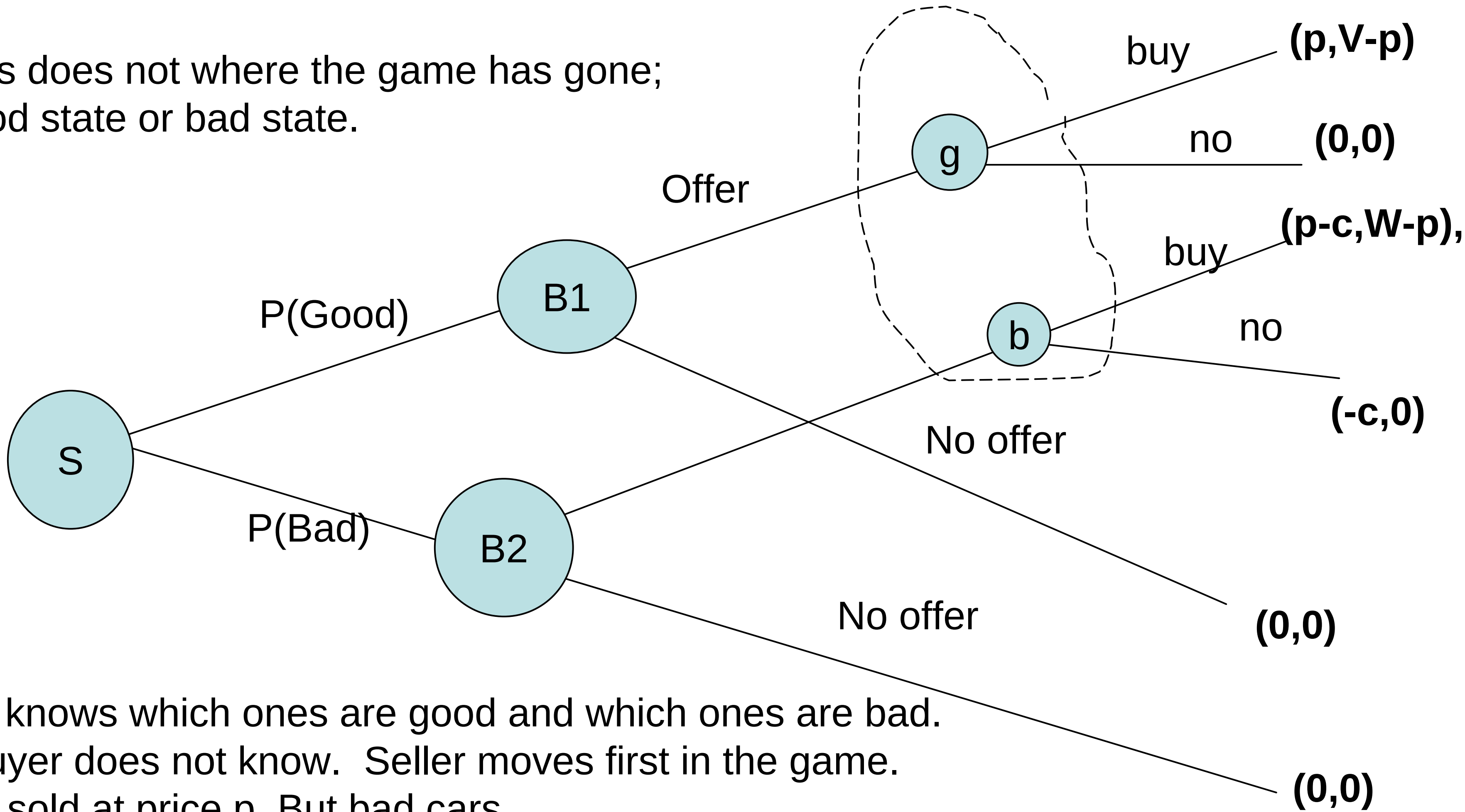
Signalling game in Markets for Used Car (Akerlof)



The buyer is uncertain about where the game is going- Need to resolve this- then the game can be solved by backward induction.

Informed Player Moves First in a Signalling Game (Akerlof)

Buyers does not where the game has gone;
At good state or bad state.



Seller knows which ones are good and which ones are bad.
But buyer does not know. Seller moves first in the game.
Car is sold at price p. But bad cars
need amount c to make repair and look just like a good one.
V is value of good car is V and W for bad car for the buyer.

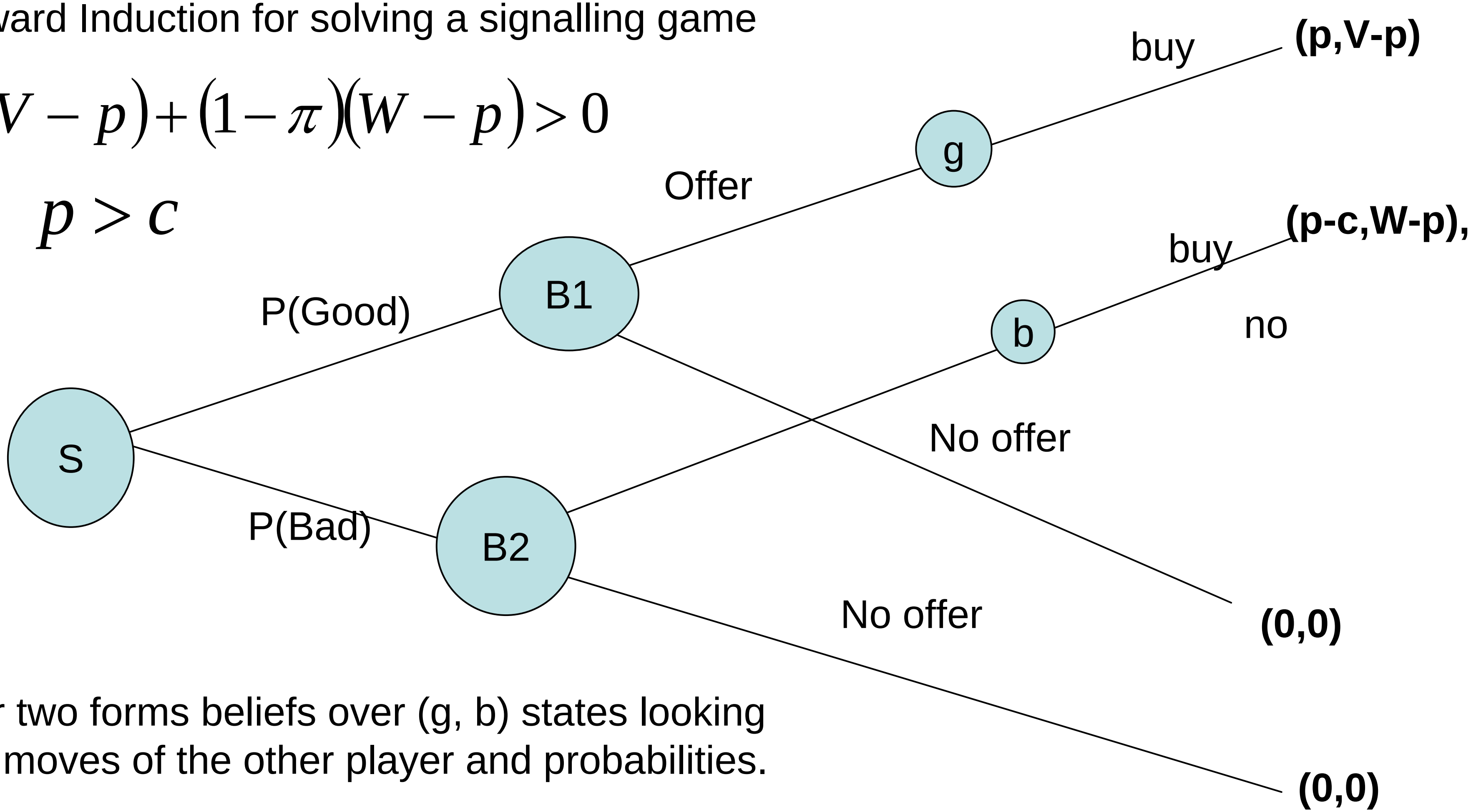
Partial Success of Market with asymmetric information.

Signals need to be credible

Backward Induction for solving a signalling game

$$\pi(V-p)+(1-\pi)(W-p)>0$$

$$p>c$$



Player two forms beliefs over (g, b) states looking at the moves of the other player and probabilities.

Tries to find a sub-game perfect equilibrium maximising expected utility.

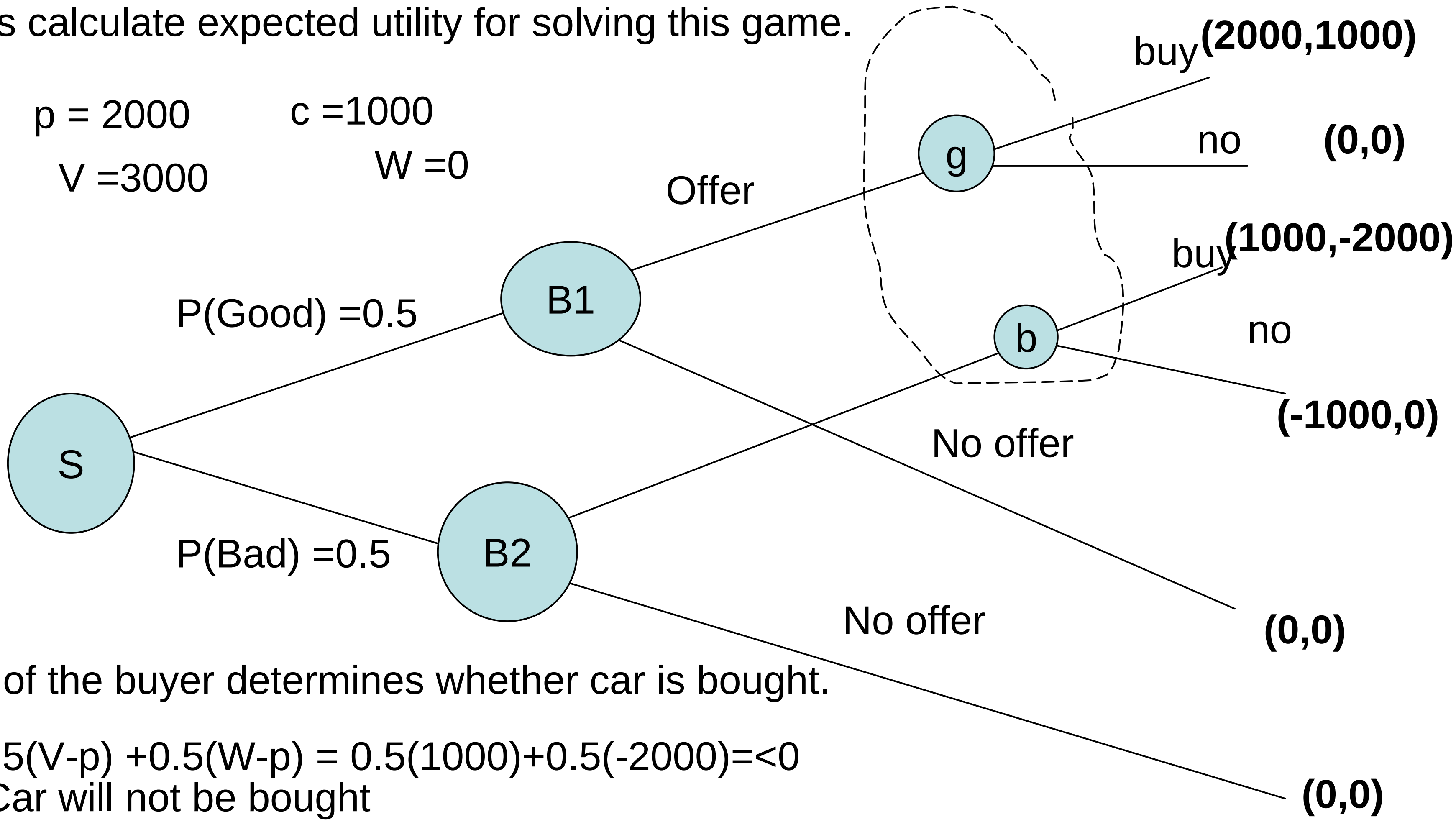
Both good and bad cars will be sold; Some buyers who get bad cars feel cheated

Mixed Strategy Sequential Equilibrium

Buyers calculate expected utility for solving this game.

p = 2000 c =1000

V =3000 W =0



Belief of the buyer determines whether car is bought.

0.5(V-p) +0.5(W-p) = 0.5(1000)+0.5(-2000)=<0

Car will not be bought

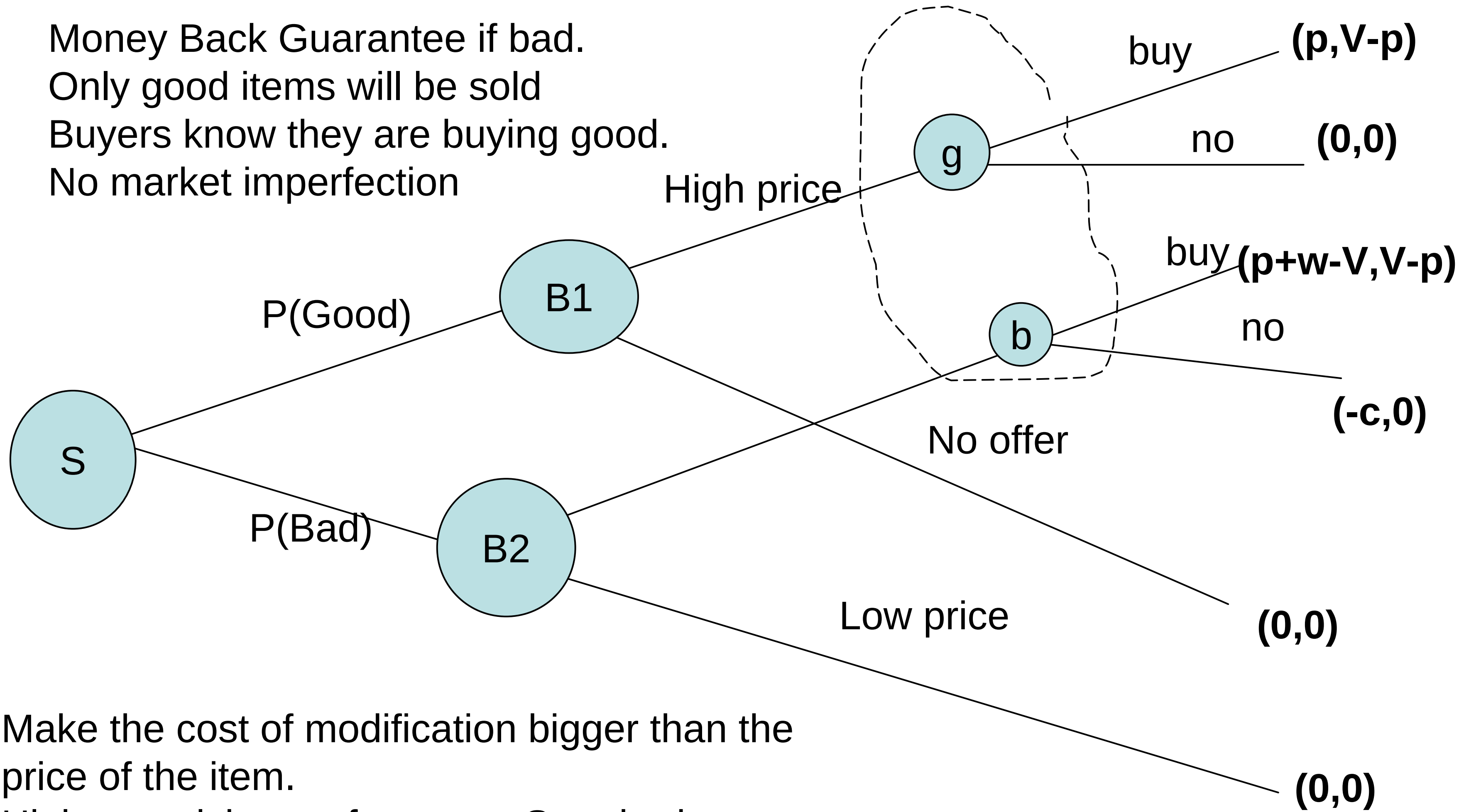
0.5(V-p) +0.5(W-p) = 0.66(1000)+0.33(-2000)=0

EU(good) = 0.5(2000)+0.5(0) = 1000>0

EU(bad) = 0.5(1000)+0.5(-1000) = 0

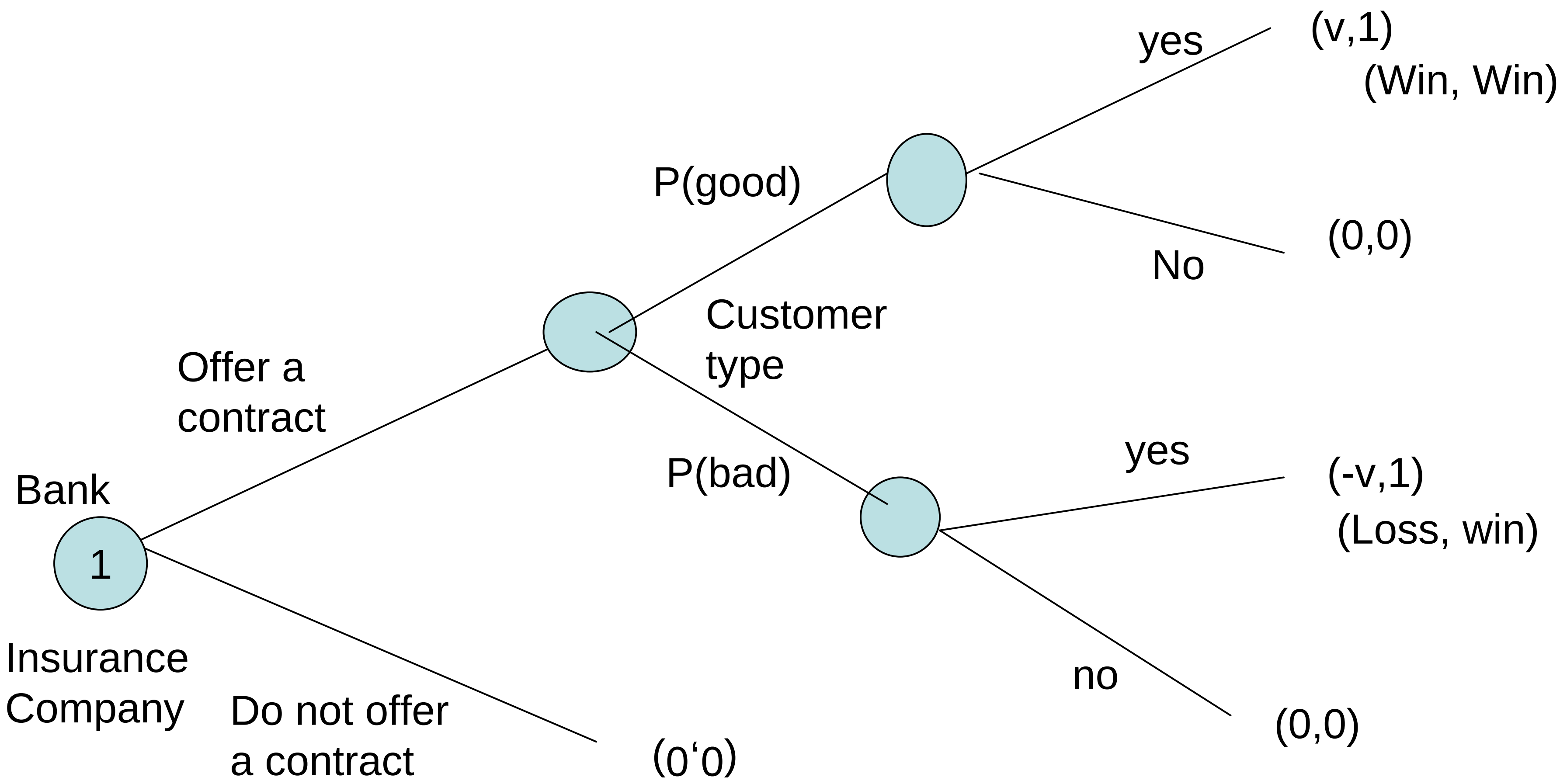
Costly Commitment for Separating Equilibrium: Complete Market Success

Money Back Guarantee if bad.
Only good items will be sold
Buyers know they are buying good.
No market imperfection



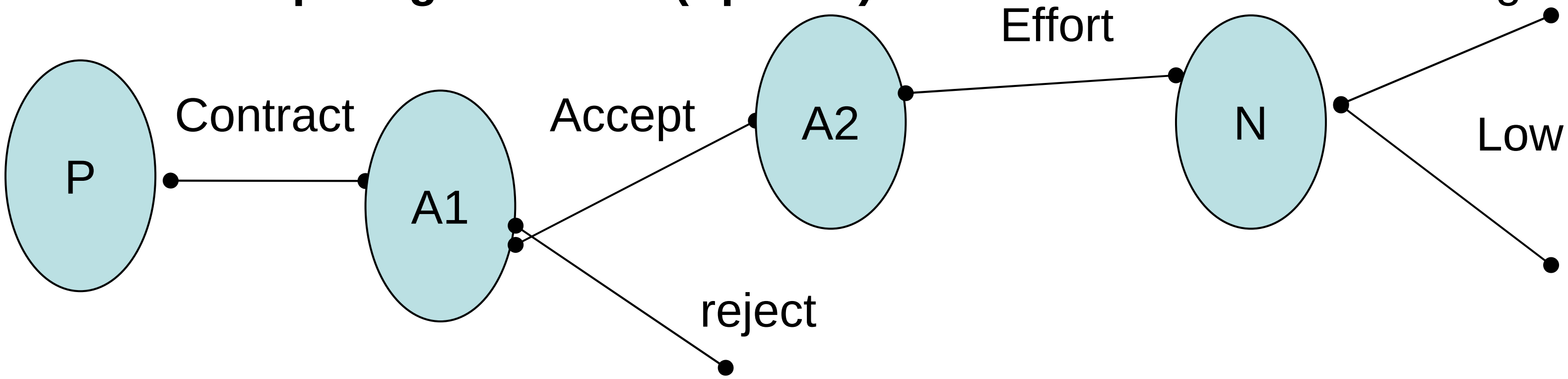
Make the cost of modification bigger than the price of the item.
Higher punishment for wrong Standards.

A Screening Game (Stiglitz)
Uninformed Players Moves First in

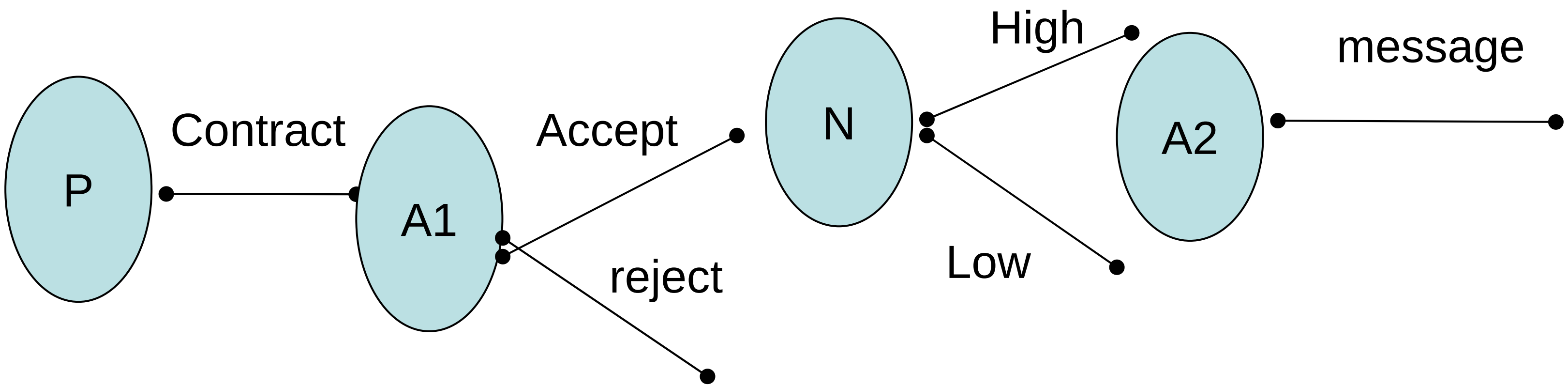


Credit Rationing or Insurance Market

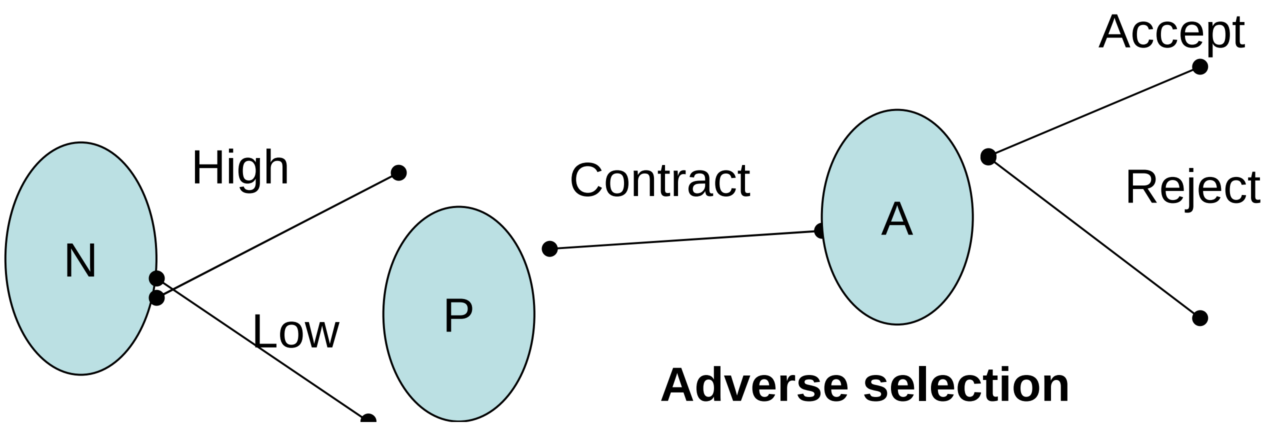
Principal Agent Model (Spence)



Moral Hazard game with Hidden action



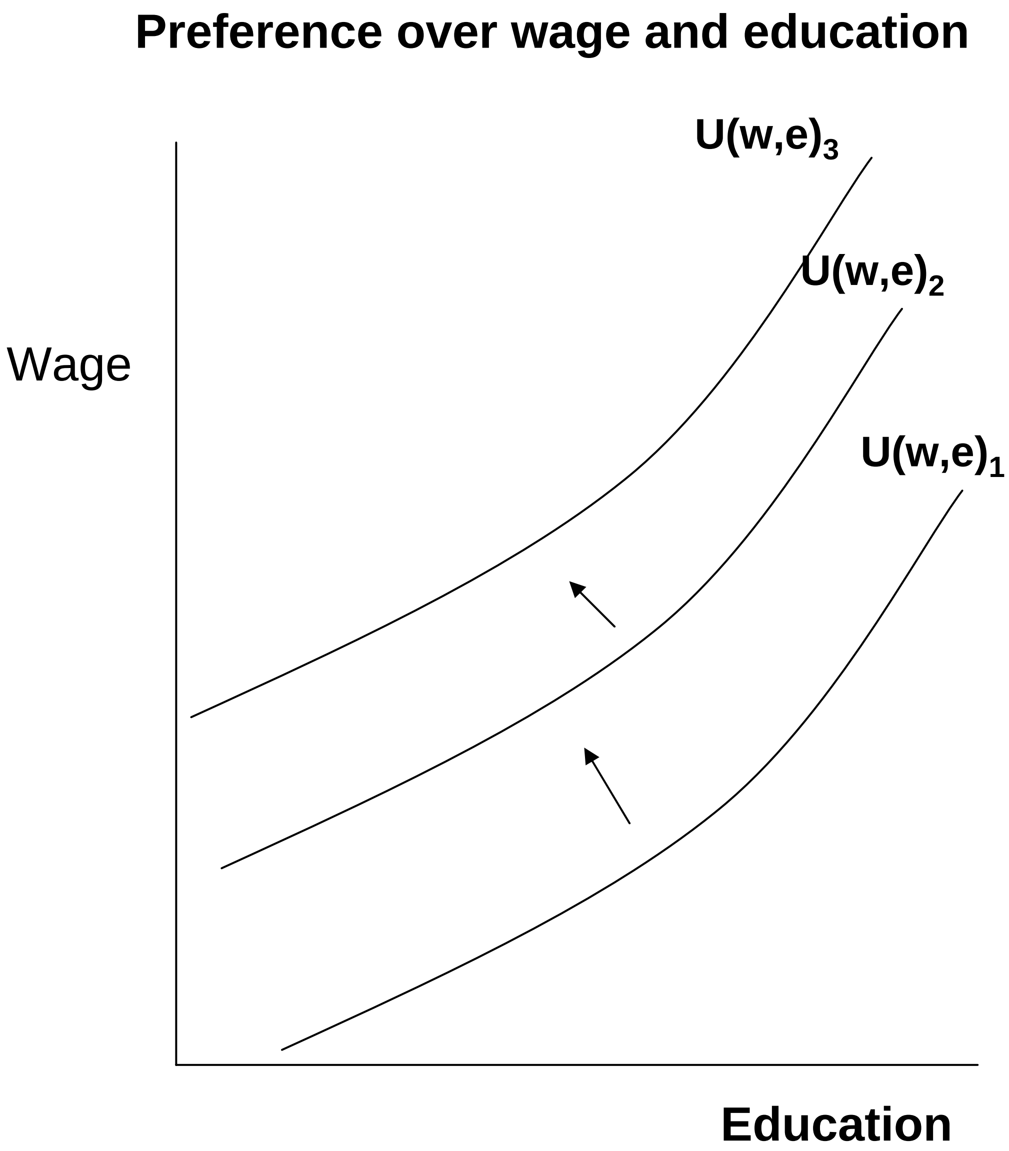
Moral Hazard with Post contractual hidden knowledge

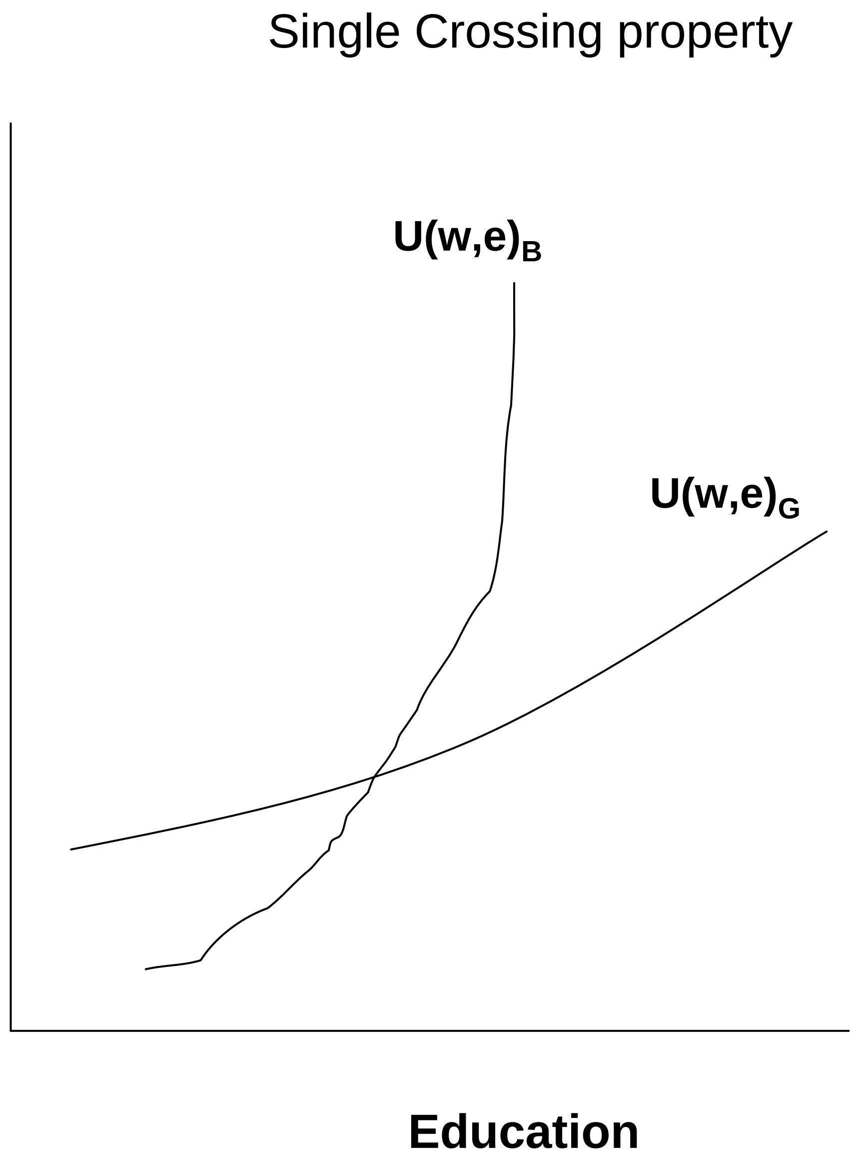


Adverse selection

Education as Productivity Signalling Game

- Players consisting of {workers, firms and nature}.
- There are two types of workers [t = {1, 2}].
- Type 1 has less productive and type 2 more productive.
- Employer does not know which one is low or high quality worker but sees level of education
- Nature decides whether a worker is high or low productivity type.
- Level of education signals the quality of worker





More Productive Worker Can Get Education Easily than Less Productive Worker

Preferences over Wage and Education

Workers choose level of education according to their beliefs about its impact on wage offer: $w_t(e)$.

Utility from wage and education is given by

$$u_t(w, e)$$

Utility is rising in wage received

$$\partial u_t(w, e)/\partial w > 0$$

Utility falls in work efforts

$$\partial u_t(w, e)/\partial e < 0$$

It is costly to get education.

The utility function satisfies the single-crossing property

Specification of Utility Function

$$U_t(w,e) = f(w) - k_t g(e) \quad \text{for } k_t > 0 \text{ for } t = 1,2.$$

k_t indicates the cost of education for the worker type *t*.

It is more expensive for less productive worker to produce education signal

$$k_1 > k_2.$$

More Specifically

$$U_t(w,e) = 42\sqrt{w} - k_t e^{1.5} \qquad k_1 = 2 \quad k_2 = 1$$

Level of Education Chosen by Low Productive Worker

In perfect information equilibrium, firms pay according to the marginal productivity

$$w_1(e) = e$$

$$w_2(e) = 2e$$

The type 1 workers optimisation problem

$$U_t(w,e) = 42\sqrt{w} - k_1 e^{1.5} = 42\sqrt{e} - 2e^{1.5}$$

$$\frac{\partial U_t(w,e)}{\partial e} = 42 \times \frac{1}{2\sqrt{e}} - 3e^{0..5} = 0 \qquad e^* = 7$$

Level of Education Chosen by More Productive Worker

The type 2 worker's optimisation problem

$$U_t(w,e) = 42\sqrt{w} - k_1 e^{1.5} = 42\sqrt{2e} - e^{1.5}$$

$$\frac{\partial U_t(w,e)}{\partial e} = 42 \times \frac{1}{2\sqrt{2e}} - 1.5e^{0.5} = 0$$

$$42 \times \frac{1}{\sqrt{2e}} = 1.5e^{0..5}; \quad \frac{42}{1.5\sqrt{2}} = e;$$

$$e^* = 19.8$$

Popular Principal Agent Games

Principal	Agent	Action
Plantation owner	Share cropper	Labour input
Insurance company	Policy holder	Careful behaviour
Patient	Doctor	Intervention
Owner	Renter	Maintenance
Firms	Workers	Work effort

Both principal and agents have their objective functions they like to maximize.

The principal is interested in maximising profit from the business,

Agents aim to maximise utility(payoff) choosing the best contract available from the principal with proper allowances for its efforts.

Rasmusen (2007) has interesting examples on this topic.

Incomplete Information and Adverse Selection

Principal wants to produce output employing workers with a scheme of wage contract that matches efforts put by a worker to produce.

Worker knows his type but the principal does not.

Principal knows the distribution of quality of workers *F(s)*, where s denotes either good or bad state such as probability of observing good is 0.5 and of bad 0.5.

Principal offers the agent a wage contract W(q).

Worker accepts or rejects this contract based on self-selection and participation constraints.

Objectives of Principal and Agents

Basically worker evaluates the utility from the wage and disutility from work and decides the amount of work to put in.

Output from good workers is $q(e, good) = 3e$ and from bad state is $q(e, bad) = e$

If agent rejects the contract there is no work both worker and principal get zero payoff.

If worker accepts the contract

Agent's utility: $\pi_{agent} = U(e, w, s) = w - e^3$

Principal's profit: $\pi_{principal} = V(q - w) = q - w$.

Objectives and Optimal Efforts for Agents

Good worker maximises

$$\underset{e_g}{Max} \quad 3e_g - e_g^2$$

The first part is wage income and the second part of disutility of work.

The optimal level of efforts by good agent is:

$$3 - 2e_g = 0$$

$$e_g = 1.5$$

Bad worker's Objective and Optimal Efforts

$$\underset{e_b}{Max} \quad e_b - e_b^2$$

$$1 - 2e_b = 0 \; ; \; e_g = 0.5$$

The principal does not know what levels of efforts are appropriate for good and bad workers.

Principal's Objective and Contracts

Principal maximises expected profit

$$\underset{q_g, q_b, w_g, w_b}{Max} \quad \left[0.5\left(q_g - w_g\right) + 0.5\left(q_b - w_b\right)\right]$$

by designing separate contracts for good and bad worke
$\left(q_g, w_g\right)$ and $\left(q_b, w_b\right)$.

Wage for good worker: $q(e, good) = 3e$ or $e = \frac{q_g}{3}$

Wage for bad worker: $q_b = e$

Incentive Compatibility Constraints for Agents

Self selection constraint for good worker

$$\pi_{agent}\left(q_g, w_g / good\right) = w_g - \left(\frac{q_g}{3}\right)^2 \geq \pi_{agent}\left(q_b, w_b / good\right) = w_b - \left(\frac{q_b}{3}\right)^2$$

Self selection constraint for bad worker

$$\pi_{agent}\left(q_b, w_b / bad\right) = w_b - \left(q_b\right)^2 \geq \pi_{agent}\left(q_g, w_g / bad\right) = w_g - \left(q_g\right)^2$$

Participation constraints for good worker

$$\pi_{agent}\left(q_g, w_g / good\right) = w_g - \left(\frac{q_g}{3}\right)^2 \geq 0$$

Participation constraint for good worker

$$\pi_{agent}\left(q_b, w_b / bad\right) = w_b - \left(q_b\right)^2 \geq 0$$

Binding Constraints

Participation constraint of bad worker

$$w_b = (q_b)^2$$

Self selection constraint for good worker

$$w_g = \left(\frac{q_g}{3}\right)^2 + w_b - \left(\frac{q_b}{3}\right)^2 = \left(\frac{q_g}{3}\right)^2 + (q_b)^2 - \left(\frac{q_b}{3}\right)^2$$

Solving the Principal Agent Game

Principal's objective function

$$\underset{q_g, q_b, w_g, w_b}{Max} \quad \left\lfloor 0.5\left(q_g - w_g\right) + 0.5\left(q_b - w_b\right)\right\rfloor$$

Including binding constraints of agents:

$$\underset{q_g, q_b}{Max} \quad \left[0.5\left(q_g - \left(\frac{q_g}{3}\right)^2 - (q_b)^2 + \left(\frac{q_b}{3}\right)^2\right) + 0.5\left(q_b - (q_b)^2\right)\right]$$

First order conditions with respect to q_g and q_b

$$0.5\left(1 - \frac{2q_g}{9}\right) = 0 \quad \text{or} \quad q_g = 4.5$$

$$0.5\left(-2q_b + \frac{2q_b}{9}\right) + 0.5\left(1 - 2q_b\right) = 0 \quad \text{or} \quad \left(-2q_b + \frac{2q_b}{9}\right) + \left(1 - 2q_b\right) = 0$$

$$\left(-4q_b + \frac{2q_b}{9} + 1\right) = 0; \quad 34q_b = 9 \quad \text{or} \quad q_b = 0.265$$

Incentive Compatible First Best Choices of Good and Bad Worker

Now wages can be found from the constraints

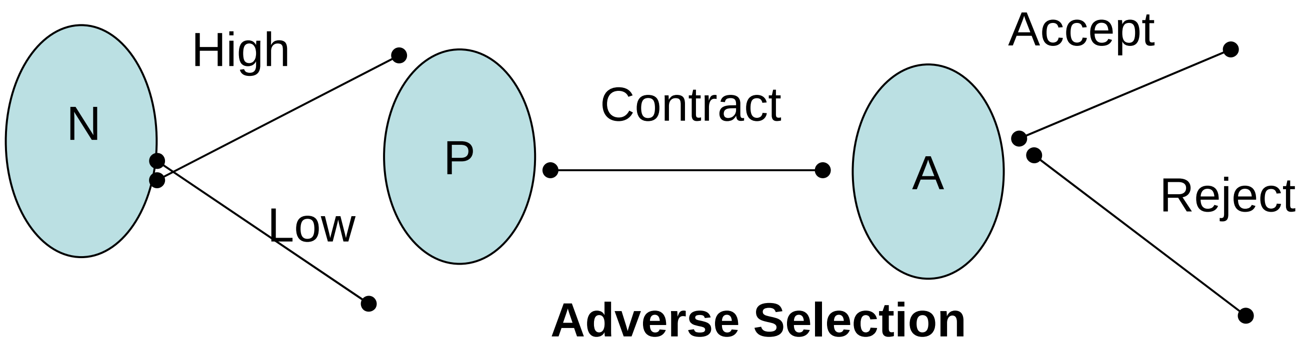
$$w_b = (q_b)^2 = (0.265)^2 = 0.07$$

$$w_g = \left(\frac{q_g}{3}\right)^2 + (q_b)^2 - \left(\frac{q_b}{3}\right)^2 = \left(\frac{4.5}{3}\right)^2 + (0.265)^2 - \left(\frac{0.265}{3}\right)^2 = 2.32$$

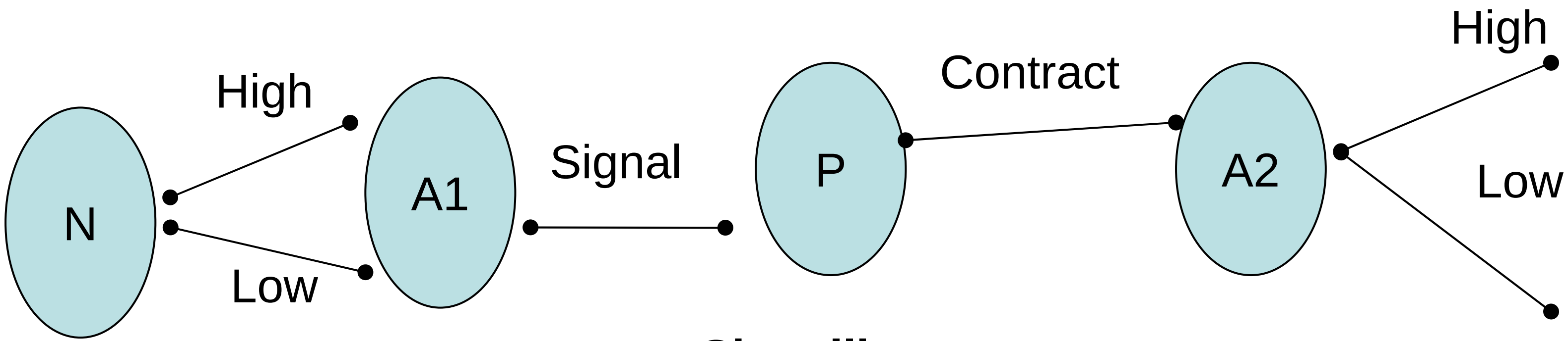
Thus in the presence of information asymmetry , the efforts by the good worker is at **the first best level** as the bad effort by him is not as attractive as the good effort.

It is not profitable for good worker to pretend as a bad worker. Good worker is not attracted by the contract for bad worker.

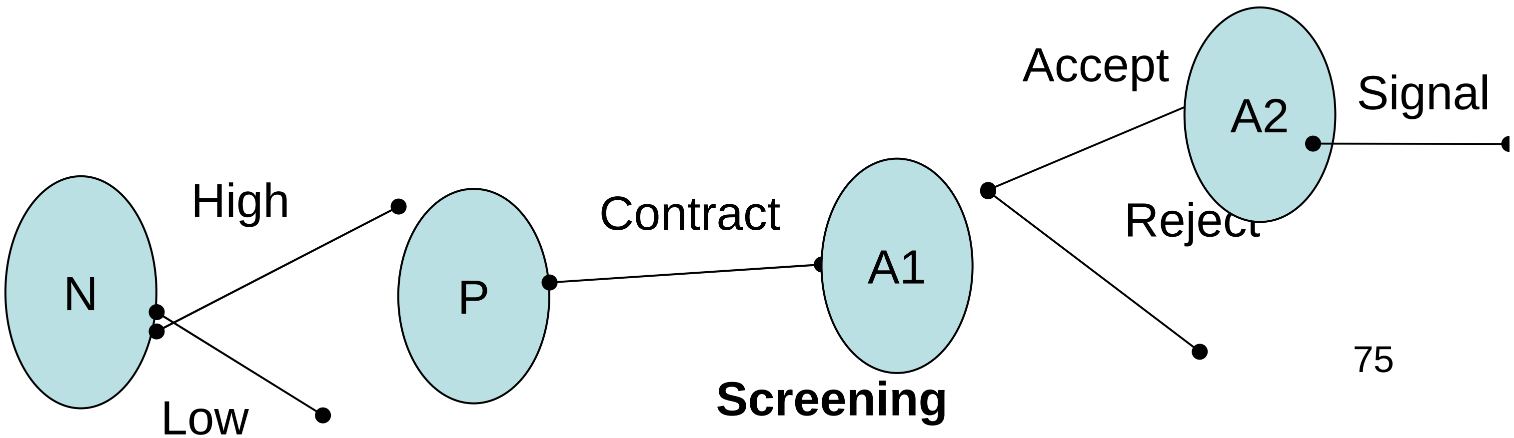
It is very costly for the bad worker to accept the contract of good worker. Bad worker’s **first best to put low effort**.



Adverse Selection



Signalling



Screening

Game theory:
Repeated Game
Strategic Behaviour Under Uncertainty
Moral Hazard and Adverse Selection

Dr. Keshab Bhattarai, Business School, University of Hull

What is a repeated game?

	E	SO
S	0,0	5,1
A	2,2	5,1

	E	SO
S	0,0	5,1
A	2,2	5,1

	E	SO
S	0,0	5,1
A	2,2	5,1

	E	SO
S	0,0	5,1
A	2,2	5,1

	E	SO
S	0,0	5,1
A	2,2	5,1

E = enter
SO = Stay out

S=smash
A =Accommodate

Nash Solution: Cournot

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

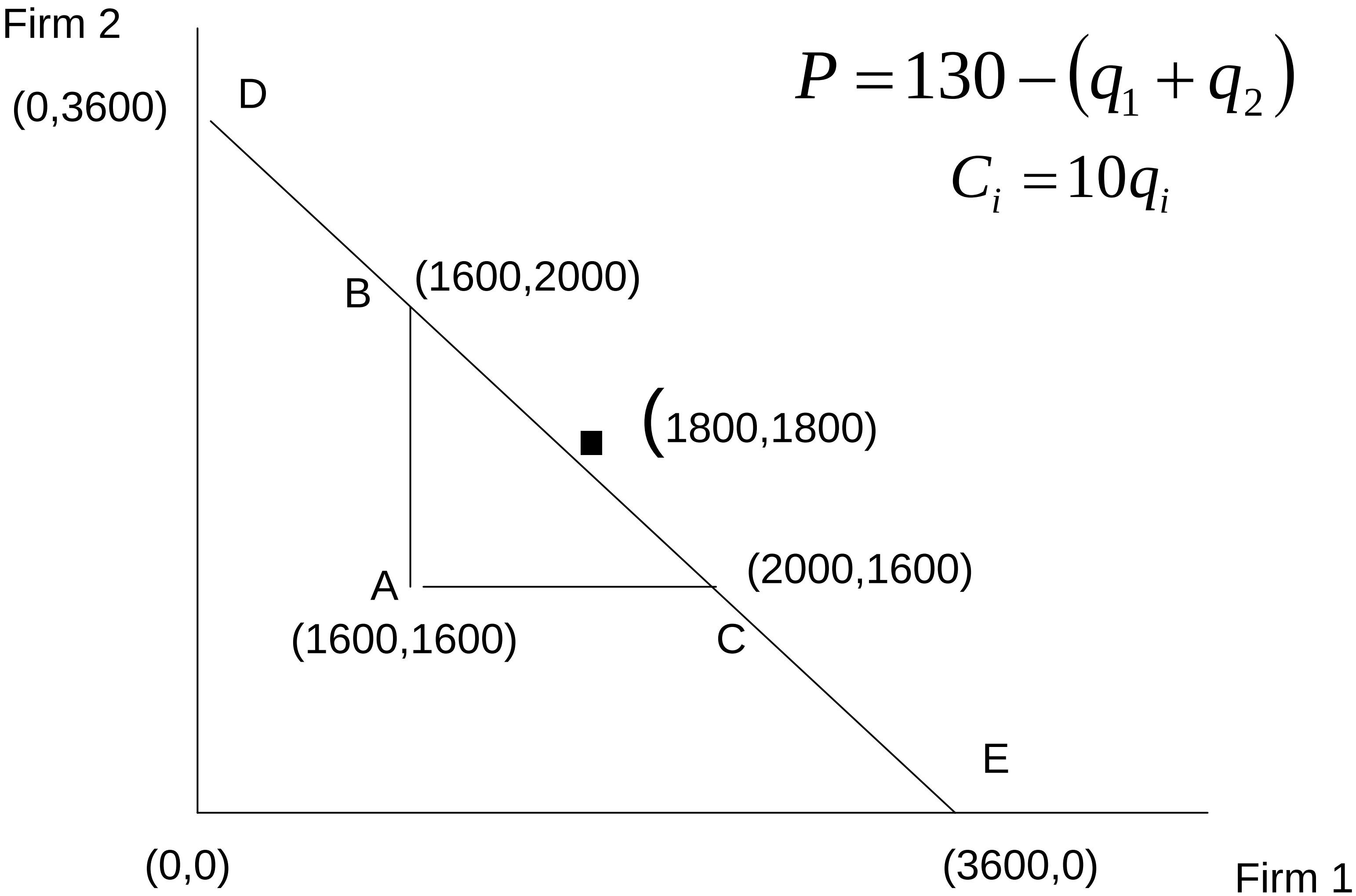
	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	E	SO
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

A = stick to agreement
Ch = Cheat

Infinitely Repeated Game in a Duopoly
Profits for firm 1 and 2



Cooperative Solution in Infinitely Played Repeated Game

Market demand for a product is $P = 130 - (q_1 + q_2)$

Cost of production for each of two firms is $C_i = 10q_i$.

If played infinite number of time two firms form a cartel and monopolise the market.

Each will supply only 30, set market price to monopoly level at £70 and divide total profit £3600 equally; each getting £1800.

This is shown by (1800,1800) point in the diagram.

It pays to cooperate in the long run; it is sub-game perfect equilibrium.

$$\Pi = (130 - Q)Q - 10Q \ ; \ \frac{\partial \Pi}{\partial Q} = 130 - 2Q - 10 = 0; \quad Q = \frac{120}{2} = 60;$$

$$P = 130 - Q = 130 - 60 = 70; \quad C = 10Q = 600;$$

$$\Pi = PQ - C = 70 \times 60 - 10 \times 60 = 4200 - 600 = 3600$$

Non-Cooperative Nash Equilibrium

If any one firm cheats and tries to supply more in order to get more profit; it will be found out by another firm.

It will react to this.

Game will be non-cooperative with resulting in a Cournot Nash equilibrium.

with each firm producing 40 units, market price of 50 and each getting £1600 profits.

$$\Pi_1 = (130 - (q_1 + q_2))q_1 - 10q_1 \text{ and } \Pi_2 = (130 - (q_1 + q_2))q_2 - 10q_2$$

with reaction functions $2q_1 + q_2 = 120$ and $q_1 + 2q_2 = 120$

Total supply is 80, each supplying 40 and making profit is 1600 and market price 50.

Trigger Strategy and Perpetual Punishment

If firm 1 plays Cournot game but firm 2 still plays cartel and supply just 30.

Then from the firm 1' reaction function $2q_1 + q_2 = 120$

$$q_1 = 60 - \frac{1}{2}q_2 = 60 - \frac{1}{2}(30) = 45.$$

If firm 1 supplies 45, market price will be $P = 130 - (q_1 + q_2) = 130 - 45 - 30 = 55.$

This makes profit margin of firm 1 to be 45 and its profit $\Pi_1 = 45 \times 45 = 2025.$

Firm 2 will find out that firm 1 has cheated.

It will also produce according to its reaction curve.

Thus the Nash equilibrium will result with each firm producing 40 and earning 1600 profit for the rest of the periods and the market price will be 50.

Cooperation or Cheating?

Cartel Solution and Agreement

	A	Ch
A	1800,1800	1800,1800
CH	1800,1800	1800,1800

	A	Ch
A	1800,1800	1800,1800
CH	1800,1800	1800,1800

	A	Ch
A	1800,1800	1800,1800
CH	1800,1800	1800,1800

	E	SO
A	1800,1800	1800,1800
CH	1800,1800	1800,1800

	A	Ch
A	1800,1800	1800,1800
CH	1800,1800	1800,1800

A = stick to agreement
Ch = Cheat

Trigger strategy and perpetual punishment for all

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	A	Ch
A	1800,1800	1600,2000
CH	2000,1600	1600,1600

	E	SO
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

	A	Ch
A	1600,1600	1600,1600
CH	1600,1600	1600,1600

A = stick to agreement
Ch = Cheat

For whom is it profitable to Cheat?

Does firm 1 gain or lose by deviation from the agreement. For this evaluate the infinite series of profits in deviation and in compliance with agreement.

Present value of profit in case of cheating

$\Pi_1 = (1-\delta)\left[2025 + 1600\delta + 1600\delta^2 + ... + ...\right]$ by adding and subtracting 1600 and applying the formula for infinite series

$$\Pi_1 = (1-\delta)\left[2025 - 1600 + 1600 + 1600\delta + 1600\delta^2 + ... + ...\right] = (1-\delta)\left[425 + \frac{1600}{1-\delta}\right]$$

$$\Pi_1 = 425(1-\delta) + 1600 = 425 - 425\delta + 1600 = 2025 - 425\delta$$

By comparing profits with and without cheating

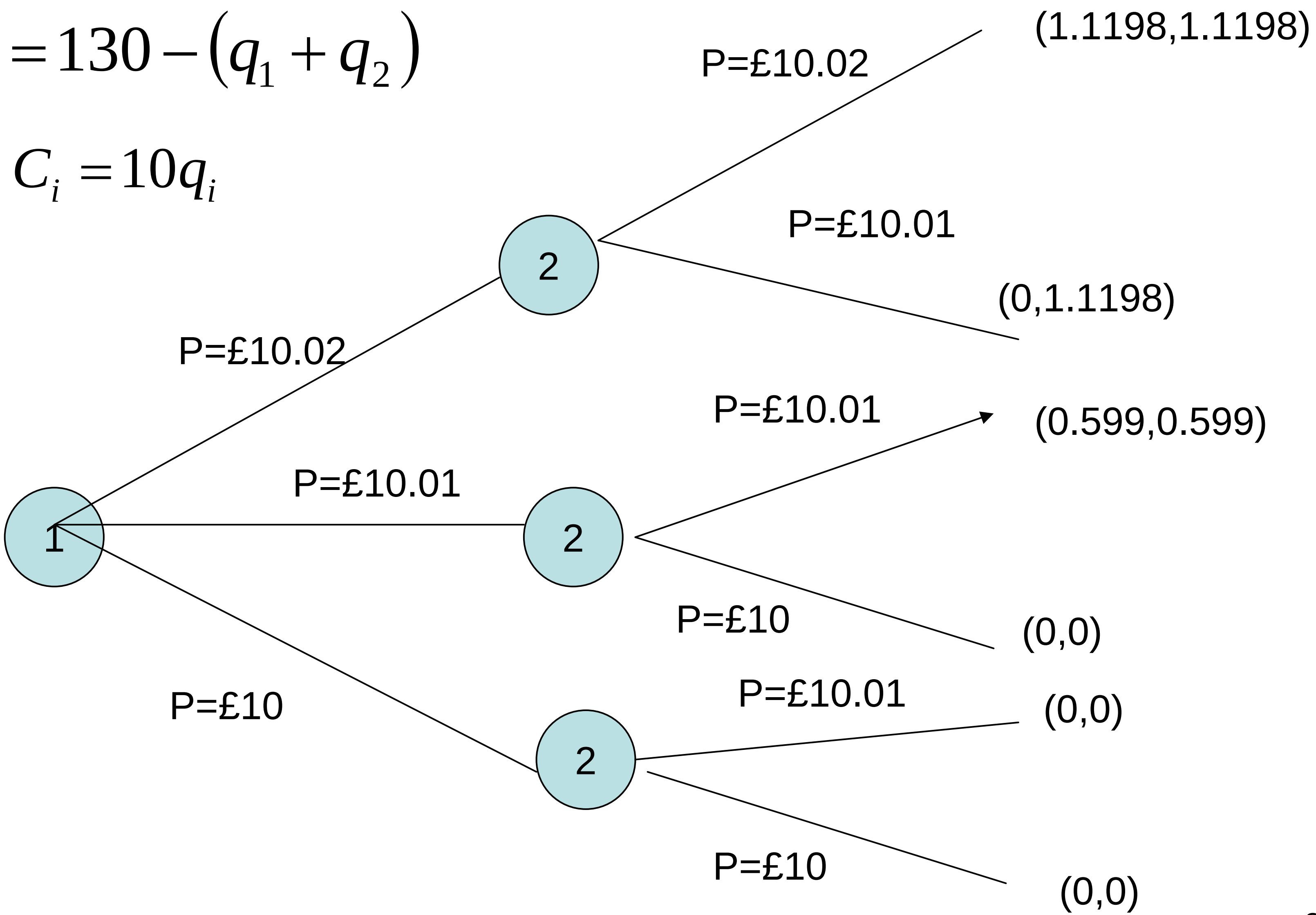
$$2025 - 425\delta < 1800 \text{ or } 425\delta > 2025 - 1800; \ \delta > \frac{225}{425}; \ \delta > \frac{9}{17}$$

Whether the firm 1 will stick to agreement or not depends on whether its discount factor if greater than $\delta > \frac{9}{17}$. For discount factor $\delta < \frac{9}{17}$ it benefits from sticking to the agreement.

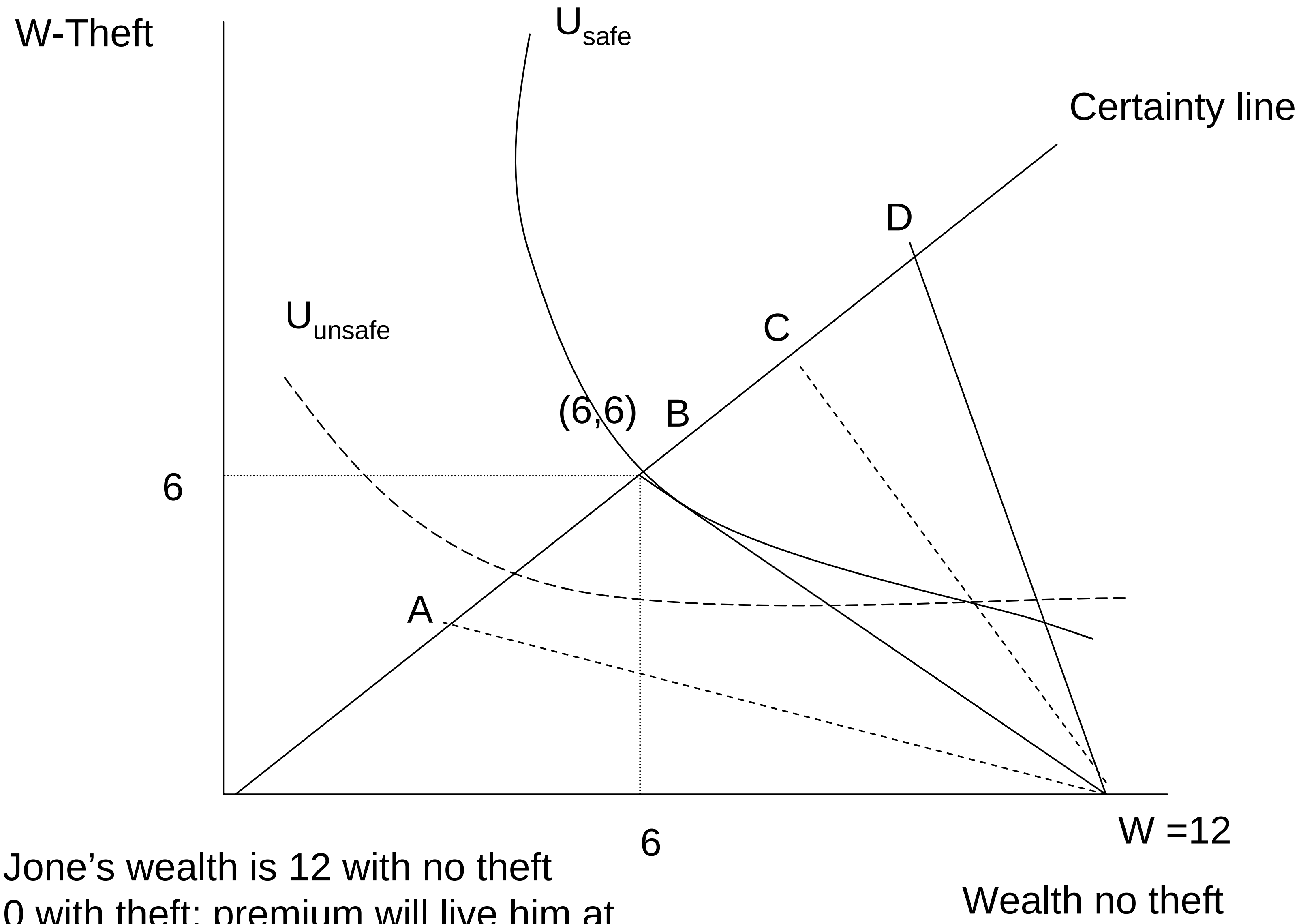
Bertrand-Stackelberg Cut-Throat Competition

$$P = 130 - (q_1 + q_2)$$

$$C_i = 10q_i$$



Theft Insurance



Jone's wealth is 12 with no theft
0 with theft; premium will live him at
(6,6) in both states.

Insuring Safe and Unsafe Customers?

Probability of theft reduces with safety precautions of individual.

Theft insurance company does not know who is safe and who is unsafe customer, therefore it aims to maximise its expected profit from the business.

Nature decides 0.5 chances of theft for safe and 0.75 for unsafe.

Property value: 12   Pooling insurance: 6

Safe customers with full insurance get expected utility:

0.5 [12-x] +0.5 [12-x] = 6

Unsafe customer with full insurance get:

0.25 [12-x] +0.75 [12-x] =6

Expected Profit of the Insurance Company

Insurance company's expected profit with safe customer:

$$0.5\,[x] + 0.5\,[x-12] = 3 - 3 = 0$$

Insurance company's expected profit with unsafe customer:

$$0.25\,[x] + 0.75\,[x-12] = 1.5 - 4.5 = -3.0$$

Insurance company believes that 60 percent of consumers are safe and 40 percent are unsafe.

Adjusting for probability of safe and unsafe customer, insurance companies expected profit is:

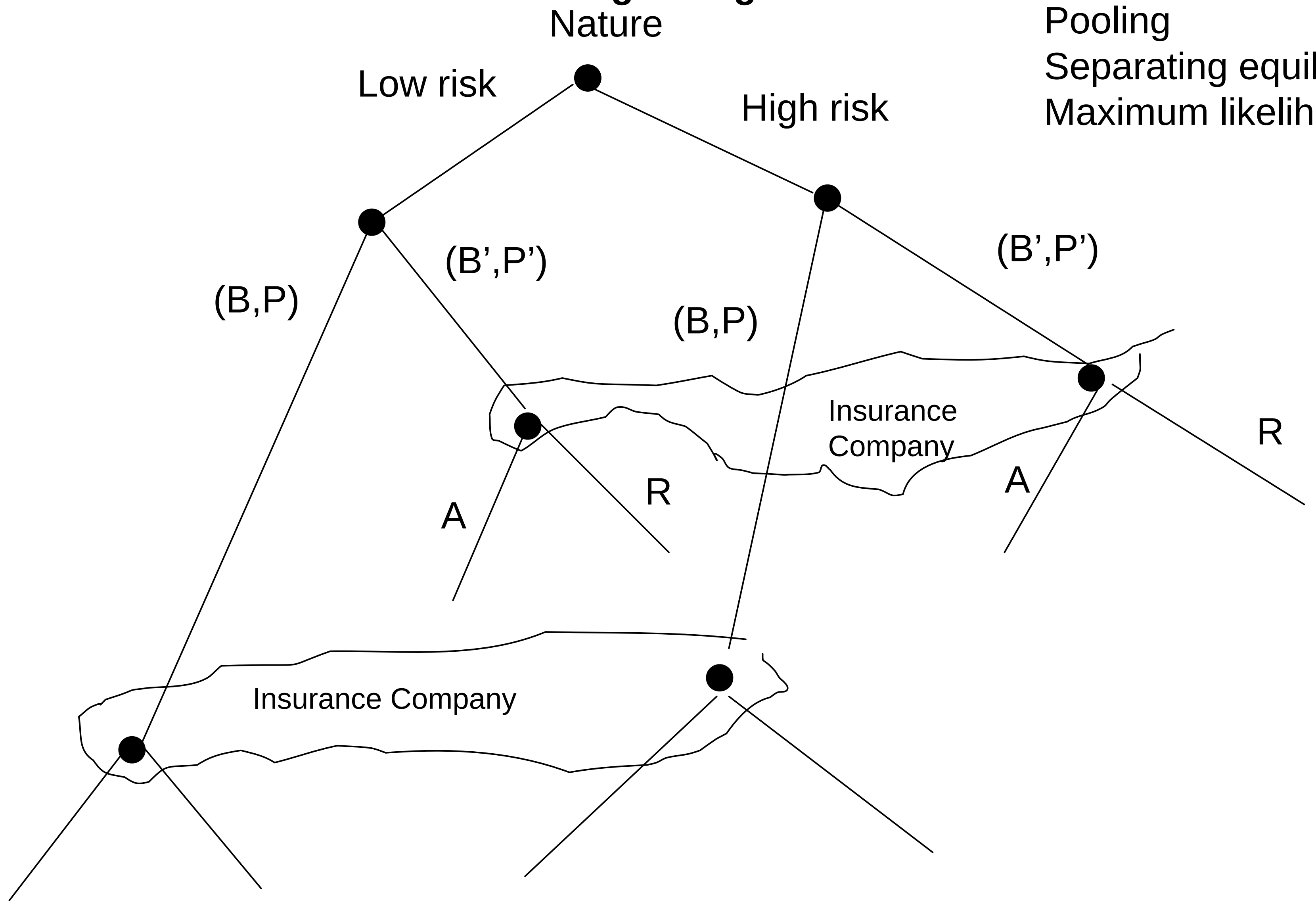
$$0.6\{0.5\,[x] + 0.5\,[x-12]\} + 0.4\{0.25\,[x] + 0.75\,[x-12]\}$$

$$0.6\,(0) – 0.4(-3) = -1.2$$

Calculate for separating insurance: x1 =6 for safe and x2 =8 for unsafe.

Insurance Signalling Game

Pooling
Separating equilibrium
Maximum likelihood ratio



Nature decides whether a customer is high or low risk type but the insurance company does not know which is of which type. It offers insurance schemes suitable for low and high risk individual. Choice of the customer signals the insurance company of its type. Then the insurance company can make an incentive mechanism so that it is beneficial to buy high risk type insurance for high risks and low risk type for low risks.

Insurance Game with Moral Hazard

Let p be insurance premium.

$\pi(e)$ probability of accident with effort e, this diminishes with greater care (higher e).

Level of benefit offered in case of accident is B_L specific to losses L =1,2,….L .

The Moral hazard problem is for insurance company to set the premium according to efforts

$$\underset{e,p,B_0....B_L}{Max} \quad p-\sum_{l=0}^{L}\pi_l(e)B_l \qquad (1)$$

subject to participation constraint:

$$\sum\pi_l(e)u(W-p-l+B_l)-d(e)\geq\bar{u} \qquad (2)$$

Lagrangian function

$$L=p-\sum_{l=0}^{L}\pi_l(e)B_l+\lambda\left[\sum_l\pi_l(e)u(W-p-l+B_l)-d(e)-\bar{u}\right] \quad (3)$$

Solving Moral Hazard for Insurance

$$\frac{\partial L}{\partial p} = 1 - \lambda\left[\sum \pi_l(e)u'(W - p - l + B_l)\right] = 0 \quad (4)$$

$$\frac{\partial L}{\partial B_l} = -\pi_l(e) + \lambda \pi_l(e)u'(W - p - l + B_l) = 0 \qquad (5)$$

$$\frac{\partial L}{\partial \lambda} = \sum_l \pi_l(e)u(W - p - l + B_l) - d(e) - \bar{u} = 0 \qquad (6)$$

From (5) $u(W - p - l + B_l) = d(e) + \bar{u}$

Under full insurance $B_l = l$

this implicitly defines the insurance premium for effort level

$$u(W - p(e)) = d(e) + \bar{u} \; .$$

Since low effort is less costly than more effort for the costumer
$d(0) \leq d(1)$;

the premium under lower effort must be set higher than for the higher effort: $p(0) \geq p(1)$ f or profit maximisation

$$p - \sum_{l=0}^{L} \pi_l(e) \cdot l$$

This is the prediction of moral hazard with complete information but uncertainty with consumer's hidden action.

Insurance company cannot observe the consumer's choice of accident prevention efforts.

But the insurance company continues to seek maximize he expected profit.

It now need to add incentive compatibility constraint.

$$\underset{e,p,B_0....B_L}{Max} \quad p - \sum_{l=0}^{L} \pi_l(e) B_l \quad (1)$$

subject to participation constraint:

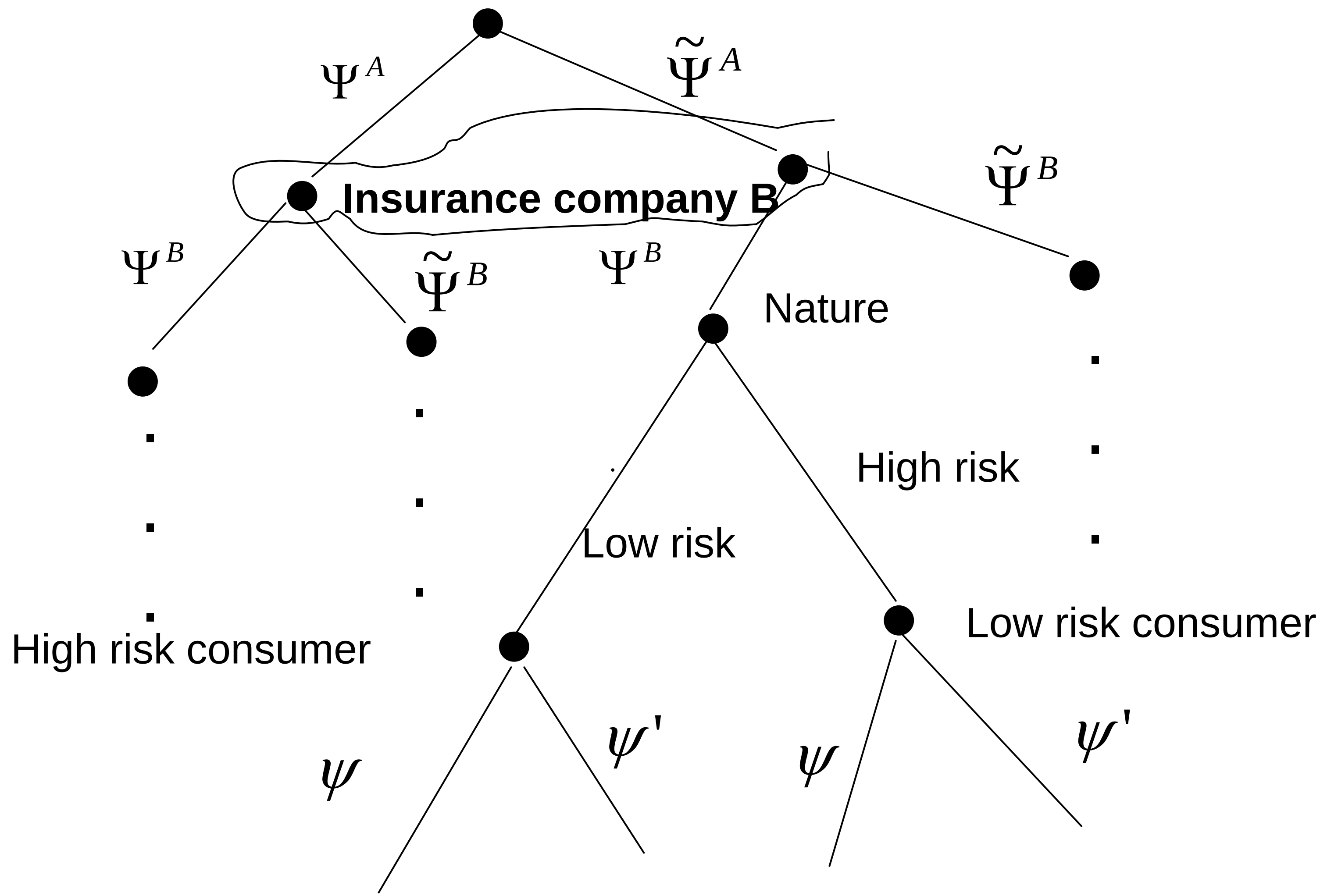
$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \sum_l \pi_l(e') u(W - p - l + B_l) - d(e')$$

$$\sum \pi_l(e) u(W - p - l + B_l) - d(e) \geq \bar{u} \quad (2)$$

Incentive compatibility constraint

$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \sum_l \pi_l(e') u(W - p - l + B_l) - d(e')$$

Solving Screening Game for an Insurance Company



Insurance company A and B move simultaneously. They do not know who is risky and who is not risky consumer. Consumers reveal their choice by choosing the insurance policy. Then companies guess who is risky and who is safe.

Optimisation Conditions

$$L = p - \sum_{l=0}^{L} \pi_l(e) B_l + \lambda \left[\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) - \bar{u} \right]$$
$$+ \beta \left[\left\{ \sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \right\} - \left\{ \sum_l \pi_l(e') u(W - p - l + B_l) - d(e') \right\} \right] \quad (3)$$

$$\frac{\partial L}{\partial p} = 1 - \left[\sum \left(\lambda \pi_l(1) + \beta(\pi_l(1) - \pi_l(0)) (u'(W - p - l + B_l)) \right) \right] = 0 \quad (4)$$

$$\frac{\partial L}{\partial B_l} = -\pi_l(1) + [\lambda \pi_l(1) + \beta(\pi_l(1) - \pi_l(0))] u'(W - p - l + B_l) = 0 \quad (5)$$

$$\frac{\partial L}{\partial \lambda} = \sum_l \pi_l(e) u(W - p - l + B_l) - d(e) - \bar{u} \geq 0$$

$$\frac{\partial L}{\partial \beta} = \left\{ \sum_l (\pi_l(1) - \pi_l(0)) u(W - p - l + B_l) + d(0) - d(1) \right\} \geq 0$$

$$\frac{1}{u'(W - p - l + B_l)} = \lambda + \beta \left[1 - \frac{\pi_l(0)}{\pi_l(1)} \right]$$

$$\frac{1}{u'(W-p-l+B_l)}=\lambda+\beta\left[1-\frac{\pi_l(0)}{\pi_l(1)}\right]$$

$$\sum_l \pi_l(e)u(W-p-l+B_l)-d(e)\geq\sum_l \pi_l(e')u(W-p-l+B_l)-d(e')$$

since $\beta>0$ the RHS is strictly decreasing, this implies that $u'(W-p-l+B_l)$ must be strictly increasing for this to happen $l-B_l$ be must increase with effort levels and losses

$l=0,1,2,......L$. Optimal high policy does not provide full insurance but the deductible payment increases size of loss (Jehle and Reny (2001, Chapter 8).

Optimal policy is crafted so that the utility of benefit from high efforts equals higher costs.

Incentive System: "How can I get someone to do something for me?" :Spence Model

- If a worker puts *x* amount of effort, the land produces $y = f(x)$
- Then the land owner pays worker s(y) .
- The land owner wants to maximise profit

$$\pi = f(x) - s(y) = f(x) - s(f(x))$$

- Worker has cost of putting effort *c*(*x*) and has a reservation utility, $\overline{u}$
- The participation constraint is given by .

$$s(f(x)) - c(x) \geq \overline{u}$$

- Including this constraint, the maximisation problem becomes $Max \quad f(x) - s(f(x))$
- subject to $s(f(x)) - c(x) \geq \overline{u}$

Incentive compatible contract (Varian Chapter 36)

- (a) renting the land where the workers pays a fixed rent R to the owner and takes the residual amount of output, at equilibrium $f\left(x^*\right) - c\left(x^*\right) - R = \overline{u}$
- (b) Take it or leave it contract where the owner gives some amount such as,, $B^* - c\left(x^*\right) = \overline{u}$
- (c) hourly contract $s\left(f\left(x\right)\right) = wx + K$
- (d) sharecropping, in which both worker and owner divide the output in a certain way.
- In (a)-(c) burden of risks due to fluctuations in the output falls on the worker but it is shared by both owner and worker in (d).
- Which of these incentives work best depends on the situation.