

Questions and Answers

Advanced Economic Theory
Review for the Final

Noeclassical model for the long run

Preference:
$$\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt$$

Technology:
$$Y_t = A_t K_t^\alpha N_t^{1-\alpha} \quad \text{assume } A_t = 1 \quad N_t = 1$$

Capital accumulation:
$$\dot{K}_t = Y_t - N_t C_t - \delta K_t$$

Current value Hamiltonian of this problem

$$H(c, K, \theta) = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta [K_t^\alpha - C_t - \delta K_{t-1}]$$

C is control, K is state variable, θ is co-state variable.

Optimality and Boundary Conditions

First order conditions

$$\frac{\partial H}{\partial C_t} = 0 \rightarrow C_t^{-\sigma} = \theta_t \quad (2)$$

$$\dot{\theta}_t = \rho\theta_t - \frac{\partial H_t}{\partial K_t} \rightarrow \dot{\theta}_t = \rho\theta_t - \theta_t \left[\alpha K_t^{\alpha-1} - \delta \right] \quad (3)$$

$$\dot{K}_t = K_t^\alpha - N_t C_t - \delta K_t \quad (4)$$

Transversality condition

$$\lim_{n \rightarrow \infty} e^{-\rho t} \theta_t K_t = 0 \quad (5)$$

Characterisation of the Balanced Growth Path

Capital stock, consumption and the shadow price of capital remain constant in the

balanced growth path $\frac{\dot{C}}{C} = g_c$; $\frac{\dot{K}}{K} = g_K$ and $\frac{\dot{\theta}_t}{\theta_t} = g_\theta$. From (3)

$$\frac{\dot{\theta}_t}{\theta_t} = \rho - [\alpha K_t^{\alpha-1} - \delta] \rightarrow \alpha K_t^{\alpha-1} = \rho - \frac{\dot{\theta}_t}{\theta_t} + \delta \quad (6)$$

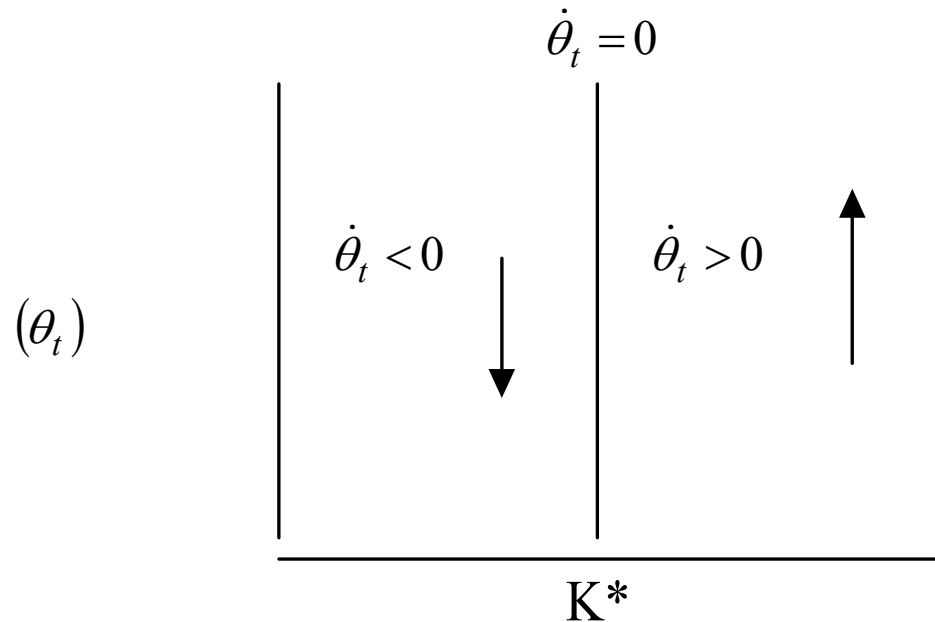
Since the RHS is constant, therefore LHS also should be constant $\frac{\dot{K}}{K} = 0$. If capital stock

is not growing output is not growing $\frac{\dot{Y}}{Y} = 0$ and consumption is not growing $\frac{\dot{C}}{C} = 0$.

$$\text{From (2)} \quad \frac{\dot{\theta}_t}{\theta_t} = -\sigma \frac{\dot{C}_t}{C_t} \rightarrow \frac{\dot{\theta}_t}{\theta_t} = 0 \quad (7)$$

Transitional Dynamics-1

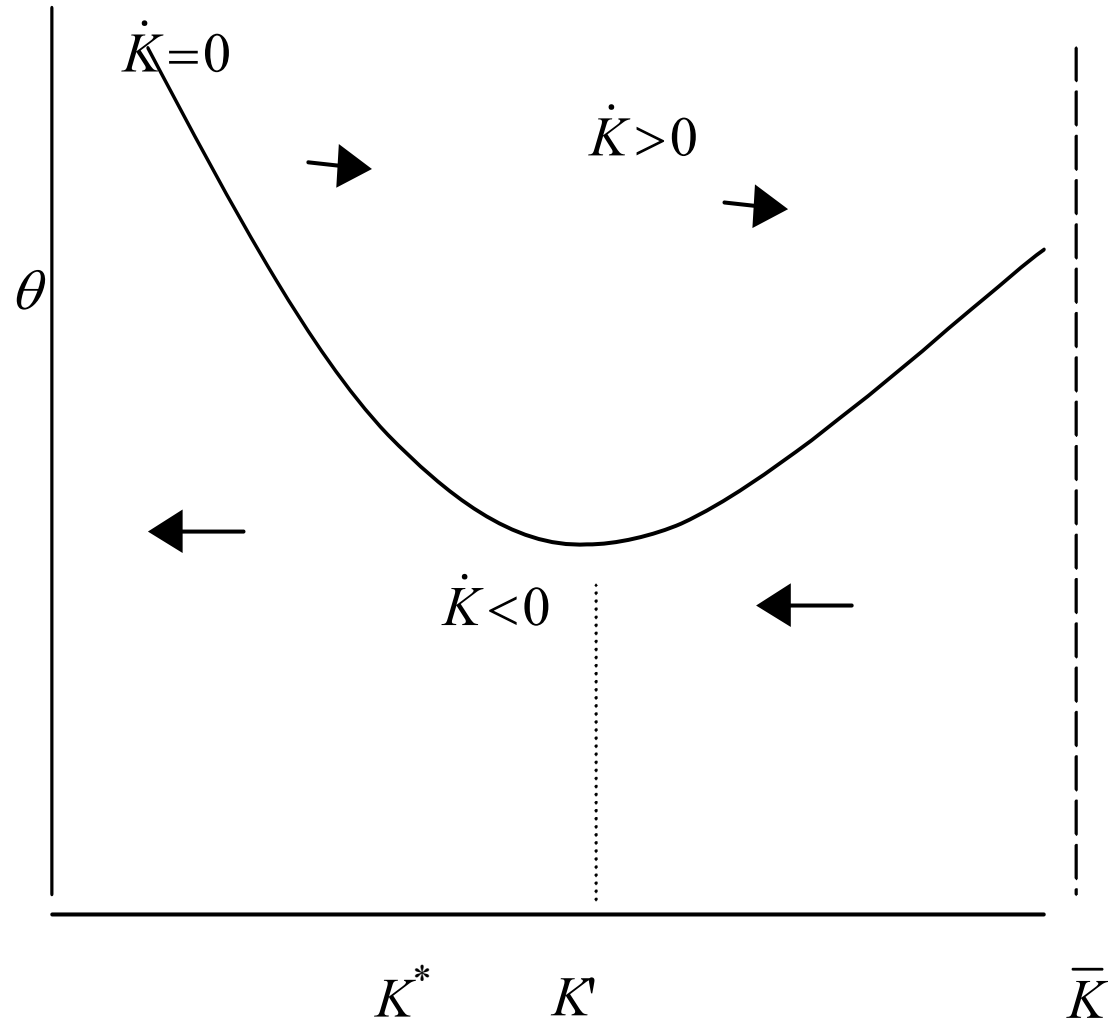
In (θ_t, K_t) space the transition dynamics of the shadowprice θ_t relative to the steady state capital stock is that



$$K^* = \left[\frac{\alpha}{\rho + \delta - \frac{\dot{\theta}}{\theta}} \right]^{\frac{1}{1-\alpha}}$$

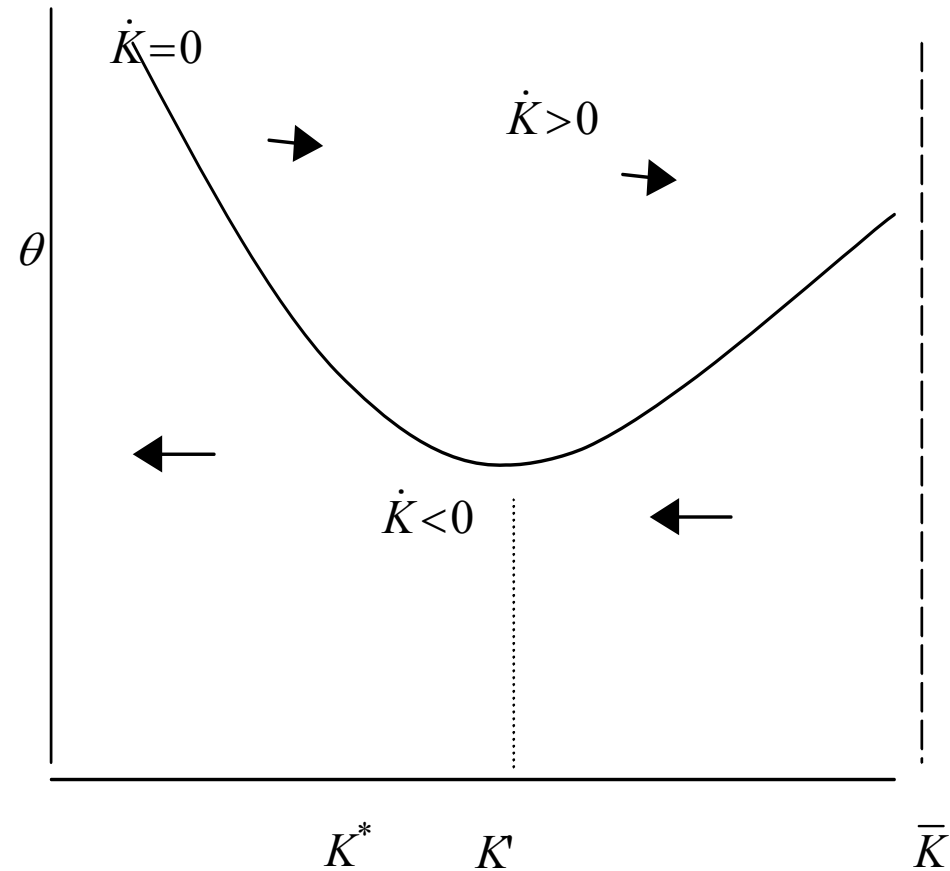
Transitional Dynamics-2

$$\bar{K} > K^* > K.$$



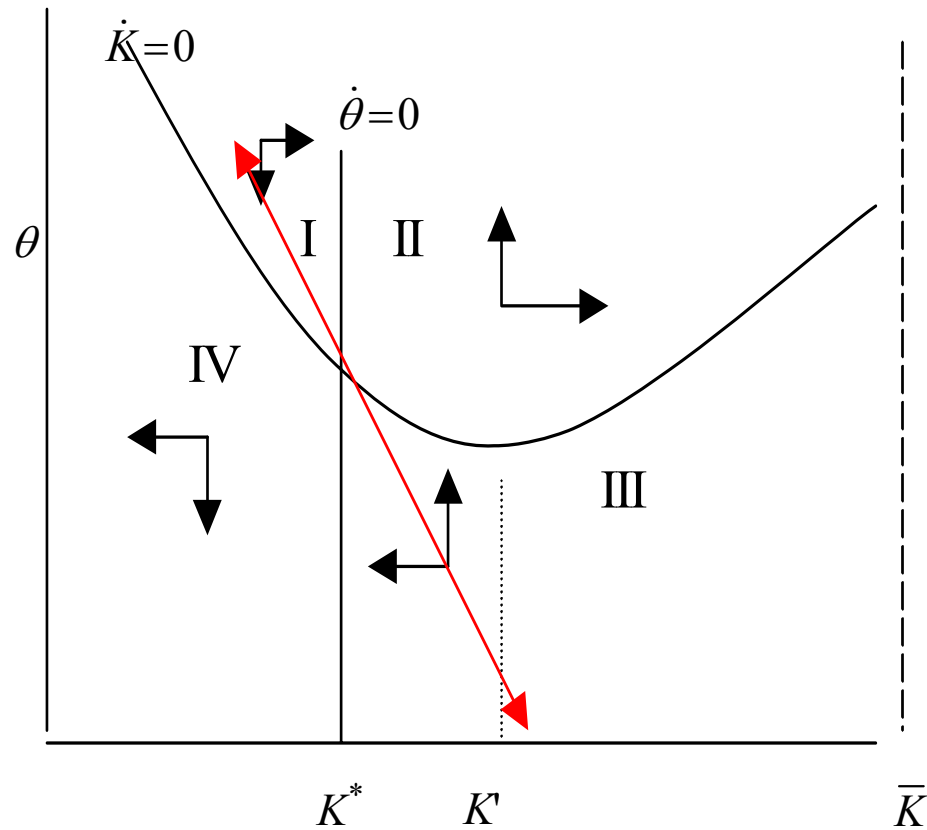
Transitional Dynamics-2

$$\bar{K} > K^* > K.$$



Saddle Point Solution

Putting all these things together the convergence to the steady state can be summarised in the following diagram.



Convergence to the steady state lies in region I and III as shown by the double arrow red line.

Brock-Mirman(1972)dynamic programming problem

Bellman's Equations

$$\text{Max } U = \sum_t^{\infty} \beta^t \ln C_t \quad 0 < \beta < 1$$

Subject to

$$K_{t+1} + C_t = AK_t^\alpha \quad 0 < \alpha < 1$$

Value function

$$V_1(K) = \ln C + \beta V_0(K)$$

Solution by Iteration

First and Second Iteration of the Value function

$$K_{t+1} = 0$$

$$V_1(K) = \ln C = \ln(AK^\alpha) = \ln A + \alpha \ln K$$

$$V_2(K) = \ln(AK^\alpha - K') + \beta(\ln A + \alpha \ln K)$$

max,k

$$\frac{\partial V_2(K)}{\partial K} = -\frac{1}{AK^\alpha - K'} + \frac{\beta\alpha}{K} = 0 \qquad \frac{1}{AK^\alpha - K'} = \frac{\beta\alpha}{K}$$

$$K' = \beta\alpha(AK^\alpha - K') \qquad K'(1 + \beta\alpha) = \beta\alpha AK^\alpha$$

$$V_2(K) = \ln\left[\frac{1}{(1 + \beta\alpha)} A\right] + \beta \ln A + \beta\alpha \ln\left(\frac{\beta\alpha}{(1 + \beta\alpha)} A\right) + \alpha(1 + \alpha\beta) \ln K'$$

Third Iteration of the Value function

$$V_3(K) = \ln(AK^\alpha - K') + \beta(\alpha(1 + \alpha\beta)\ln K')$$

$\max_{c, k}$

$$\frac{\partial V_3(K)}{\partial K} = -\frac{1}{AK^\alpha - K'} + \frac{\beta\alpha(1 + \alpha\beta)}{K'} = 0 \qquad \frac{1}{AK^\alpha - K'} = \frac{\beta\alpha(1 + \alpha\beta)}{K'}$$

$$K' = \beta\alpha(1 + \alpha\beta)(AK^\alpha - K')$$

$$K' = \frac{(\beta\alpha + \alpha^2\beta^2)}{(1 + \beta\alpha + \alpha^2\beta^2)} AK^\alpha$$

$$C = AK^\alpha - K'$$

$$C = AK^\alpha - \frac{(\beta\alpha + \alpha^2\beta^2)}{(1 + \beta\alpha + \alpha^2\beta^2)} AK^\alpha$$

$$C = \frac{1}{(1 + \beta\alpha + \alpha^2\beta^2)} AK^\alpha$$

$$V_3(K') = \beta \ln \left[\frac{A}{(1 + \beta\alpha)} \right] + \beta^2 \ln A + \beta^2 \alpha \ln \left(\frac{\beta\alpha A}{(1 + \beta\alpha)} \right) + \ln \left(\frac{A}{(1 + \beta\alpha + \alpha^2\beta^2)} \right) + \beta\alpha(1 + \alpha\beta) \ln \left[\frac{(\beta\alpha + \alpha^2\beta^2)A}{(1 + \beta\alpha + \alpha^2\beta^2)} \right] + \alpha(1 + \beta\alpha + \alpha^2\beta^2) \ln K'$$

Fourth Iteration of the Value function

$$V_4(K') = \ln C + \beta V_3(K')$$

$$V_4(K) = \ln(AK^\alpha - K') + \alpha\beta(1 + \beta\alpha + \alpha^2\beta^2) \ln K'$$

max c, k

$$\frac{1}{(AK^\alpha - K')} = \frac{\alpha\beta(1 + \beta\alpha + \alpha^2\beta^2)}{K'}$$

$$K' = \frac{(\beta\alpha + \alpha^2\beta^2 + \alpha^3\beta^3)}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3} AK^\alpha \quad C = \frac{1}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3} AK^\alpha$$

$$\begin{aligned} V_4(K') = & \ln \left[\frac{1}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3} A \right] + \beta \ln \left(\frac{A}{(1 + \beta\alpha + \alpha^2\beta^2)} \right) + \beta^2 \left[\frac{\ln A}{(1 + \beta\alpha)} \right] + \beta^3 \ln A \\ & + \beta\alpha(1 + \beta\alpha + \alpha^2\beta^2) \ln \left[\frac{(\beta\alpha + \alpha^2\beta^2 + \alpha^3\beta^3)\alpha\beta A}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3} \right] + \beta \left\{ \beta\alpha(1 + \alpha\beta) \ln \left[\frac{(\beta\alpha + \alpha^2\beta^2)A}{(1 + \beta\alpha + \alpha^2\beta^2)} \right] \right\} + \beta^2 \left\{ \beta\alpha \ln \left[\frac{\alpha\beta A}{(1 + \beta\alpha)} \right] \right\} \\ & + \alpha [1 + \beta\alpha(1 + \beta\alpha + \alpha^2\beta^2 + \alpha^3\beta^3)] \ln K \end{aligned}$$

Limits of the Value Function in Infinite Iterations

$$v_4(k) = v_0^4 + v_1^4 \ln k$$

$$\lim_{j \rightarrow \infty} v_1^j = \alpha \left[1 + \beta\alpha(1 + \beta\alpha + \alpha^2\beta^2 + \alpha^3\beta^3 + \dots + \alpha^{j-1}\beta^{j-1}) \right] = \frac{\alpha}{1 - \alpha\beta}$$

$$x_t^j = \ln \left[\frac{1}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3} A \right] \quad a^j = \sum_{t=0}^{j-1} \beta^t x_t^j \quad b^j = \sum_{t=0}^{j-2} \beta^t y_t^j$$

$$y_t^j = \beta\alpha(1 + \beta\alpha + \alpha^2\beta^2 + \dots + \alpha^{j-2}\beta^{j-2}) \ln \left[\frac{(\beta\alpha + \alpha^2\beta^2 + \alpha^3\beta^3 + \dots + \alpha^{j-2}\beta^{j-2})\alpha\beta A}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3 + \dots + \alpha^{j-2}\beta^{j-2}} \right]$$

$$\lim_{j \rightarrow \infty} x_1^j = \ln[A(1 - \alpha\beta)]$$

$$\lim_{j \rightarrow \infty} y_t^j = \frac{\beta\alpha}{1 - \beta\alpha} \ln(A\beta\alpha)$$

Limits of the Value Function in Infinite Iterations

$$\lim_{j \rightarrow \infty} a^j = \lim_{j \rightarrow \infty} \sum_{t=0}^{j-1} \beta^t x_t^j = \lim_{j \rightarrow \infty} \sum_{t=0}^{j-1} \beta^t \ln[A(1 - \beta\alpha)] = (1 - \beta)^{-1} \ln[A(1 - \beta\alpha)]$$

$$\lim_{j \rightarrow \infty} b^j = \lim_{j \rightarrow \infty} \sum_{t=0}^{j-1} \beta^t y_t^j = \lim_{j \rightarrow \infty} \sum_{t=0}^{j-2} \beta^t \frac{\beta\alpha}{1 - \beta\alpha} \ln[A\beta\alpha] = (1 - \beta)^{-1} \frac{\beta\alpha}{1 - \beta\alpha} \ln[A\beta\alpha]$$

$$\lim_{j \rightarrow \infty} v_0^j = (1 - \beta)^{-1} \left\{ \ln[A(1 - \beta\alpha)] + \frac{\beta\alpha}{1 - \beta\alpha} \ln[A\beta\alpha] \right\}$$

$$\lim_{j \rightarrow \infty} v(k) = (1 - \beta)^{-1} \left\{ \ln[A(1 - \beta\alpha)] + \frac{\beta\alpha}{1 - \beta\alpha} \ln[A\beta\alpha] \right\} + \frac{\alpha}{1 - \alpha\beta} \ln k$$

$$v(k) = v_0 + v_1 \ln k$$

References

- Bellman, R (1957) Dynamic Programming, Princeton University Press.
- Brock W and L Mirman (1972) Optimal Economic Growth and Uncertainty: the Discounted Case, *Journal of Economic Theory* 4(3):479-513.
- Cass, D. (1965): Optimum Growth in Aggregative Model of Capital Accumulation, *Review of Economic Studies*, 32:233-240.
- Ljungqvist L and T.J. Sargent (2000), *Recursive Macroeconomic theory*, MIT Press
- Parente S.L.(1994) Technology Adoption, Learning-by-Doing, and Economic Growth, *Journal of Economic Theory*, 63, pp. 346-369.
- Sargent TJ (1987) *Dynamic Macroeconomic Theory*, Chapter 1, Harvard University Press.
- Solow, R.M. (1956) "A Contribution to the Theory of Economic Growth." *Quarterly Journal of Economics* 70, 65-94.
- Stokey, N. L. and R.E. Lucas (1989) *Recursive Methods in Economic Dynamics*, Harvard UP, Cambridge, MA.
- Uzawa, H. (1962) "On a Two-Sector Model of Economic Growth," *Review of Economic Studies* 29, 40-47.

Optimal Growth Models

One Sector Ramsey Model

Role of the Financial Sector

Ramsey (1928), Cass (1965) Koopmans(1965) type Optimal Growth Model

$$U = \sum_{t=0}^{\infty} \beta^t \ln(C_t) \quad 0 < \beta < 1$$

$$Y_t = A_t K_t^\alpha \quad 0 < \alpha < 1$$

$$K_{t+1} = I_t$$

$$C_t + I_t = Y_t \quad K_0 = K_0$$

$$K_{t+1} = I_t \quad K_{t+1} = K_t(1 - \delta) + I_t$$

$$C_t = Y_t - I_t \quad C_t = AK_t^\alpha - \phi\{K_{t+1} - K_t(1 - \delta)\}$$
$$0 < \phi < 1$$

Steady State in an Optimal Growth Model

$$U = \sum_{t=0}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1}) \quad \delta=1$$

$$U_t = +\beta^t \ln(AK_t^\alpha - K_{t+1}) + \beta^{t+1} \ln(AK_{t+1}^\alpha - K_{t+2}) + \dots$$

$$\frac{\partial U_t}{\partial K_{t+1}} = -\frac{\beta^t}{C_t} + \frac{\beta^{t+1}}{C_{t+1}} \alpha AK_{t+1}^{\alpha-1} = 0 \quad \frac{C_{t+1}}{C_t} = \frac{\beta^{t+1}}{\beta^t} \alpha AK_{t+1}^{\alpha-1}$$

$$\frac{C_{t+1}}{C_t} = \beta \alpha AK_{t+1}^{\alpha-1}$$

$$U = \ln(A\bar{K}^\alpha - \bar{K}) \sum_{t=0}^{\infty} \beta^t$$

$$\frac{C_{t+1}}{C_t} = \frac{\bar{C}}{\bar{C}} = \beta \alpha A \bar{K}^{\alpha-1}$$

$$\dots = K_{t-1} = K_t = K_{t+1} = \dots = \bar{K}$$

$$\dots = C_{t-1} = C_t = C_{t+1} = \dots = \bar{C}$$

$$\bar{K} = \left(\frac{1}{\beta \alpha A} \right)^{\frac{1}{\alpha-1}} = (\beta \alpha A)^{\frac{1}{1-\alpha}}$$

$$\bar{Y} = A\bar{K}^\alpha \quad \bar{Y} = A(\beta \alpha A)^{\frac{\alpha}{1-\alpha}}$$

$$\bar{C} = \bar{Y} - \bar{I} = A\bar{K}^\alpha - \bar{K} = A(\beta \alpha A)^{\frac{\alpha}{1-\alpha}} - (\beta \alpha A)^{\frac{1}{1-\alpha}} = A^{\frac{1}{1-\alpha}} (\beta \alpha)^{\frac{1}{1-\alpha}} [(\beta \alpha)^\alpha - 1]$$

Optimal Growth Model with less than 100% depreciation

$$0 < \delta < 1$$

$$K_{t+1} = K_t(1 - \delta) + I_t \quad K_{t+1} - K_t(1 - \delta) = I_t$$

$$C_t = Y_t - I_t \quad C_t = AK_t^\alpha - K_{t+1} - K_t(1 - \delta)$$

$$U_t = \sum_{t=0}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1} + K_t(1 - \delta))$$

$$U = +\beta^t \ln(AK_t^\alpha - K_{t+1} + K(1 - \delta)) + \beta^{t+1} \ln(AK_{t+1}^\alpha - K_{t+2} + K_{t+1}(1 - \delta)) + \dots +$$

$$\frac{\partial U_t}{\partial C_t} \frac{\partial C_t}{\partial C_{t+1}} \frac{\partial C_{t+1}}{\partial K_{t+1}} = -\frac{\beta^t}{C_t} + \frac{\beta^{t+1}}{C_{t+1}} (\alpha AK_{t+1}^{\alpha-1} + (1 - \delta)) = 0$$

$$\frac{C_{t+1}}{C_t} = \beta (\alpha AK_{t+1}^{\alpha-1} + (1 - \delta)) \quad \dots = K_{t-1} = K_t = K_{t+1} = \dots = \bar{K}$$

$$\dots = C_{t-1} = C_t = C_{t+1} = \dots = \bar{C} \quad U_t = \ln(A\bar{K}^\alpha - \bar{K} + \bar{K}(1 - \delta)) \sum_{t=0}^{\infty} \beta^t$$

Steady State in an Optimal Growth Model with less than 100% depreciation

$$0 < \delta < 1$$

$$\frac{C_{t+1}}{C_t} = \frac{\bar{C}}{\bar{C}} = \beta(\alpha A \bar{K}^{\alpha-1} + (1-\delta)) \quad (\alpha A \bar{K}^{\alpha-1} + (1-\delta)) = \left(\frac{1}{\beta}\right)$$

$$(\bar{K}^{\alpha-1}) = \frac{1}{\alpha A} \left(\frac{1}{\beta} - (1-\delta) \right) \quad (\bar{K}^{\alpha-1}) = \frac{1}{\alpha A} \left(\frac{1 - \beta(1-\delta)}{\beta} \right)$$

$$\bar{K} = \left(\frac{1 - \beta(1-\delta)}{\alpha A \beta} \right)^{\frac{1}{\alpha-1}} \quad \bar{K} = \left(\frac{\alpha A \beta}{1 - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

$$\bar{Y} = A \bar{K}^{\alpha} \quad \bar{Y} = A^{\frac{2-\alpha}{1-\alpha}} \left(\frac{\alpha \beta}{1 - \beta(1-\delta)} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\bar{I} = \bar{K} - (1-\delta)\bar{K} \quad \bar{I} = \delta \bar{K} \quad \bar{I} = \delta \bar{K} = \delta \left(\frac{\alpha A \beta}{1 - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

$$\bar{C} = \bar{Y} - \bar{I} \quad \bar{C} = \left(\frac{\alpha A \beta}{1 - \beta(1-\delta)} \right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha A \beta}{1 - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

Optimal Growth Model with Financial Intermediation

$$\phi S_t = I_t$$

$$C_t = AK_t^\alpha - \phi\{K_{t+1} - K_t(1 - \delta)\}$$

$$U_t = \sum_{t=0}^{\infty} \beta^t \ln[AK_t^\alpha - \phi\{K_{t+1} - K_t(1 - \delta)\}]$$

$$U = +\beta^t \ln[AK_t^\alpha - \phi\{K_{t+1} - K_t(1 - \delta)\}] + \beta^{t+1} \ln[AK_{t+1}^\alpha - \phi\{K_{t+2} - K_{t+1}(1 - \delta)\}] + \dots +$$

$$\frac{\partial U_t}{\partial C_t} = -\frac{\phi\beta^t}{C_t} + \frac{\beta^{t+1}}{C_{t+1}}(\alpha AK_{t+1}^{\alpha-1} + \phi(1 - \delta)) = 0 \quad \frac{C_{t+1}}{C_t} = \frac{\beta}{\phi}(\alpha AK_{t+1}^{\alpha-1} + \phi(1 - \delta))$$

$$\dots = K_{t-1} = K_t = K_{t+1} = \dots = \bar{K} \quad \dots = C_{t-1} = C_t = C_{t+1} = \dots = \bar{C}$$

$$U_t = \ln(A\bar{K}^\alpha - \phi\bar{K} + \bar{K}\phi(1 - \delta)) \sum_{t=0}^{\infty} \beta^t$$

Steady State in Optimal Growth Model with Financial Intermediation

$$\frac{C_{t+1}}{C_t} = \frac{\bar{C}}{\bar{C}} = \frac{\beta}{\phi} (\alpha A \bar{K}^{\alpha-1} + \phi(1-\delta))$$

$$(\alpha A \bar{K}^{\alpha-1} + \phi(1-\delta)) = \left(\frac{\phi}{\beta}\right) \quad (\bar{K}^{\alpha-1}) = \frac{1}{\alpha A} \left(\frac{\phi}{\beta} - \phi(1-\delta)\right) \quad (\bar{K}^{\alpha-1}) = \frac{1}{\alpha A} \left(\frac{\phi - \beta\phi(1-\delta)}{\beta}\right)$$

$$\bar{K} = \left(\frac{\phi - \beta\phi(1-\delta)}{\alpha A \beta}\right)^{\frac{1}{\alpha-1}} \quad \bar{K} = \left(\frac{\alpha A \beta}{\phi - \beta\phi(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

$$\bar{Y} = A \bar{K}^{\alpha} \quad \bar{Y} = \left(\frac{\alpha A \beta}{\phi - \beta\phi(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} \quad \bar{I} = \bar{K} - (1-\delta)\bar{K}$$

$$\bar{I} = \delta \bar{K} = \delta \left(\frac{\alpha A \beta}{\phi - \beta\phi(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

$$\bar{C} = \bar{Y} - \bar{I} \quad \bar{C} = \left(\frac{\alpha A \beta}{\phi - \beta\phi(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha A \beta}{\phi - \beta\phi(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

Table 5
Capital Stock, Output, Consumption and Investment in the Steady State

| Parameters of the Infinite Horizon Model | | | | | | | | | |
|--|--------|--------|--------|---------|---------|---------|-------|--------|-----------|
| | I | II | II | IV | V | VI | VII | VIII | IX |
| Technology | 44.025 | 44.025 | 44.025 | 44.025 | 44.025 | 44.025 | 100 | 100 | 100 |
| Capital share: alpha | 0.4 | 0.4 | 0.2 | 0.6 | 0.6 | 0.6 | 0.4 | 0.4 | 0.6 |
| Beta | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 |
| Initial capital K0 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Delta | 1 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 1 | 0.05 | 0.05 |
| Intermediation cost | 1 | 1 | 1 | 1 | 1.2 | 1.05 | 1 | 1 | 1.05 |
| Infinite Horizon Economy in the Steady State | | | | | | | | | |
| Capital Stock | 100 | 2,499 | 149 | 344,202 | 218,202 | 304,677 | 392 | 9,807 | 2,369,142 |
| Output | 278 | 9,750 | 2,472 | 420,017 | 319,518 | 390,376 | 1,090 | 62,607 | 4,214,584 |
| Consumption | 178 | 7,251 | 2,323 | 75,815 | 101,315 | 85,699 | 698 | 52,800 | 1,845,441 |
| Investment | 100 | 2,499 | 149 | 344,202 | 218,202 | 304,677 | 392 | 9,807 | 2,369,142 |

Input-output table for a Hypothetical Two Sector Economy

| | X_1 | X_2 | F | Total |
|---------------|-------|-------|-----|-------|
| X_1 | 10 | 20 | 70 | 100 |
| X_2 | 30 | 20 | 150 | 200 |
| Labour input | 40 | 50 | | 90 |
| Capital input | 20 | 110 | | 130 |
| Total | 100 | 200 | 220 | |

Liontief Coefficients in Input-Output Model

$$X_1 = X_{1,1} + X_{1,2} + F_1$$

$$X_2 = X_{2,1} + X_{2,2} + F_2$$

$a_{i,j}$ share of input from row sector i to sector j

$$a_{1,1} = \frac{X_{1,1}}{X_1}$$

$$a_{1,2} = \frac{X_{1,2}}{X_2}$$

$$a_{2,1} = \frac{X_{2,1}}{X_1}$$

$$a_{2,2} = \frac{X_{2,2}}{X_2}$$

$$X_1 = a_{1,1}X_1 + a_{1,2}X_2 + F_1$$

$$X_2 = a_{2,1}X_1 + a_{2,2}X_2 + F_2$$

Input-Output Model

$$(1 - a_{1,1})X_1 - a_{1,2}X_2 = F_1$$

$$-a_{2,1}X_1 + (1 - a_{2,2})X_2 = F_2$$

$$\begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} \\ -a_{2,1} & (1 - a_{2,2}) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} \\ -a_{2,1} & (1 - a_{2,2}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$X = (I - A)^{-1} F$$

Input-Output Model

$$(I - A)^{-1} = \frac{Adj(A)}{|A|} = \frac{[C_{i,j}]}{|A|} = \frac{\begin{bmatrix} (1 - a_{2,2}) & a_{2,1} \\ a_{1,2} & (1 - a_{1,1}) \end{bmatrix}}{(1 - a_{1,1})(1 - a_{2,2}) - a_{1,2}a_{2,1}}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} (1 - a_{2,2}) & a_{2,1} \\ a_{1,2} & (1 - a_{1,1}) \end{bmatrix}}{(1 - a_{1,1})(1 - a_{2,2}) - a_{1,2}a_{2,1}} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$a_{1,1} = \frac{X_{1,1}}{X_1} = \frac{10}{100} = 0.1$$

$$a_{1,2} = \frac{X_{1,2}}{X_2} = \frac{20}{200}$$

$$a_{2,1} = \frac{X_{2,1}}{X_2} = \frac{30}{100} = 0.3$$

$$a_{2,2} = \frac{X_{2,2}}{X_2} = \frac{20}{200} = 0.1$$

Input-Output Model

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}}{(0.9)(0.9) - 0.3 \times 0.1} \begin{bmatrix} 70 \\ 50 \end{bmatrix} = \frac{\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}}{0.81 - 0.03} \begin{bmatrix} 70 \\ 50 \end{bmatrix} = \begin{bmatrix} \frac{78}{0.78} \\ \frac{156}{0.78} \\ \frac{0.78}{0.78} \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

Model is calibrated

Impact of 15 percent increase in final demand

$$\begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} = \frac{\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}}{0.81 - 0.03} \begin{bmatrix} 0.15 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{0.135}{0.78} \\ \frac{0.045}{0.78} \\ \frac{0.78}{0.78} \end{bmatrix} = \begin{bmatrix} 0.1731 \\ 0.0577 \end{bmatrix}$$

Excel Exercise on 21 sector Input-Output Model of the UK Economy.

Solution of the Input-Output Model using Matrix Inverse

$$X_1 = a_{1,1}X_1 + a_{1,2}X_2 + F_1$$

$$X_2 = a_{2,1}X_1 + a_{2,2}X_2 + F_2$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} \\ -a_{2,1} & (1-a_{2,2}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$(I-A)^{-1} = \frac{Adj(A)}{|A|} = \frac{[C_{i,j}]}{|A|} = \frac{\begin{bmatrix} (1-a_{2,2}) & a_{2,1} \\ a_{1,2} & (1-a_{1,1}) \end{bmatrix}}{(1-a_{1,1})(1-a_{2,2})-a_{1,2}a_{2,1}}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} (1-a_{2,2}) & a_{2,1} \\ a_{1,2} & (1-a_{1,1}) \end{bmatrix}}{(1-a_{1,1})(1-a_{2,2})-a_{1,2}a_{2,1}} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Solution of the Input-Output Model using Cramer's rule

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} \\ -a_{2,1} & (1-a_{2,2}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} \\ -a_{2,1} & (1-a_{2,2}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$X_1 = \frac{\begin{vmatrix} F_1 & -a_{1,1} \\ F_2 & (1-a_{2,2}) \end{vmatrix}}{(1-a_{1,1})(1-a_{2,2})-a_{1,2}a_{2,1}}$$

$$X_2 = \frac{\begin{vmatrix} (1-a_{1,1}) & F_1 \\ -a_{2,1} & F_2 \end{vmatrix}}{(1-a_{1,1})(1-a_{2,2})-a_{1,2}a_{2,1}}$$

Three Sector Input-Output Model of the UK

| | Manf | Dis | Serv | Cons | Gov | In v | Exp | Total demand |
|---------|---------|---------|---------|---------|---------|---------|---------|--------------|
| Manf | 173,863 | 22,397 | 48,669 | 79,745 | 0 | 70,629 | 135,961 | 531,265 |
| Dis | 23,676 | 5,667 | 10,841 | 116,237 | 0 | 4,611 | 20,789 | 181,822 |
| Serv | 63,622 | 48,036 | 186,526 | 160,908 | 141,031 | 11,088 | 39,238 | 650,448 |
| Import | 73,133 | 10,542 | 34,791 | 51,083 | 0 | 29,947 | 7,555 | 207,052 |
| Tax | 6,715 | 6,559 | 22,744 | 51,875 | 0 | 5,564 | -33 | 93,424 |
| wage | 114,014 | 60,487 | 212,218 | 0 | 0 | 0 | 0 | 386,718 |
| capital | 76,241 | 28,134 | 134,659 | 0 | 0 | 0 | 0 | 239,034 |
| Total | 531,265 | 181,822 | 650,447 | 459,848 | 141,031 | 121,839 | 203,511 | |

Derived from 123 sectors input-output table for the UK economy.

Consistency of the Benchmark

| | |
|-----------------|---------|
| Tax | 93,424 |
| G | 141,031 |
| balance | -47,607 |
| Income | 625,752 |
| Consumption | 459,848 |
| saving | 165,904 |
| Investment | 121,839 |
| Sav - Inv | 44,065 |
| National saving | -3,542 |
| Exports | 203,511 |
| imports | 207,052 |
| Tradebal | -3.541 |

Leontief Coefficients

| | Manf | Dis | Serv |
|---------|----------|----------|----------|
| Manf | 0.327263 | 0.123182 | 0.074824 |
| Dis | 0.044565 | 0.031169 | 0.016667 |
| Serv | 0.119756 | 0.264192 | 0.286766 |
| Import | 0.137659 | 0.05798 | 0.053488 |
| Tax | 0.01264 | 0.036074 | 0.034966 |
| wage | 0.214608 | 0.332669 | 0.326264 |
| capital | 0.143508 | 0.154734 | 0.207026 |
| Total | 1 | 1 | 1 |

$$X = (I - A)^{-1} F \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1 - a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1 - a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

A- Matrix

| | Manf | Dis | Serv |
|------|----------|----------|----------|
| Manf | 0.327263 | 0.123182 | 0.074824 |
| Dis | 0.044565 | 0.031169 | 0.016667 |
| Serv | 0.119756 | 0.264192 | 0.286766 |

I-Matrix

| | | |
|---|---|---|
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

$(I - A)$

| | | |
|----------|----------|----------|
| 0.672737 | -0.12318 | -0.07482 |
| -0.04457 | 0.968831 | -0.01667 |
| -0.11976 | -0.26419 | 0.713234 |

$(I - A)^{-1}$

| | | |
|----------|----------|----------|
| 1.531982 | 0.240141 | 0.166328 |
| 0.075375 | 1.050606 | 0.032458 |
| 0.285149 | 0.429481 | 1.442014 |

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1-a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1-a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 286335.5 \\ 141637.7 \\ 352264.3 \end{bmatrix} = \begin{bmatrix} 531265.3 \\ 181821.7 \\ 650448.9 \end{bmatrix}$$

Thus the model calibrated the benchmark economy.

Impact of Change in Final Demand on Gross Output

$$\begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \end{bmatrix} = \begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1 - a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1 - a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}$$

Impact of 20 percent Change in Final Demand on Gross Output

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 57267.1 \\ 28327.54 \\ 70452.86 \end{bmatrix} = \begin{bmatrix} 106253.1 \\ 36364.33 \\ 130089.8 \end{bmatrix}$$

Employment Impacts of 20 percent increase in the Final Demand

| | | | |
|------|----------|----------|----------|
| wage | 0.214608 | 0.332669 | 0.326264 |
|------|----------|----------|----------|

$$\begin{bmatrix} \Delta L_1 \\ \Delta L_2 \\ \Delta L_3 \end{bmatrix} = \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix} \begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1 - a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1 - a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.214608 & 0 & 0 \\ 0 & 0.332669 & 0 \\ 0 & 0 & 0.326264 \end{bmatrix} \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 57267.1 \\ 28327.54 \\ 70452.86 \end{bmatrix} = \begin{bmatrix} 22802.8 \\ 12097.28 \\ 42443.64 \end{bmatrix}$$

Capital Accumulation Impacts of 20 percent increase in the Final Demand

| | | | |
|---------|----------|----------|----------|
| Capital | 0.143508 | 0.154734 | 0.207026 |
|---------|----------|----------|----------|

$$\begin{bmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1-a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1-a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.143508 & 0 & 0 \\ 0 & 0.154734 & 0 \\ 0 & 0 & 0.207026 \end{bmatrix} \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 57267.1 \\ 28327.54 \\ 70452.86 \end{bmatrix} = \begin{bmatrix} 15248.16 \\ 5626.782 \\ 26931.94 \end{bmatrix}$$

Revenue Impacts of 20 percent increase in the Final Demand

| | | | |
|-----|---------|----------|----------|
| Tax | 0.01264 | 0.036074 | 0.034966 |
|-----|---------|----------|----------|

$$\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \\ \Delta T_3 \end{bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} (1 - a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1 - a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1 - a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.01264 & 0 & 0 \\ 0 & 0.036074 & 0 \\ 0 & 0 & 0.034966 \end{bmatrix} \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 57267.1 \\ 28327.54 \\ 70452.86 \end{bmatrix} = \begin{bmatrix} 1343.013 \\ 1311.797 \\ 4548.727 \end{bmatrix}$$

Import Impacts of 20 percent increase in the Final Demand

| | | | |
|--------|----------|---------|----------|
| Import | 0.137659 | 0.05798 | 0.053488 |
|--------|----------|---------|----------|

$$\begin{bmatrix} \Delta M_1 \\ \Delta M_2 \\ \Delta M_3 \end{bmatrix} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} (1-a_{1,1}) & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & (1-a_{2,2}) & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & (1-a_{3,3}) \end{bmatrix}^{-1} \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.137659 & 0 & 0 \\ 0 & 0.05798 & 0 \\ 0 & 0 & 0.053488 \end{bmatrix} \begin{bmatrix} 0.672737 & -0.12318 & -0.07482 \\ -0.04457 & 0.968831 & -0.01667 \\ -0.11976 & -0.26419 & 0.713234 \end{bmatrix}^{-1} \begin{bmatrix} 57267.1 \\ 28327.54 \\ 70452.86 \end{bmatrix} = \begin{bmatrix} 14626.7 \\ 2108.41 \\ 6958.249 \end{bmatrix}$$

Theoretical Aspects of General Equilibrium Analysis

Economic system is described by a system of $n(n-1)$ relative prices that clear all goods and factor markets. It is often stated in terms of vectors of prices, demand and supply and excess demand functions for inputs and outputs.

Vector of prices $p = (p_1, p_2, \dots, p_j, \dots, p_n)$

Demand for commodities are expressed in terms of the price vector

$$X_j^d = X_j^d(p) = X_j^d(p_1, p_2, \dots, p_j, \dots, p_n)$$

Supply functions defined similarly

$$X_j^s = X_j^s(p) = X_j^s(p_1, p_2, \dots, p_j, \dots, p_n)$$

The excess demand functions reflect the gap between demand and supply for each commodity

$$E_j(p) = X_j^d(p) - X_j^s(p) \quad \text{for } j = 1, 2, \dots, n$$

Economy has n excess demand functions

$$E(p) = (E_1(p), E_2(p), \dots, E_j(p), \dots, E_n(p)).$$

The general equilibrium is a price vector, p^* , such that

$$\begin{aligned} p^* &\geq 0 \\ E(p^*) &\leq 0 \\ \text{if } E(p^*) < 0 & \quad p^* = 0 \end{aligned}$$

Properties of Excess Demand Functions and Pure Exchange

The excess demand functions fulfil a number of properties:

- 1) it is single valued continuous function
- 2) it is bounded from below $E(p) \geq b$ for all p
- 3) it is homogenous of degree zero in all prices $E(\alpha p) = E(p)$ for all α ; only relative price matters
- 4) it satisfies the Walras' law; $p \cdot E(p) = \sum_{i=1}^n p_i E_i(p) = 0$ for all $p \geq 0$

If the excess demand functions satisfy above properties then, the existence of the general equilibrium is guaranteed by fixed point theorems.

An example of general equilibrium Model in Pure Exchange and in an Economy with Production (see Richard Cornwall for details):

Pure exchange:

Consumption set: $x_i \in X_i$ (point below the budget line with $m = 2$).

Feasibility condition: $\sum_{i=1}^m x_i = \sum_{i=1}^m e_i$

Equilibrium can be represented using a budget constraint for consumer i and Edgeworth box diagram in case of two consumers and Euclidian space for m consumers.

Prove existence using the Fixed point theorem such that equilibrium price vector \bar{p} and the allocation \bar{x} each of $1 \times m$ dimension and equilibrium allocation (\bar{p}, \bar{x}) are feasible and maximise consumer's preferences over commodities.

Equilibrium with production

A possible allocation $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ for $i = 1, \dots, m$ consumers and $j = 1, \dots, n$ producers satisfying following:

Consumption set: $x_i \in X_i$

Production possibility set: $y_j \in Y_j$

Resource balance condition:
$$\sum_{i=1}^m x_i = \sum_{i=1}^m e_i + \sum_{j=1}^n y_j$$

Wealth of the consumer: $w_i(p) = p \cdot e_i + \sum_j \theta_{i,j} \pi_j(p)$

Supply correspondence: $s_j(p) = \{y \in Y_j : y' \in Y_j \Rightarrow p \cdot y \geq p \cdot y'\}$

Excess demand correspondence: $z(p) = \sum_i (d_i(p) - e_i) - \sum_i s_i(p)$

A competitive equilibrium is a pair of prices, demand and supply $(\bar{p}, (\bar{x}, \bar{y}))$ with \bar{p} a vector in R^l and $\bar{x}_i \in d_i(\bar{p})$ for consumer i to m , and $\bar{y}_j \in s_j(\bar{p})$ for firms j to n and where the excess demand $z(\bar{p})$ is zero in equilibrium.

Robinson Crusoe Economy: Labour Leisure Choice and Consumption

Commodity space: R^2 (leisure and food)

Consumer characteristic: $X_i = R_+^2$

Endowment: $e_i = (24, 0)$

Preference relation: $\succsim_i \quad U(L, F) = LF$

Producer characteristics: $Y_i = \{(-L, F) : L \geq 0, F \leq \sqrt{L}\}$;

where $F = \sqrt{L}$ is the production function.

$$\text{Profit : } \pi = P_F F - P_L F^2 \rightarrow F = \frac{P_F}{2P_L} \text{ and } \pi = P_F \frac{P_F}{2P_L} - P_L \frac{1}{4} \left(\frac{P_F}{P_L} \right)^2 = \frac{1}{4} \frac{P_F^2}{P_L}$$

$$\text{Supply of Food } s(P_L, P_F) = \left(-\frac{1}{4} \left(\frac{P_F}{P_L} \right)^2, \frac{P_F}{2P_L} \right)$$

$$\text{Value of endowment: } w_1(p) = p \cdot e_1 + \sum_j \theta_{i,j} \pi_j(p) = 24P_L + \frac{1}{4} \frac{P_F^2}{P_L}$$

Equilibrium Prices and Allocations in Robinson Crusoe Economy

$$\text{Demand: } d(P_L, P_F) = \left(\frac{w_1(p)}{2P_L}, \frac{w_1(p)}{2P_F} \right) = \left(12 + \frac{1}{8} \left(\frac{P_F}{P_L} \right)^2, 12 \frac{P_L}{P_F} + \frac{1}{8} \frac{P_F}{P_L} \right)$$

$$\begin{array}{l} \text{Leisure:} \end{array} \quad \begin{array}{c} \text{Demand} \\ 12 + \frac{1}{8} \left(\frac{P_F}{P_L} \right)^2 \end{array} = \begin{array}{c} \text{supply} \\ 24 - \frac{1}{4} \left(\frac{P_F}{P_L} \right)^2 \end{array}$$

$$\text{Food:} \quad 12 \frac{P_L}{P_F} + \frac{1}{8} \frac{P_F}{P_L} = \frac{P_F}{2P_L}$$

Both of above markets give the same solution; by the Walras' law when labour market clears food market clears as well. This model determines the only relative

$$\text{price } 12 + \frac{1}{8} \left(\frac{P_F}{P_L} \right)^2 = 24 - \frac{1}{4} \left(\frac{P_F}{P_L} \right)^2 \rightarrow \frac{3}{8} \left(\frac{P_F}{P_L} \right)^2 = 12 \rightarrow \left(\frac{P_F}{P_L} \right)^2 = 32 \quad \frac{P_F}{P_L} = 4\sqrt{2}$$

Normalise $P_L = 1$

Equilibrium solution:

$$\text{price vector: } \bar{p} = (1, 4\sqrt{2})$$

$$\text{demand vector: } \bar{x} = \left(\left(16, 2\sqrt{2} \right) \right)$$

$$\text{output vector: } \bar{y} = \left(\left(-8, 2\sqrt{2} \right) \right)$$

Brouwer's Fixed Point Theorem

A continuous function from a compact convex set into itself has a fixed point. Prices can be normalised to made their sum equal to one using the homogeneity assumption as

$$S = \left\{ p / \sum_{i=1}^n p_i = 1, p \geq 0 \right\}$$

Consider a set of the excess demand functions evaluated a p . Update or adjust this price according to following rules for each of j commodities:

$$\left. \begin{aligned} \bar{p}_j &= p_j \quad \text{if} \quad E_j(p) = 0 \\ \bar{p}_j &= p_j + \Delta \quad \text{if} \quad E_j(p) > 0 \\ \bar{p}_j &= \max(0, p_j - \Delta_j) \quad \text{if} \quad E_j(p) < 0 \end{aligned} \right\} \quad \text{for} \quad j = 1, 2, \dots, n$$

Here Δ represents a very small positive constant. Following above rule in each iteration find new prices as

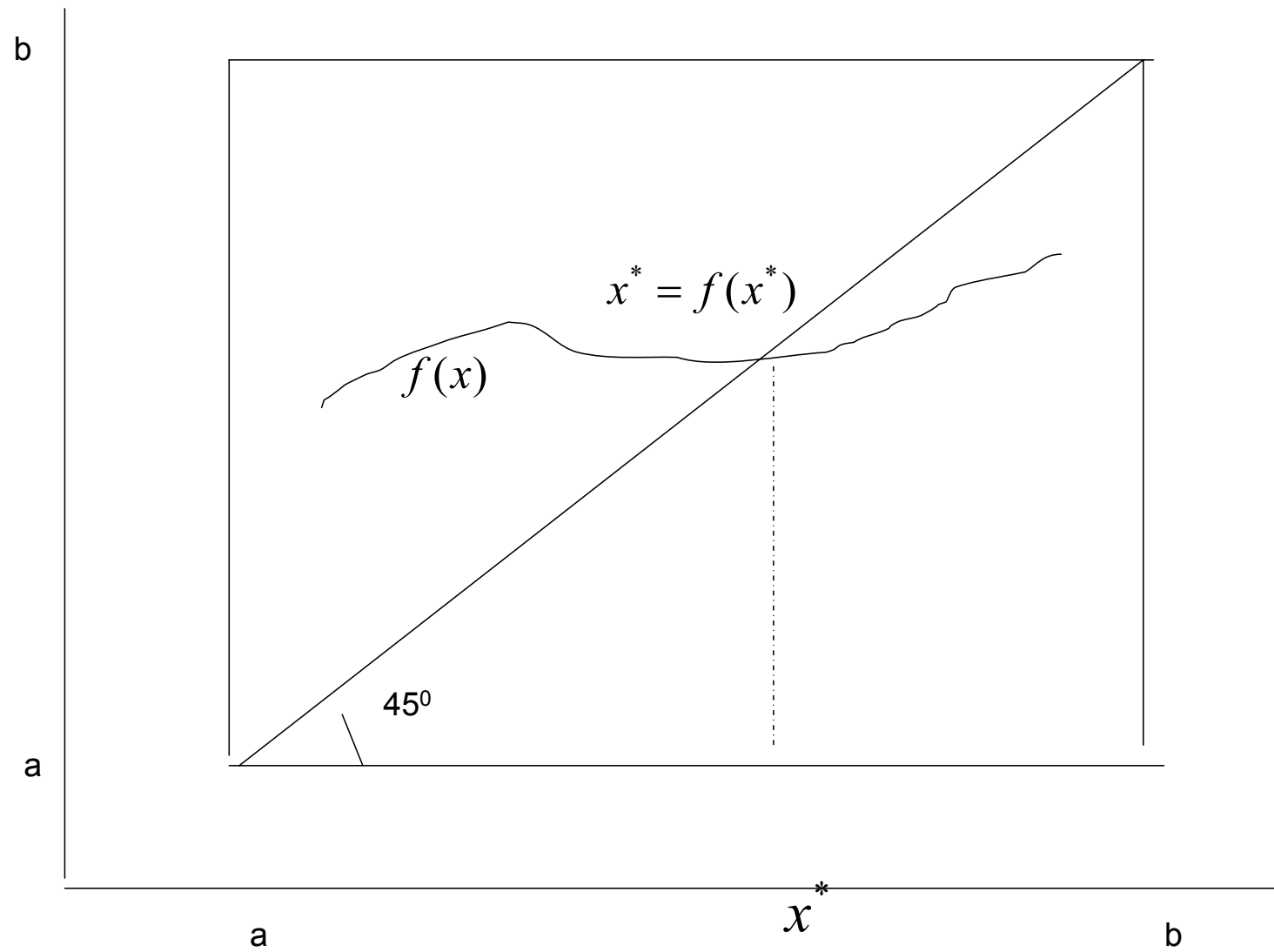
$$p \rightarrow E(p) \rightarrow \bar{p}$$

\bar{p} remains unchanged if excess demand is zero, $E(p) = 0$; \bar{p} rises if $E(p) > 0$ and \bar{p} falls if $E(p) < 0$ and if $E(p) < 0$ then $p = 0$.

The fixed equilibrium point is found by continuous transformation of the nonempty convex set onto itself

$$p^* \rightarrow E(p^*) \rightarrow p^*$$

Graphical Illustration of Brouwer fixed point theorem



More explicit proof of the Brouwer's theorem requires use of Sperner's Lemma and Kanster-Kuatowski-Mazzurkiewicz (KKM) theorem which can be found at Ross Starr (1997) on fixed point algorithm and convex hull.

Uniqueness of Equilibrium

One approach to establish the uniqueness of the equilibrium is based on evaluation of Jacobian matrix of excess demand functions. Taking n th commodity as a numeraire and differentiability of the excess demand functions Jacobian matrix is normalised as following (Hicksian method):

$$J = \begin{pmatrix} \frac{\partial E_1}{\partial p_1} & \frac{\partial E_1}{\partial p_2} & \cdots & \frac{\partial E_1}{\partial p_{n-1}} \\ \frac{\partial E_2}{\partial p_1} & \frac{\partial E_2}{\partial p_2} & \cdots & \frac{\partial E_2}{\partial p_{n-1}} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial E_{n-1}}{\partial p_1} & \frac{\partial E_{n-1}}{\partial p_2} & \cdots & \frac{\partial E_{n-1}}{\partial p_{n-1}} \end{pmatrix}$$

The equilibrium is unique if the principal minors of J alternate in sign, its values is positive for even number of rows and columns; and negative for uneven number of rows and columns.

Stability of Equilibrium and Walrasian Auctioneer

Intrilligator (1971) suggests decentralised iterative computation of equilibrium, leading to time paths for quantities and prices. If these time paths eventually converge to equilibrium values then the process is stable.

$$\begin{aligned}
 x &= (x_1, x_2, \dots, x_j, \dots, x_n) \\
 p &= (p_1, p_2, \dots, p_j, \dots, p_n)
 \end{aligned}
 \quad
 \frac{\partial p_j}{dt} = \dot{p}_j \begin{cases} > \\ < \\ = \end{cases} 0 \quad \text{if } E_j(p) \begin{cases} > \\ < \\ = \end{cases} 0 \quad j = 1, 2, \dots, n.$$

Equilibrium is locally stable if

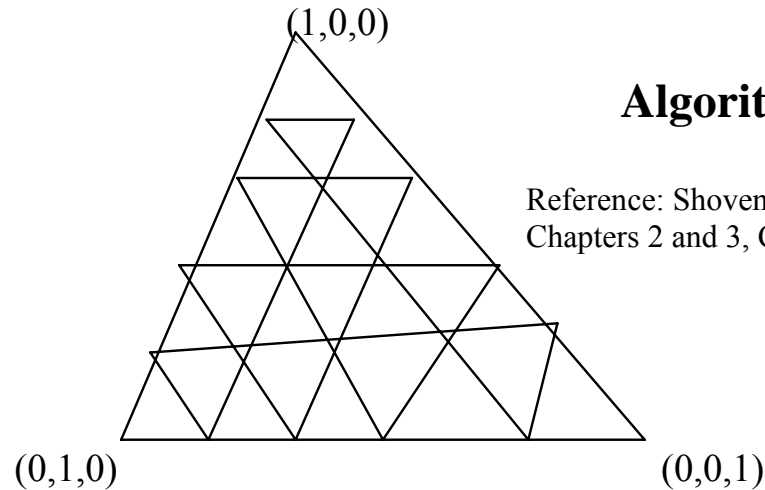
$$\lim_{t \rightarrow \infty} p(t) = p^* \quad \text{given} \quad |p(t_0) - p^*| < \delta$$

where t_0 denote the initial starting point and $|p(t_0) - p^*|$ the Euclidean norm in the price space.

Equilibrium is globally stable if it is reached regardless of any starting point

$$\lim_{t \rightarrow \infty} p(t) = p^* \quad \text{for any } p(t_0)$$

Scarf algorithm (triangulation of simplex)



Algorithms to Compute Equilibrium

Reference: Shoven J. B. and J. Whalley (1992) Applying General Equilibrium, Chapters 2 and 3, Cambridge University Press.

Triangulation of a unit simplex with grid size of 5.

Steps:

1. select an initial simplex and grid size D (5 in above simplex).
2. calculate the labels of vertices of the initial simplex by applying the labelling rule.
3. If the simplex is completely labelled go to step 5 else to step 4.
4. generate a new vertex applying the replacement rule. Determine the label of new vertex.
5. Since the vertex is completely labelled an approximation to competitive equilibrium has been found where the excess demand is less than a selected ε

Exercise (i) find the co-ordinates for any five sub-simplices (ii) show division of this unit simplex for grid size of 10.

There are many other algorithm to compute equilibrium: Merrill's algorithm, Van der Laan and Talman algorithm, MCP algorithm and Judd algorithms (See Handbooks of Computational Economics, North Holland for details on these).

Newton method for Computing General Equilibrium

As stated by Shoven J. B. and J. Whalley (1992) the system of non-linear equations are solved by using successive linear approximations to the non-linear system. Point estimates of derivatives of market excess demand functions are repeatedly evaluated to estimate adjustments necessary to the initial guess of the equilibrium prices. The problem is essentially finding zero for excess demand functions:

$$G_i = G_i(P_1, \dots, P_j, \dots, P_N)$$

If the excess demand functions are not zero then the Jacobians from the linearised excess demand functions can be evaluated $\frac{\partial G_i}{\partial P_j}$ (for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N$) to find out the change necessary in prices to make the excess demand zero

$$\Delta P_i^* = \sum_{j=1}^N \frac{\partial G_i}{\partial P_j} G_i(P)$$

Public Goods and Private Choices

$$\text{Max } u_1 = u_1(x_1, G)$$

$$\text{subject to } 1) \bar{u}_2 = u_1(x_2, G)$$

$$2) x_1 + x_2 + c(G) = w_1 + w_2$$

Lagrangian of the problem:

$$L = u_1(x_1, G) - \lambda (u_1(x_2, G) - \bar{u}_2) - \mu (x_1 + x_2 + c(G) - w_1 - w_2)$$

Three first order conditions:

$$1. \frac{\partial L}{\partial x_1} = \frac{\partial u_1(x_1, G)}{\partial x_1} - \mu = 0 \quad \text{or} \quad \mu = \frac{\partial u_1(x_1, G)}{\partial x_1}$$

$$2. \frac{\partial L}{\partial x_2} = -\lambda \frac{\partial u_2(x_2, G)}{\partial x_2} - \mu = 0 \quad \text{or} \quad -\frac{\partial u_2(x_2, G)}{\partial x_2} = \frac{\mu}{\lambda}$$

$$3. \frac{\partial L}{\partial G} = \frac{\partial u_1(x_1, G)}{\partial G} - \lambda \frac{\partial u_2(x_2, G)}{\partial G} - \mu \frac{\partial c(G)}{\partial G} = 0 \quad \text{or}$$

$$\frac{1}{\mu} \frac{\partial u_1(x_1, G)}{\partial G} - \frac{\lambda}{\mu} \frac{\partial u_2(x_2, G)}{\partial G} = \frac{\partial c(G)}{\partial G}$$

Pareto optimality requires that sum of the marginal rate of substitution between private and public goods by two individuals should equal the marginal cost of provision of public goods (see Varian 35).

$$\frac{\frac{\partial u_1(x_1, G)}{\partial G}}{\frac{\partial u_1(x_1, G)}{\partial x_1}} + \frac{\frac{\partial u_2(x_2, G)}{\partial G}}{\frac{\partial u_2(x_2, G)}{\partial x_2}} = \frac{\partial c(G)}{\partial G} \quad \text{or} \quad MRS_1 + MRS_2 = MC(G) \quad \text{Q.E.D.}$$

Global Economy with Two Countries

Country A

Technologies: $X_{1,A} = 5L_{1,A}$; $X_{2,A} = 2L_{2,A}$

Resources $L_{1,A} + L_{2,A} = 200$

Preferences $U_A = X_{1,A}^\alpha X_{2,A}^{(1-\alpha)}$ α is 0.4

Production possibility frontier of A:

$$\frac{1}{5}X_{1,A} + \frac{1}{2}X_{2,A} = 200$$

Country B

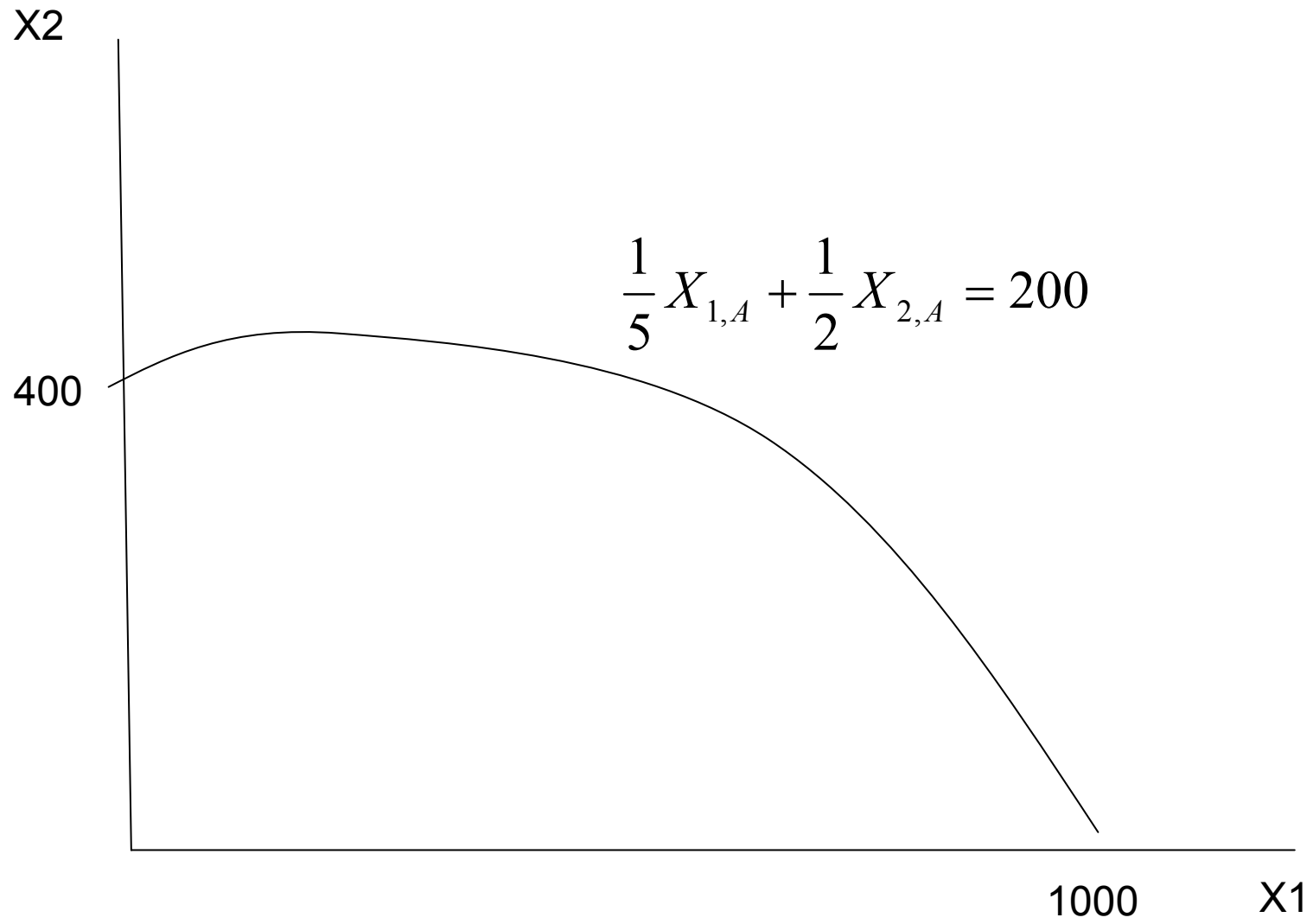
Technologies: $X_{1,B} = 2L_{1,B}$ and $X_{2,B} = 5L_{2,B}$

Resource: $L_{1,B} + L_{2,B} = 400$

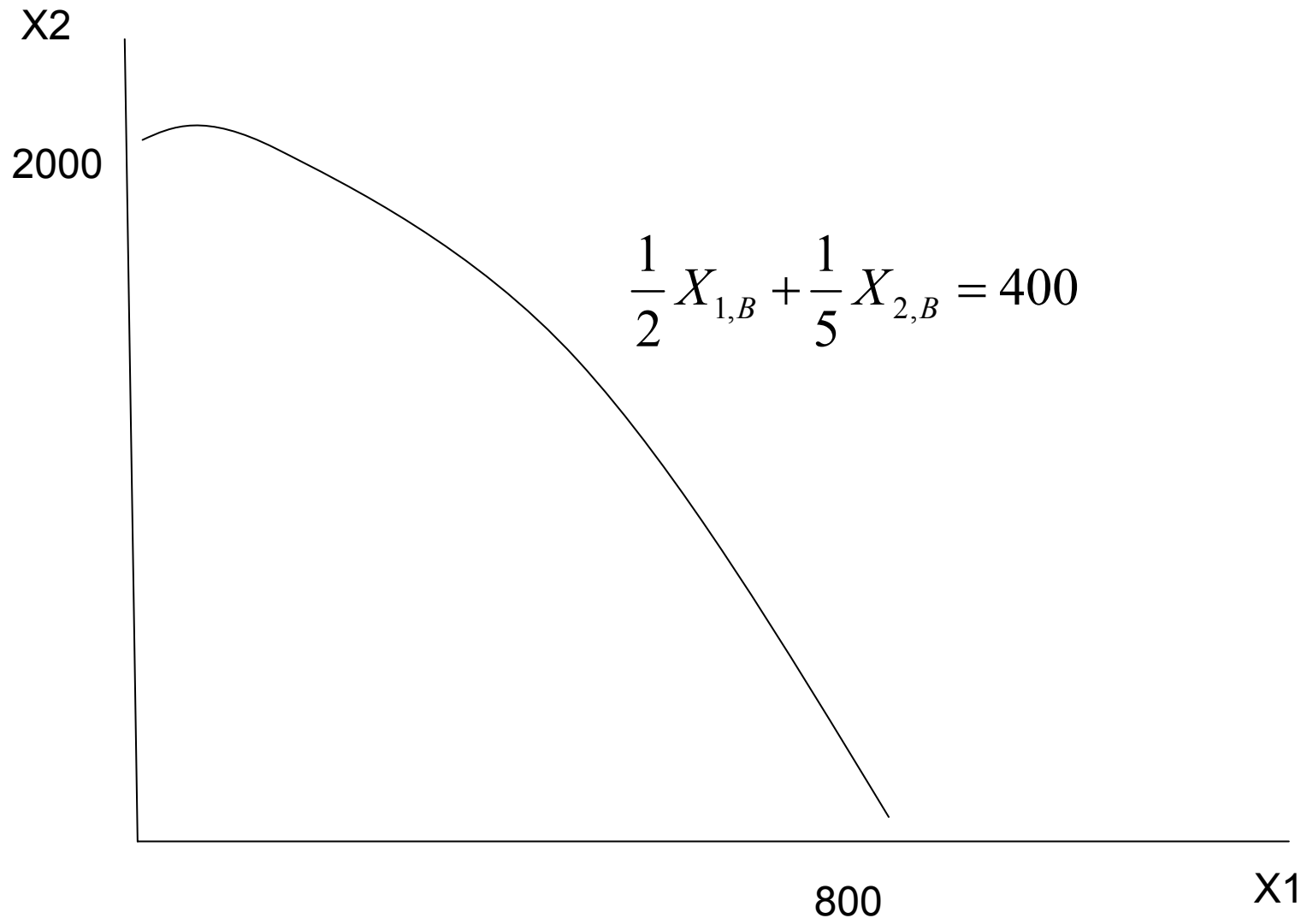
Preferences: $U_B = X_{1,B}^\beta X_{2,B}^{(1-\beta)}$ β 0.6.

Production possibility frontier: $\frac{1}{2}X_{1,B} + \frac{1}{5}X_{2,B} = 400$

Production Possibility Frontier of Country A



Production Possibility Frontier of Country B



Autarchy (no trade) Solution for Country A

$$L_A = X_{1,A}^\alpha X_{2,A}^{(1-\alpha)} + \lambda \left[200 - \frac{1}{5} X_{1,A} - \frac{1}{2} X_{2,A} \right]$$

$$\frac{\partial L_A}{\partial X_{1,A}} = \alpha X_{1,A}^{\alpha-1} X_{2,A}^{(1-\alpha)} - \frac{1}{5} \lambda = 0 ;$$

$$\frac{\partial L_A}{\partial X_{2,A}} = (1-\alpha) X_{1,A}^\alpha X_{2,A}^{(-\alpha)} - \frac{1}{2} \lambda = 0$$

$$\frac{\partial L_A}{\partial \lambda} = 200 - \frac{1}{5} X_{1,A} - \frac{1}{2} X_{2,A} = 0$$

From first two FOC, $\frac{\partial L_A}{\partial X_{1,A}} = \frac{\alpha}{(1-\alpha)} \frac{X_{2,A}}{X_{1,A}} = \frac{2}{5}$ or

$X_{2,A} = \frac{2(1-\alpha)}{5\alpha} X_{1,A}$ substituting this into the 3rd first order condition

$$200 = \frac{1}{5} X_{1,A} + \frac{1}{2} X_{2,A} = \frac{1}{5} X_{1,A} + \frac{1}{2} \left(\frac{2(1-\alpha)}{5\alpha} X_{1,A} \right) = \left[\frac{1}{5} + \frac{1}{5} \left(\frac{0.6}{0.4} \right) \right] X_{1,A}$$

$$200 = \frac{1}{5} \left[1 + \left(\frac{3}{2} \right) \right] X_{1,A} \quad X_{1,A} = 400$$

This implies $X_{2,A} = \frac{2(1-\alpha)}{5\alpha} X_{1,A} = \frac{2 \cdot 0.6}{5 \cdot 0.4} (400) = \frac{2 \cdot 3}{5 \cdot 2} (400) = 240$

This solution exhausts the total resource available in the country which can be checked putting these values in the production possibility frontier:

$$\frac{1}{5} X_{1,A} + \frac{1}{2} X_{2,A} = 200 \quad \text{and} \quad \frac{1}{5} (400) + \frac{1}{2} (240) = 80 + 120 = 200$$

Autarchy (no trade) Solution for Country B

$$L_B = X_{1,B}^\beta X_{2,B}^{(1-\beta)} + \lambda \left[400 - \frac{1}{2} X_{1,B} - \frac{1}{5} X_{2,B} \right]$$

$$\frac{\partial L_B}{\partial X_{1,B}} = \beta X_{1,B}^{\beta-1} X_{2,B}^{(1-\beta)} - \frac{1}{2} \lambda = 0$$

$$\frac{\partial L_B}{\partial X_{2,B}} = (1-\beta) X_{1,B}^\beta X_{2,B}^{(-\beta)} - \frac{1}{5} \lambda = 0$$

$$\frac{\partial L_B}{\partial \lambda} = 400 - \frac{1}{2} X_{1,B} - \frac{1}{5} X_{2,B}$$

From first two FOC, $\frac{\partial L_B}{\partial X_{1,B}} = \frac{\beta}{(1-\beta)} \frac{X_{2,B}}{X_{1,B}} = \frac{5}{2}$ or

$$X_{2,B} = \frac{5(1-\beta)}{2\beta} X_{1,B} \text{ substituting this into the 3}^{\text{rd}} \text{ first order condition}$$

$$400 = \frac{1}{2} X_{1,B} + \frac{1}{5} X_{2,B} = \frac{1}{2} X_{1,B} + \frac{1}{5} \left(\frac{5(1-\beta)}{2\beta} X_{1,B} \right) = \left[\frac{1}{2} + \frac{1}{2} \left(\frac{0.4}{0.6} \right) \right] X_{1,B}$$

$$400 = \left[\frac{1}{2} + \frac{1}{3} \right] X_{1,B} \quad X_{1,B} = \frac{400 \times 6}{5} = 480$$

This implies $X_{2,B} = \frac{5(1-\beta)}{2\beta} X_{1,B} = \frac{5}{2} \frac{4}{6} (480) = \frac{5}{3} (480) = 5 \times 160 = 800$

This satisfies the resource balance equation: $\frac{1}{2} X_{1,B} + \frac{1}{5} X_{2,B} = 400$

$$\frac{1}{2}(480) + \frac{1}{5}(800) = 240 + 160 = 400$$

Comparative Advantages of Countries A and B

Country A is more efficient in producing X_1 :

$X_{1,A} = 5L_{1,A}$ implies one unit of labour can produce five units of X_1 and 1000 units of X_1 if it only produces X_1

$X_{2,A} = 2L_{2,A}$ implies it can produce 400 units of X_2 if it produces only X_2

Country B is more efficient in producing X_2 than X_1 ;

Given its technologies $X_{1,B} = 2L_{1,B}$ and $X_{2,B} = 5L_{2,B}$

In complete specialisation B can produce 2000 units of X_2 but only 800 units of X_1 .

Global Market Clearing

A complete specialisation is desirable to reap all comparative advantage.

A competitive equilibrium implies global market clearing for both goods

$$X_{1,A} + X_{1,B} = 1000 \quad X_{2,A} + X_{2,B} = 2000$$

Competitive price can be obtained using one of these two market clearing conditions;

Let the price of good 1 be numeraire $P_1 = 1$.

$$\frac{\alpha X_1}{P_1} + \frac{\beta X_2}{P_1} = 1000; \quad 0.4(1000) + 0.6(2000P_2) = 1000 \text{ from this}$$

$$400 + 1200P_2 = 1000; \quad P_2 = \frac{600}{1200} = 0.5;$$

This is the relative price of good two that clears the global market.

Now it is possible to determine the demand for X_1 and X_2 by A and B.

$$X_{1,A} + X_{1,B} = 400 + 600 = 1000$$

Similarly market clearing in good two implies

$$X_{2,A} + X_{2,B} = 2000 \quad X_{2,A} = \frac{(1-\alpha)I_A}{P_2} = \frac{0.6(1000)}{0.5} = 1200$$

$$X_{2,B} = \frac{(1-\beta)I_A}{P_2} = \frac{0.4(2000 \times 0.5)}{0.5} = 800$$

$$X_{2,A} + X_{2,B} = 1200 + 800 = 2000$$

This satisfies the global market equilibrium.

Gains from complete specialisation is obvious;

If country A produced only good X_2 it could produce only 400 but if it specialises in X_1 it can consume 400 of X_1 and 1200 X_2 .

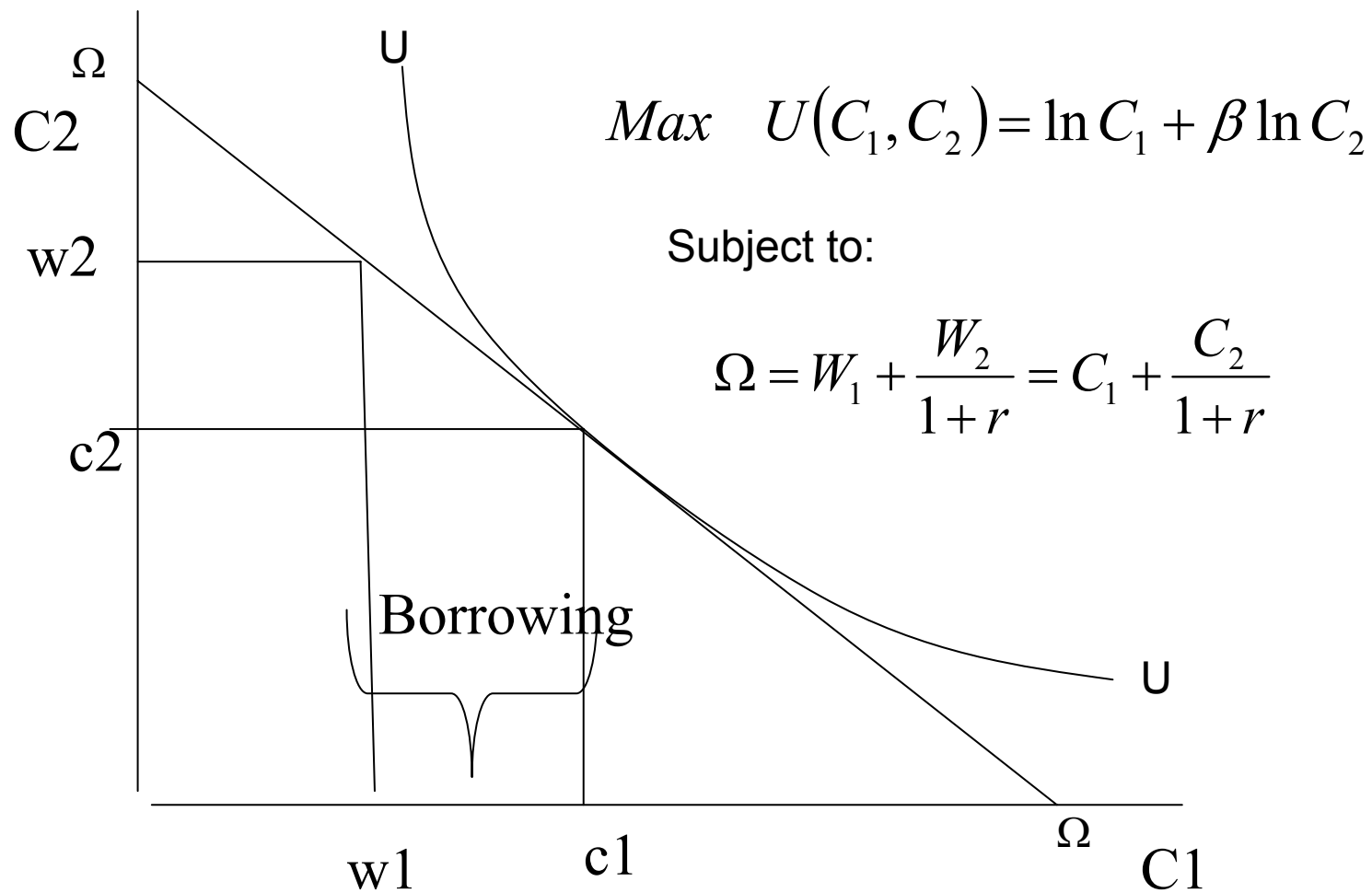
The consumption of X_2 alone is three times its production capacity.

Similarly if country B only produced X_1 it can produce only 800 of X_1 but by specialising in X_2 it can consume 600 of X_1 and 800 of X_2 .

Moral Hazard, Adverse Selection and Cooperative Bargaining in Global Trade

- **Moral hazard** arises when countries break trade agreements quietly for their benefit and harming another trading partner without any compensation.
- Anti-dumping duties and violation of copy rights are examples of moral hazard.
- Lack of information about the structure of demand in foreign country or about the technology of the foreign country would give rise to problem of **adverse selection**.
- In such situation countries will can use **signalling, screening** and optimise the expected utilities assigning probabilities various contingencies.
- **Cooperative Bargaining** a better than retaliation.

Two Period Model of Consumption and Saving



Consumption-Saving Problem in a Two Period Model

$$\text{Max } U(C_1^i, C_2^i) = \ln C_1^i + \beta \ln C_2^i \quad (1)$$

Subject to

$$C_1^i + b^i \leq W_1^i : \quad \text{budget constraint in period 1}$$

$$C_2^i \leq b^i(1+r) + W_2^i : \quad \text{budget constraint in period 2} \quad (2)$$

The inter temporal budget constraint is:
(4)

$$C_1^i + \frac{C_2^i}{1+r} = W_1^i + \frac{W_2^i}{1+r} \quad (3)$$

$$L^i = \ln C_1^i + \beta \ln C_2^i + \lambda \left[C_1^i + \frac{C_2^i}{1+r} - W_1^i - \frac{W_2^i}{1+r} \right] \quad (4)$$

Demand for period 1 and 2 consumptions

$$C_1^i = \frac{1}{1+\beta} \left[W_1^i + \frac{W_2^i}{1+r} \right] \quad (5)$$

$$C_2^i = (1+r)\beta C_1^i \Rightarrow C_2^i = (1+r) \frac{\beta}{1+\beta} \left[W_1^i + \frac{W_2^i}{1+r} \right] \quad (6)$$

Market clearing conditions in period 1 and 2

$$C_1^A + C_1^B = W_1^A + W_1^B$$

$$\frac{1}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] + \frac{1}{1+\beta} \left[W_1^B + \frac{W_2^B}{1+r} \right] = W_1^A + W_1^B \quad (7)$$

Market clearing condition for period 2:

$$C_2^A + C_2^B = W_2^A + W_2^B$$

$$\frac{\beta(1+r)}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] + \frac{\beta(1+r)}{1+\beta} \left[W_1^B + \frac{W_2^B}{1+r} \right] = W_2^A + W_2^B \quad (8)$$

The gross interest rate that clears the market

$$\frac{1}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] + \frac{1}{1+\beta} \left[W_1^B + \frac{W_2^B}{1+r} \right] = W_1^A + W_1^B$$

or

$$\frac{1}{1+\beta} \left[W_1^A + W_1^B \right] + \frac{1}{1+\beta} \left[\frac{W_2^A}{1+r} + \frac{W_2^B}{1+r} \right] = W_1^A + W_1^B$$

$$\frac{1}{1+\beta} + \frac{1}{1+\beta} \left(\frac{W_2^A + W_2^B}{W_1^A + W_1^B} \right) = 1+r \quad (9)$$

Numerical Example

If endowments and preferences were $\{W_1^A, W_2^A, W_1^B, W_2^B\} = \{100, 0, 0, 200\}$ and $\beta = 0.9$

respectively the interest rate that clears these two markets is given by:

$$\frac{1}{1+0.9} + \frac{1}{1+0.9} \left(\frac{200}{100} \right) = 1+r = 2.222 \Rightarrow r = 1.222 \quad (10)$$

Same result can be obtained using the period 2 budget constraint

$$1+r = \frac{1}{\beta} \left(\frac{W_2^A + W_2^B}{W_1^A + W_1^B} \right) = \frac{1}{0.9} \left(\frac{200}{100} \right) \Rightarrow 2.222 \quad (10)$$

Scenarios of Equilibrium Allocation in Pure Exchange Model

Supply of savings in Two period Model with Various parameters

| Parameters of Consumption Saving Model | | | | | | | |
|--|----------|----------|----------|----------|----------|----------|----------|
| | I | II | III | IV | V | VI | VII |
| w1a | 100 | 100 | 100 | 100 | 100 | 500 | 1000 |
| w2a | 0 | 0 | 0 | 50 | 200 | 200 | 200 |
| w1b | 0 | 0 | 0 | 100 | 100 | 400 | 2000 |
| w2b | 200 | 200 | 200 | 200 | 200 | 200 | 100 |
| Beta | 0.9 | 0.95 | 1 | 0.9 | 0.9 | 0.9 | 0.9 |
| Equilibrium Interest Rate and Consumption | | | | | | | |
| 1+r | 2.222222 | 2.105263 | 2 | 1.388889 | 2.222222 | 0.493827 | 0.111111 |
| c1a | 52.63158 | 51.28205 | 50 | 71.57895 | 100 | 476.3158 | 1473.684 |
| c2a | 105.2632 | 102.5641 | 100 | 89.47368 | 200 | 211.6959 | 147.3684 |
| Ua | 8.154133 | 8.336304 | 8.517193 | 8.315351 | 9.373656 | 10.98572 | 11.78916 |
| Uanb | 4.60517 | 4.60517 | 4.60517 | 8.125991 | 9.373656 | 10.98309 | 11.67624 |
| c1b | 47.36842 | 48.71795 | 50 | 128.4211 | 100 | 423.6842 | 1526.316 |
| c2b | 94.73684 | 97.4359 | 100 | 160.5263 | 200 | 188.3041 | 152.6316 |
| Ub | 7.953948 | 8.236282 | 8.517193 | 9.425926 | 9.373656 | 10.76324 | 11.85584 |
| Ubnb | 4.768486 | 5.033401 | 5.298317 | 9.373656 | 9.373656 | 10.75995 | 11.74556 |
| Equilibrium Savings (Borrowing or Lending) | | | | | | | |
| s1a | 47.36842 | 48.71795 | 50 | 28.42105 | 0 | 23.68421 | -473.684 |
| s2a | -105.263 | -102.564 | -100 | -39.4737 | 0 | -11.6959 | 52.63158 |
| s1b | -47.3684 | -48.7179 | -50 | -28.4211 | 0 | -23.6842 | 473.6842 |
| s2b | 105.2632 | 102.5641 | 100 | 39.47368 | 0 | 11.69591 | -52.6316 |

Consumption Saving Decisions of Agent A

$$C_1^A = \frac{1}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] = \frac{1}{1.9} (100) = 52.63$$

$$C_2^A = (1+r)\beta \frac{1}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] = 2.222 \left(\frac{0.9}{1.9} \right) 100 = 105.25$$

Amount of savings in period 1 and period 2 respectively are

$$S_1^A = W_1^A - C_1^A = W_1^A - \frac{1}{1+\beta} \left[W_1^A + \frac{W_2^A}{1+r} \right] = 100 - 52.63 = 47.37$$

Consumption Saving Decisions of Agent B

$$C_1^B = \frac{1}{1+\beta} \left[W_1^B + \frac{W_2^B}{1+r} \right] = \frac{1}{1.9} \left(\frac{200}{2.222} \right) = 47.37$$

$$C_2^B = (1+r) \frac{\beta}{1+\beta} \left[W_1^B + \frac{W_2^B}{1+r} \right] = 2.222 \left(\frac{0.9}{1.9} \right) \frac{200}{2.222} = 94.73$$

$$S_1^B = W_1^B - C_1^B = -47.37; \quad S_2^B = W_2^B - C_2^B = 200 - 94.73 = 105.3$$

Steady State and Transitions Dynamics in an Overlapping Generation model

The transition dynamics in the neoclassical model is given by the law of motion of the capital stock. Consider an economy inhabited by N number of individuals.

At period 0 each of them is endowed by k_0 capital stock and aggregate capital stock is K_0 . The level of technical know how is denoted by A .

Production technology is standard Cobb-Douglas production function; $Y_t = AK_t^\beta L_t^{1-\beta}$. This implies per capita output to be $y_t = A_t k_t^\beta$. Let the labour force L_t be fixed to N in each period.

The remuneration to capital is according to its marginal productivity;

$$r_t = \frac{\partial y_t}{\partial k_t} = \beta A_t k_t^{\beta-1}. \text{ Labour is paid the residual amount } w_t = \frac{\partial y_t}{\partial L_t} = (1-\beta)A_t k_t^\beta.$$

There are two types of people living in this economy, young and old. Young work and earn labour income and consume a α fraction of income $c_{yt} = \alpha w_t$ and save $(1-\alpha)$ for their old age.

Consumption Saving and Capital Accumulation of Young and Old

There are two types of people living in this economy, young and old. Young work and earn labour income and consume a α fraction of income $c_{yt} = \alpha w_t$ and save $(1 - \alpha)$ for their old age.

The life time budget constraint is given by $C_{Y,t} + \frac{C_{O,t}}{(1 + r_{t+1})} = W_t$. The old people earn interest in their asset and consume all of their income, $c_{ot} = a_t(1 + r_t)$.

The capital stock of period t is result of the saving of old people; $a_{t+1} = (1 - \alpha)w_t$. Next periods capital stock equals the assets saved today as given by the equation of accumulation;

$K_{t+1} = (1 - \alpha)w_t = (1 - \alpha)(1 - \beta)A_t k_t^\beta$. Aggregate saving equals total output minus the consumption of young and old, this also it the market clearing condition in this model $S_t = Y_t - Nc_{yt} - Nc_{ot}$.

Saving equals investment in each period $S_t = I_t$ and investment adds to the capital stock $I_t = K_{t+1} - K_t$.

Numerical Results of Steady State and Transitional Dynamics in an Overlapping Generation Model

Growth, Capital Accumulation and Consumption in an Overlapping Generation Model

| Time | K | K | Y | w | r | cy | co | S | I |
|------|-------|---------|----------|--------|-------|-------|--------|---------|---------|
| 0.0 | 3.000 | 300.000 | 1390.389 | 9.733 | 1.390 | 4.866 | 7.171 | 186.636 | 186.636 |
| 1.0 | 4.866 | 486.636 | 1607.538 | 11.253 | 0.991 | 5.626 | 9.689 | 76.002 | 76.002 |
| 2.0 | 5.626 | 562.638 | 1679.070 | 11.753 | 0.895 | 5.877 | 10.664 | 25.036 | 25.036 |
| 3.0 | 5.877 | 587.675 | 1701.144 | 11.908 | 0.868 | 5.954 | 10.980 | 7.726 | 7.726 |
| 4.0 | 5.954 | 595.400 | 1707.822 | 11.955 | 0.861 | 5.977 | 11.077 | 2.338 | 2.338 |
| 5.0 | 5.977 | 597.738 | 1709.831 | 11.969 | 0.858 | 5.984 | 11.107 | 0.703 | 0.703 |
| 6.0 | 5.984 | 598.441 | 1710.434 | 11.973 | 0.857 | 5.987 | 11.116 | 0.211 | 0.211 |
| .. | .. | .. | .. | .. | .. | .. | .. | .. | .. |
| 40.0 | 5.987 | 598.742 | 1710.693 | 11.975 | 0.857 | 5.987 | 11.120 | 0.000 | 0.000 |
| 41 | 5.987 | 598.742 | 1710.693 | 11.975 | 0.857 | 5.987 | 11.120 | 0.000 | 0.000 |

| | | | | |
|----------|-----|-------------------------|--------------------------|---------------------------|
| α | 0.5 | Mpc | k = capital labour ratio | cy = consumption of young |
| N | 100 | Population | K = capital stock | co = consumption of old |
| β | 0.3 | productivity of capital | Y = aggregate output | S=aggregate saving |
| A | 10 | technological index | w=wage rate | I= net investment |
| K0 | 300 | Initial capital stock | r= interest rate | a = asset position |

For details on this type of model see Auerbach and Kotlikoff (1998) *Macroeconomics: An Integrated Approach*, MIT Pres p.63-71.

Probability models with Dummy dependent variables

- the labour force participation (1 if a person participates in the labour force, 0 otherwise)
- yes or no vote on a particular issue
- to marry or not to marry
- to study further or to start a job
- to buy or not to buy a particular stock, or commodity
- choice of transportation mode to work (1 if a person drives to work, 0 otherwise)
- Union membership (1 if one is a member of the union, 0 otherwise)
- Owning a house (1 if one owns 0 otherwise)
- Multinomial choices: work as a teacher, or as a clerk, or as a self employed or professional or as a factory worker
- Multinomial ordered choices: strongly agree, agree, neutral, disagree, strongly disagree

Probability Models

$$Y_i = \begin{cases} 1 & \text{if event occurs} \\ 0 & \text{otherwise} \end{cases}$$

There are mainly three types of limited dependent variable models

Linear probability model

Probit model

Logit model or multinomial logit or nested logit

Tobit Model basically Probit with Truncation or censoring

Probability in Linear Probability Model

$$Y_i = \begin{cases} 1 & \text{if event occurs} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$Y_i = 1$ if a person owns a house, $Y_i = 0$ otherwise

X_i can be both continuous or dichotomous variables
(e.g. family income; or male / female]

$$\begin{aligned} E[(Y_i = 1)/X_i] &= 0(1 - P_i) + 1(P_i) = P_i \\ E[(Y_i = 1)/X_i] &= \beta_1 + \beta_2 X_i = P_i \end{aligned} \quad (3)$$

In theory $0 \leq P_i \leq 1$

$$0 \leq E[(Y_i = 1)/X_i] \leq 1$$

This is not always guaranteed in linear model like this.

Estimation of the Linear Probability Model

Does attendance to PSI courses improve grade of a student how has attended this? Is there any difference in Grade of those who have taken PSI and those who have not taken it?

Result from the estimation as given in the LIMDEP output in the next page show that there were significant effect of PSI attendance; It improves grade by 37 percent.

```

+-----+-----+-----+-----+-----+
| Ordinary least squares regression      Weighting variable = none
| Dep. var. = GRADE      Mean=      .3437500000      , S.D.=      .4825587044
| Model size: Observations =      32, Parameters =      4, Deg.Fr.=      28
| Residuals: Sum of squares= 4.216473951      , Std.Dev.=      .38806
| Fit:      R-squared=      .415900, Adjusted R-squared =      .35332
| Model test: F[ 3,      28] =      6.65,      Prob value =      .00157
| Diagnostic: Log-L =      -12.9782, Restricted(b=0) Log-L =      -21.5812
|      LogAmemiyaPrCrt.=      -1.775, Akaike Info. Crt.=      1.061
| Autocorrel: Durbin-Watson Statistic =      2.34645,      Rho =      -.17322
+-----+-----+-----+-----+
+-----+-----+-----+-----+-----+
| Variable | Coefficient | Standard Error | t-ratio | P[|T|>t] | Mean of X |
+-----+-----+-----+-----+-----+
Constant -1.498017120      .52388862      -2.859      .0079
GPA      .4638516793      .16195635      2.864      .0078      3.1171875
TUCE     .1049512224E-01      .19482854E-01      .539      .5944      21.937500
PSI      .3785547879      .13917274      2.720      .0111      .43750000

```

Problem with Linear Probability Model

$u_i = 1 - \beta_1 - \beta_2 X_i$ with probability $(1 - P_i)$ and

$u_i = -\beta_1 - \beta_2 X_i$ with probability P_i

Variance of the error term is not constant

$$\text{var}(u_i) = (-\beta_1 - \beta_2 X_i)^2 (1 - P_i) + (1 - \beta_1 - \beta_2 X_i)^2 P_i$$

$$\text{var}(u_i) = (-\beta_1 - \beta_2 X_i)^2 (1 - \beta_1 - \beta_2 X_i) + (1 - \beta_1 - \beta_2 X_i)^2 (\beta_1 + \beta_2 X_i)$$

$$\text{var}(u_i) = (\beta_1 + \beta_2 X_i)(1 - \beta_1 - \beta_2 X_i) = P_i(1 - P_i) \quad (4)$$

Thus the variance depends on X_i .

It is possible to transform this model to make it homoscedastic by dividing the

original variables by $\sqrt{(\beta_1 + \beta_2 X_i)(1 - \beta_1 - \beta_2 X_i)} = \sqrt{P_i(1 - P_i)} = \sqrt{w_i}$

$$\frac{Y_i}{\sqrt{w_i}} = \frac{\beta_1}{\sqrt{w_i}} + \beta_2 \frac{X_i}{\sqrt{w_i}} + \frac{u_i}{\sqrt{w_i}} \quad (5)$$

Limitations of a linear probability model

- It does not guarantee that the probability lies inside $(0,1)$ bands.
- Probability in non-linear phenomenon:
 - at very low level of income a family does not own a house;
 - at very high level of income **every** one owns a house; marginal effect of income is very negligible.
- The linear probability model does not explain this fact very well; it violates the fundamental property of probability ,
- it is possible to have negative as well as greater than 1 value of with a linear model.

Probit Model

Probit model guarantees that the limiting case of a probability remains between 0 and 1; $0 \leq P_i \leq 1$;

Probability distribution

$$\Pr(Y_i = 1) = \Pr(Z_i^* \leq Z_i) = F(Z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_i} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta_i + \beta_2 X} e^{-\frac{t^2}{2}} dt$$

Here t is standardised normal variable, $t \sim N(0,1)$.

The Unknown parameters of this model β and σ are estimated by the maximum likelihood estimator.

Likelihood Functions for Modelling Probability Models

$$L = \text{Pr}(Y_1)\text{Pr}(Y_2)\dots\text{Pr}(Y_n)$$

$$L = \prod_{i=1}^{n1} P_i \prod_{i=n1}^N (1-P_i) \quad L = \prod_{i=1}^{n1} P_i^{y_i} \prod_{i=n1+1}^N P_i^{1-y_i}$$

Log likelihood functions for modelling probability

$$\text{Log}L = \sum_{i=1}^{n1} \log P_i + \sum_{i=1}^N \log(1-P_i)$$

First order conditions for ML estimates of β and σ

$$\frac{\partial \text{Log}L}{\partial \alpha} = \sum_{i=1}^{n1} \frac{\partial P_i / \partial \alpha}{P_i} - \sum_{i=1}^{n1} \frac{\partial P_i / \partial \alpha}{(1-P_i)} = 0; \quad \frac{\partial \text{Log}L}{\partial \beta} = \sum_{i=1}^{n1} \frac{\partial P_i / \partial \beta}{P_i} - \sum_{i=1}^{n1} \frac{\partial P_i / \partial \beta}{(1-P_i)} = 0$$

ML takes the OLS estimates as starting value for computations of optimal β and σ .

$1 - L_0 / L_{\max}$ gives the indication of goodness of fit of ML estimates.

Comparison of Maximum Likelihood and OLS Estimator

$$L(\alpha, \beta, \sigma) = f(y_1, y_1, \dots, y_n) = \prod_{i=1}^n \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{1}{2}} \exp \left[-\frac{1}{2\sigma^2} (y_i - \alpha - \beta x_i)^2 \right] \quad (1)$$

Take log of this function to get a log-likelihood function.

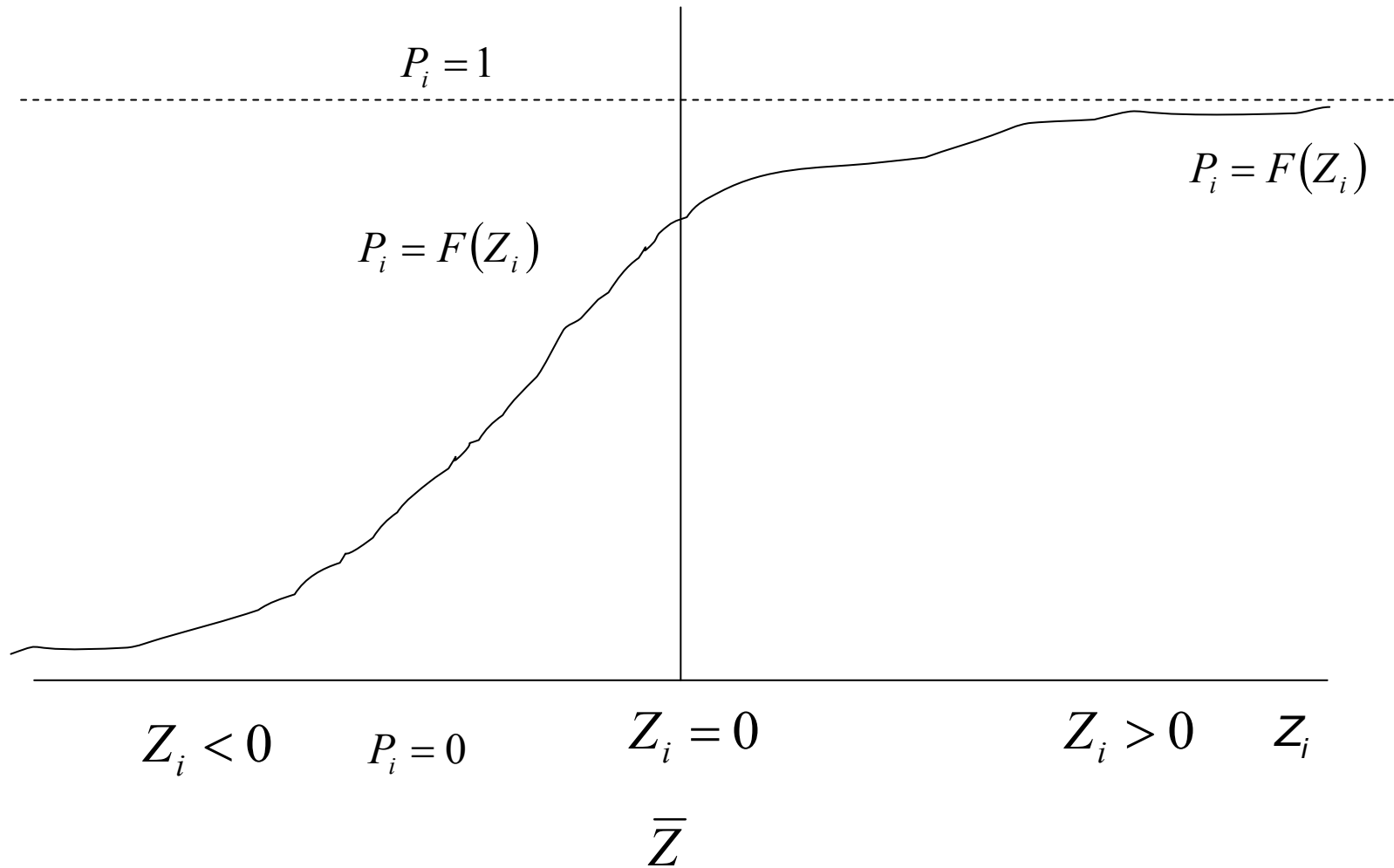
$$\text{Log}(\alpha, \beta, \sigma) = \left[\sum_{i=1}^n -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} (y_i - \alpha - \beta x_i)^2 \right] = c - \frac{N}{2} \log(\sigma^2) - \frac{Q}{2\sigma^2} \quad (2)$$

where $c = -\frac{n}{2} \log(2\pi)$ and $Q = \sum (y_i - \alpha - \beta x_i)^2$

Maximising (2) w.r.t. α, β, σ is equivalent to $Q = \sum (y_i - \alpha - \beta x_i)^2$

minimizing which is the negative term in the likelihood function. Therefore the estimators of α and β under the ML method are the same as in the OLS method.

Main Characteristics of a Probit Model



$$Z_i = \beta_1 + \beta_2 X_i + u_i$$

Summary on effect of explanatory variable upon probability

The effect of observed variables on probability

$$\frac{\partial P_i}{\partial x_{i,j}} = \begin{cases} \beta_j & \text{for the linear probability model} \\ \beta_j P_j (1 - P_j) & \text{for the logit probability model} \\ \beta_j \phi(Z_i) & \text{for the probit probability model} \end{cases}$$

where $Z_i = \beta_0 + \sum_{i=1}^k \beta_i x_{i,j}$ and $\phi(\cdot)$ is the standard normal density function

Note: for the logit model:

$$\ln\left(\frac{P_i}{1 - P_i}\right) = Z_i = \beta_1 + \beta_2 X_i + u_i \quad \frac{\partial P_i}{\partial X_i} \left[\ln\left(\frac{P_i}{1 - P_i}\right) \right] = \beta_2;$$

$$\frac{\partial P_i}{\partial X_i} \left[\ln\left(\frac{P_i}{1 - P_i}\right) \right] = \frac{\Delta P_i}{P_i} + \frac{\Delta P_i}{(1 - P_i)} = \frac{\Delta P_i}{P_i(1 - P_i)} = \beta_2 ; \Delta P_i = \beta_2 P_i(1 - P_i)$$

Initial Values and Final Estimates: Explaining Probit Results

```

+-----+
| Dependent variable is binary, y=0 or y not equal 0
| Ordinary least squares regression Weighting variable = none
| Dep. var. = GRADE Mean= .3437500000 , S.D.= .4825587044
| Model size: Observations = 32, Parameters = 4, Deg.Fr.= 28
| Residuals: Sum of squares= 4.216473951 , Std.Dev.= .38806
| Fit: R-squared= .415900, Adjusted R-squared = .35332
| Model test: F[ 3, 28] = 6.65, Prob value = .00157
| Diagnostic: Log-L = -12.9782, Restricted(b=0) Log-L = -21.5812
| LogAmemiyaPrCrt.= -1.775, Akaike Info. Crt.= 1.061
+-----+

```

| Variable | Coefficient | Standard Error | b/St.Er. | P[Z >z] | Mean of X |
|----------|-----------------|----------------|----------|----------|-----------|
| Constant | -1.498017120 | .52388862 | -2.859 | .0042 | |
| GPA | .4638516793 | .16195635 | 2.864 | .0042 | 3.1171875 |
| TUCE | .1049512224E-01 | .19482854E-01 | .539 | .5901 | 21.937500 |
| PSI | .3785547879 | .13917274 | 2.720 | .0065 | .43750000 |

```

+-----+
| Binomial Probit Model
| Maximum Likelihood Estimates
| Dependent variable GRADE
| Weighting variable ONE
| Number of observations 32
| Iterations completed 6
| Log likelihood function -12.81880
| Restricted log likelihood -20.59173
| Chi-squared 15.54585
| Degrees of freedom 3
| Significance level .1404896E-02
+-----+

```

| Variable | Coefficient | Standard Error | b/St.Er. | P[Z >z] | Mean of X |
|--------------------------------|-----------------|----------------|----------|----------|-----------|
| Index function for probability | | | | | |
| Constant | -7.452319647 | 2.5424723 | -2.931 | .0034 | |
| GPA | 1.625810039 | .69388249 | 2.343 | .0191 | 3.1171875 |
| TUCE | .5172894549E-01 | .83890261E-01 | .617 | .5375 | 21.937500 |
| PSI | 1.426332342 | .59503790 | 2.397 | .0165 | .43750000 |

Explaining Probit Results

```

+-----+
| Partial derivatives of E[y] = F[*] with |
| respect to the vector of characteristics. |
| They are computed at the means of the Xs. |
| Observations used for means are All Obs. |
+-----+

```

```

+-----+-----+-----+-----+-----+-----+
| Variable | Coefficient | Standard Error | b/St.Er. | P[|Z|>z] | Mean of X |
+-----+-----+-----+-----+-----+-----+

```

Index function for probability

| Variable | Coefficient | Standard Error | b/St.Er. | P[Z >z] | Mean of X |
|----------|-----------------|----------------|----------|----------|-----------|
| Constant | -2.444733653 | .75885194 | -3.222 | .0013 | |
| GPA | .5333470255 | .23246407 | 2.294 | .0218 | 3.1171875 |
| TUCE | .1696968191E-01 | .27119788E-01 | .626 | .5315 | 21.937500 |
| PSI | .4679083617 | .18764238 | 2.494 | .0126 | .43750000 |

Frequencies of actual & predicted outcomes
 Predicted outcome has maximum probability.

| Actual | Predicted | | Total |
|--------|-----------|----|-------|
| | 0 | 1 | |
| 0 | 18 | 3 | 21 |
| 1 | 3 | 8 | 11 |
| Total | 21 | 11 | 32 |

Features of a logit model

- Probability goes from 0 to 1 as the index variable Z_i goes from $-\infty$ to $+\infty$. Probability lies between 0 and 1.
- Log of the odds is linear in x, characteristic variables but probabilities themselves are not linear but non linear function of the parameters. Probabilities are estimated using the maximum likelihood method.
- Any explanatory variable that determines the value of Z_i , measures how the log of odds of an event (i.e. probability of owning a house rather than not owning it) changes as a result of change in explanatory variable such as income.
- We can calculate P_i for given estimates of $\hat{\beta}_1$ and $\hat{\beta}_2$ or all other $\hat{\beta}_i$.
- Limiting case when $P_i=1$; $\ln\left(\frac{P_i}{1-P_i}\right)$ or when $P_i=0$ $\ln\left(\frac{0}{1-0}\right)$; OLS cannot be applied in such case but the maximum likelihood method may be used to estimate the parameters.

Logit Probability Model

$$P_i = \frac{1}{1 + e^{-Z_i}} ;$$

$$Z_i = \beta_1 + \beta_2 X_i + u_i$$

$$1 - P_i = 1 - \frac{1}{1 + e^{-Z_i}} = \frac{1 + e^{-Z_i} - 1}{1 + e^{-Z_i}} = \frac{e^{-Z_i}}{1 + e^{-Z_i}} = \frac{1}{1 + e^{Z_i}}$$

[e = exp = 2.718].

Ratio of odds:

$$\left[\frac{P_i}{1 - P_i} = \frac{1 + e^{Z_i}}{1 + e^{-Z_i}} = \frac{e^{Z_i} (1 + e^{Z_i})}{e^{Z_i} (1 + e^{-Z_i})} = \frac{e^{Z_i} (1 + e^{Z_i})}{(e^{Z_i} + 1)} = e^{Z_i} \right]$$

Taking log of the odds $\ln\left(\frac{P_i}{1 - P_i}\right) = Z_i$

Initial Values for Logit Estimates

```

+-----+
| Multinomial logit model
| There are 2 outcomes for LH variable GRADE
| These are the OLS start values based on the
| binary variables for each outcome Y(i) = j.
| Coefficients for LHS=0 outcome are set to 0.0
+-----+

```

| Variable | Coefficient | Standard Error | b/St.Er. | P[Z >z] | Mean of X |
|---|-----------------|----------------|----------|----------|-----------|
| Characteristics in numerator of Prob[Y = 1] | | | | | |
| Constant | -1.498017120 | .52388862 | -2.859 | .0042 | |
| GPA | .4638516793 | .16195635 | 2.864 | .0042 | 3.1171875 |
| TUCE | .1049512224E-01 | .19482854E-01 | .539 | .5901 | 21.937500 |
| PSI | .3785547879 | .13917274 | 2.720 | .0065 | .43750000 |

Normal exit from iterations. Exit status=0.

```

+-----+
| Multinomial Logit Model
| Maximum Likelihood Estimates
| Dependent variable          GRADE
| Weighting variable         ONE
| Number of observations      32
| Iterations completed        6
| Log likelihood function     -12.88963
| Restricted log likelihood    -20.59173
| Chi-squared                 15.40419
| Degrees of freedom          3
| Significance level           .1501878E-02
+-----+

```

Final Estimates and Prediction: Explaining Logit Results

```

+-----+-----+-----+-----+-----+
|Variable | Coefficient | Standard Error |b/St.Er.|P[|Z|>z] | Mean of X|
+-----+-----+-----+-----+-----+
                Characteristics in numerator of Prob[Y = 1]
Constant -13.02134648      4.9313241      -2.641      .0083
GPA       2.826112525       1.2629411       2.238      .0252      3.1171875
TUCE      .9515765670E-01      .14155420       .672      .5014      21.937500
PSI       2.378687596       1.0645642       2.234      .0255      .43750000
    
```

```

+-----+
| Partial derivatives of probabilities with |
| respect to the vector of characteristics. |
| They are computed at the means of the Xs. |
| Observations used for means are All Obs. |
+-----+
    
```

```

+-----+-----+-----+-----+-----+
|Variable | Coefficient | Standard Error |b/St.Er.|P[|Z|>z] | Mean of X|
+-----+-----+-----+-----+-----+
                Marginal effects on Prob[Y = 1]
Constant -2.459760743      .81771031      -3.008      .0026
GPA       .5338588183      .23703797       2.252      .0243      3.1171875
TUCE      .1797548884E-01      .26236909E-01      .685      .4933      21.937500
PSI       .4493392735      .19676264       2.284      .0224      .43750000
    
```

Frequencies of actual & predicted outcomes
 Predicted outcome has maximum probability.

| | | Predicted | | |
|--------|----|-----------|--|-------|
| Actual | 0 | 1 | | Total |
| 0 | 18 | 3 | | 21 |
| 1 | 3 | 8 | | 11 |
| Total | 21 | 11 | | 32 |

Multiple choice models

- choice of different brands of a particular goods such as cereals
- different subjects by students in the business school such as economics, marketing, finance, business, accounting, management or
- in the university such as science, engineering, medicine, mathematics, philosophy, politics or history, arts or like that.
- ordered probit or ordered logit for choice of bonds such as AAA BBB; orders are used to rank the outcome.
- survey questions with ranking.

Multinomial Logit Model

Let $P_{i,j}$ denote the probability of choosing alternative J by individual i,.

There are J alternatives and N individuals.

$$\sum_i^J y_{i,j} = \sum_i^J P_{i,j} = 1$$

McFadden (1974)

$$L = \prod_{i=1}^T P_{i,1}^{y_{i,1}} P_{i,2}^{y_{i,2}} P_{i,3}^{y_{i,3}} \dots P_{i,J}^{y_{i,J}}$$

Individuals make a choice to maximise utility; such utilities are specific to individual and unobserved factors $U_{i,j} = \bar{U}_{i,j} + e_{i,j} = X'_{i,j} \beta + e_{i,j}$.

Errors $e_{i,j}$ need to have Weibull distribution a multinomial logit.

Multinomial Logit with Random Utility Model

Probability that the choice 1 es made when J alternatives were available implies

$$P_{i,1} = \Pr[U_{i,1} > U_{i,2} \text{ and } U_{i,1} > U_{i,3} \dots \text{and } U_{i,1} > U_{i,J}] =$$

$$P_{i,1} = \Pr[e_{i,2} < U_{i,1} - U_{i,2} + e_{i,1} \text{ and } e_{i,3} < U_{i,1} - U_{i,3} + e_{i,1} \dots \text{and } e_{i,J} < U_{i,1} - U_{i,J} +$$

$$P_{i,1} = \Pr[e_{i,2} < U_{i,1} - U_{i,2} + e_{i,1} \text{ and } e_{i,3} < U_{i,1} - U_{i,3} + e_{i,1} \dots \text{and } e_{i,J} < U_{i,1} - U_{i,J} +$$

$$P_{i,1} = \Pr[e_{i,2} - e_{i,1} < U_{i,1} - U_{i,2} \text{ and } e_{i,3} - e_{i,1} < U_{i,1} - U_{i,3} \dots \text{and } e_{i,J} - e_{i,1} < U_{i,1} - U$$

Weibull errors: $\Pr(e_{i,j} < \varepsilon) = \exp(-\exp(-\varepsilon))$

$$P_{i,j} = \frac{\exp(X'_{i,j}\beta)}{\sum_{j=1}^J \exp(X'_{i,j}\beta)}$$

Multinomial Logit and Independence of Irrelevant Alternatives

Choices are independent of irrelevant alternatives (IIA); only odds between two choices need to be compared when one is confronted with choosing one or another.

$$P_{i,j} = \frac{\exp(X'_{i,j}\beta)}{\sum_{j=1}^J \exp(X'_{i,j}\beta)}$$

$$\frac{P_{i,1}}{P_{i,2}} = \frac{\exp(X'_{i,1}\beta) / \sum_{j=1}^J \exp(X'_{i,j}\beta)}{\exp(X'_{i,1}\beta) / \sum_{j=1}^J \exp(X'_{i,j}\beta)} = \frac{\exp(X'_{i,1}\beta)}{\exp(X'_{i,1}\beta)}$$

$$\frac{P_{i,k}}{P_{i,1}} = \frac{\exp(X'_{i,k}\beta_k)}{\exp(X'_{i,1}\beta_1)} = \exp(X'_{i,k}\beta_k - X'_{i,1}\beta_1) \text{ for } k = 2, \dots, J$$

Normalisation in Multinomial Logit Model

with normalisation $\beta_1 = 0$ $P_{i,1} = \frac{\exp(X'_{i,j}\beta_1)}{\sum_{j=1}^J \exp(X'_{i,j}\beta_j)} = \frac{1}{1 + \sum_{j=2}^J \exp(X'_{i,j}\beta_j)}$

$$P_{i,1} = \frac{\exp(X'_{i,j}\beta_j)}{\sum_{j=1}^J \exp(X'_{i,j}\beta_j)} = \frac{\exp(X'_{i,j}\beta_j)}{1 + \sum_{j=2}^J \exp(X'_{i,j}\beta_j)} \quad \text{or } j = 2, \dots, J$$

Parameters in a multinomial logit models are estimated by a maximum likelihood method (see McFadden (1974). Judge , Griffiths, Hill, Lutkepohl and. Lee(1990). Similarly one can define the nested logit models.

Nested Logit Model

If there are J alternatives and k subgroups a multiple logit evaluation of probability of choosing j in k subgroup is given by:

$$P_{j,k} = P_j / {}_k P_k = \left(\frac{e^{\beta'X|_k}}{\sum_{j=1}^J e^{\beta'X|_k}} \right) \left(\frac{e^{\gamma'z|_k}}{\sum_{k=1}^K e^{\gamma'z|_k}} \right) \frac{\left(\sum_{j=1}^J e^{\beta'X|_k} \right) \sum_{k=1}^K e^{\gamma'z|_k}}{\left(\sum_{k=1}^K \sum_{j=1}^J e^{\beta'X|_k + \gamma'z|_k} \right)}$$

This can be estimated using a two step Maximum likelihood procedure.

It can be applied to model the choices of **travel modes** in which an individual chooses first whether to fly or take a ground transport.

Once a ground transport is chosen then similar choice is made for train, bus or car transport.

Model can be estimated with limited information maximum likelihood (LIML) or full information maximum likelihood (FIML). (see Greene: 19)

Count Data Model: Poisson Distribution

$$\text{Poisson random variable } P(Y = y) = \frac{e^{-\lambda} \lambda^y}{y!}$$

where λ denotes the intensity of occurrence or the rate parameter and $y = 0, 1, 2, 3, \dots$, denote the counts of events in a given time interval. Mean and variance are the same $E(Y) = \lambda$ and variance $Var(Y) = \lambda$.

$$\lambda_i = \exp(X_i' \beta)$$

$$\text{Likelihood function } \ln L(\beta) = \sum \{y_i X_i' \beta - \exp(X_i' \beta) - \ln y_i\}$$

Parameters β_p is estimated by k number of first order conditions as

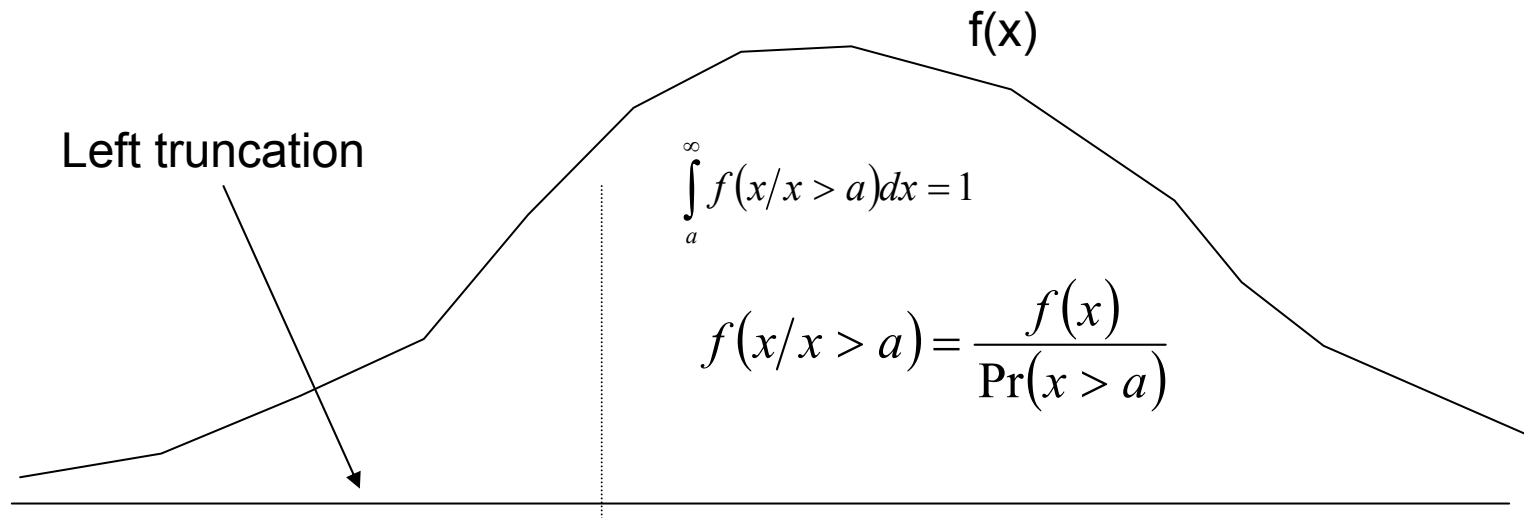
$$\sum_{i=1}^n \{y_i X_i - \exp(X_i' \beta)\} X_i = 0$$

β_p is asymptotically normal with the sample covariance matrix

$$V[\hat{\beta}_p] = \left(\sum_{i=1}^n \lambda_i X_i X_i' \right)^{-1}$$

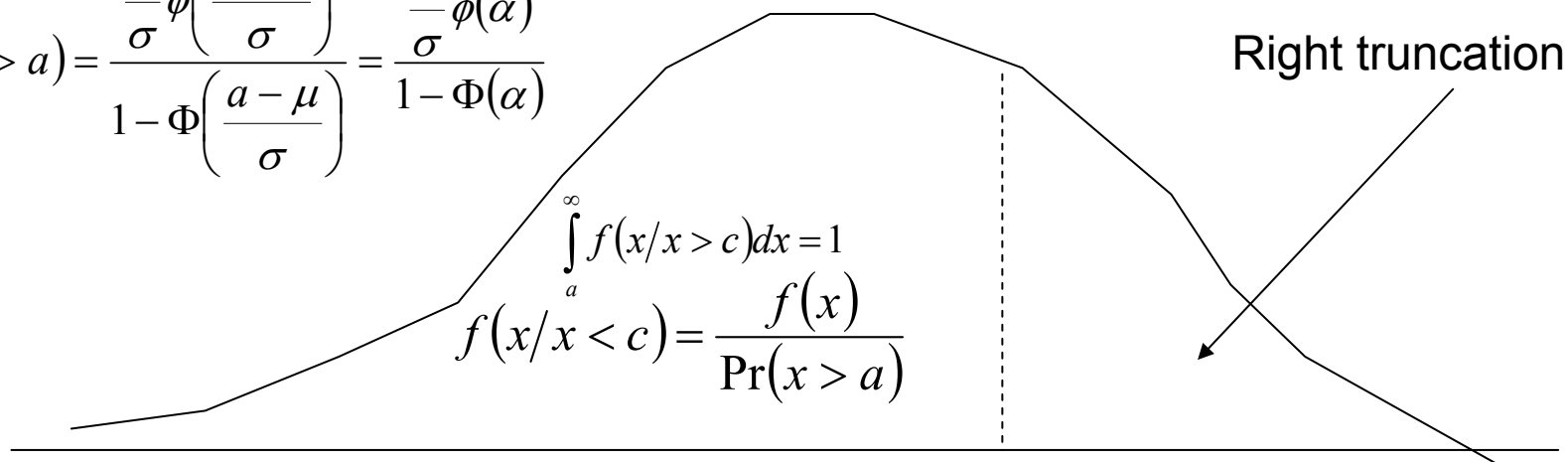
Gauss-Newton or Newton-Raphson iterative algorithm is used to find unique

Truncated Distribution



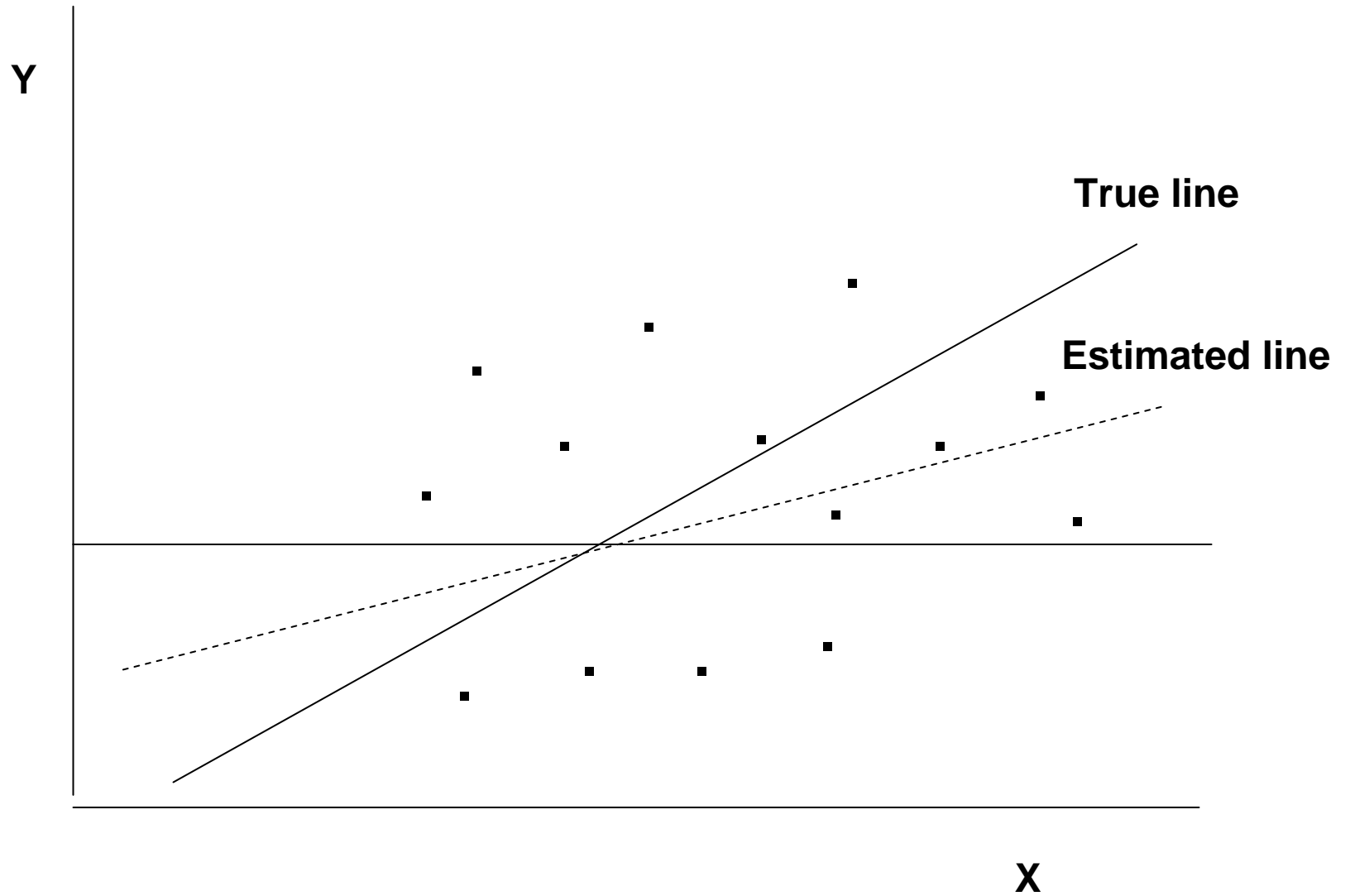
$$X > X' \leq a$$

$$f(x/x > a) = \frac{\frac{1}{\sigma} \phi\left(\frac{a - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{a - \mu}{\sigma}\right)} = \frac{\frac{1}{\sigma} \phi(\alpha)}{1 - \Phi(\alpha)}$$



$$X > X' = c$$

How Does Truncation and Censoring Affect a Regression Line?



Mean and Variance of a Truncated Distribution

The mean of truncated variable is $E(x/x > a) = \mu + \sigma\lambda(\alpha)$ and

its variance is $\text{var}(x/x > a) = \mu + \sigma^2(1 - \delta(\alpha))$,

where $0 \leq \delta(\alpha) = \lambda(\alpha)[\lambda(\alpha) - \alpha] < 1$.

Variance is also called a hazard function of this distribution.

$\lambda(\alpha)$ is called inverse Mill's ratio. Here $0 < \delta(\alpha) < 1$ for all α .

$$\lambda(\alpha) = \left. \begin{array}{l} \frac{\phi(\alpha)}{(1 - \Phi(\alpha))} \\ \frac{-\phi(\alpha)}{\Phi(\alpha)} \end{array} \right\} \begin{array}{l} \text{if } x > a \\ \text{if } x < a \end{array}$$

Regression For A Variable with Truncated Distribution

$$Y_i = \beta X_i + \varepsilon_i \quad \text{where } \varepsilon_i \sim N(0, \sigma^2) \quad (Y_i/X_i) \sim N(\beta_1 X_i, \sigma^2)$$

$$E(Y_i/X_i) = X_i \beta$$

$$\text{if truncated } (Y_i/y_i > a) = X_i \beta + \sigma \lambda_i(\alpha_i)$$

$$E(Y_i/y_i > a) = X_i \beta + E(\varepsilon_i/\varepsilon_i > a) > a - X_i \beta = X_i \beta + \sigma \frac{\phi\left(\frac{a - X_i \beta}{\sigma}\right)}{1 - \Phi\left(\frac{a - X_i \beta}{\sigma}\right)} = X_i \beta + \sigma \lambda_i$$

This is a non-linear function and the estimates of β and σ by the OLS technique is neither efficient nor consistent. Because the OLS ignores the

$$\lambda(\alpha) = \frac{\frac{1}{\sigma} \phi\left(\frac{a - X_i \beta}{\sigma}\right)}{1 - \Phi\left(\frac{a - X_i \beta}{\sigma}\right)} = \frac{1}{\sigma} \frac{\phi(\alpha)}{1 - \Phi(\alpha)} \text{ part;}$$

$\lambda(\alpha)$ obviously depends on X_i .

Maximum Likelihood Estimator for Truncated Distribution

The unknown parameters β and σ can be estimated consistently by the Maximum likelihood technique as:

$$\log L = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (Y_i - \beta X_i)^2 - \sum \log \left(1 - \Phi \left(\frac{a - \beta' X_i}{\sigma} \right) \right)$$

First order conditions for estimation of this ML are:

$$\frac{\partial \log L}{\partial \begin{pmatrix} \beta \\ \sigma^2 \end{pmatrix}} = \sum_{i=1}^n \begin{bmatrix} \left(\frac{(Y_i - \beta X_i)}{\sigma^2} - \frac{\lambda_i}{\sigma} \right) X_i \\ -\frac{1}{2\sigma^2} + \frac{(Y_i - \beta X_i)^2}{\sigma^2} - \frac{\alpha_i \lambda_i}{\sigma^2} \end{bmatrix} = \sum_i^n g_i = 0$$

Maximise this likelihood by choosing β and σ iteratively taking the OLS estimates as the initial starting point.

Censored Distribution

$$y_i = \begin{cases} y_i^* & \text{if } y_i^* > 0 \\ 0 & y_i^* \leq 0 \end{cases}$$

If $y_i^* \sim N(\mu, \sigma^2)$; $\Pr(y_i = 0) = \Pr(y_i^* \leq 0) = \Phi\left(-\frac{\mu}{\sigma}\right) = 1 - \Phi\left(\frac{\mu}{\sigma}\right)$

This the distribution of the transformed variable is

$$g(y_i) = \begin{cases} N(\mu, \sigma^2) & \text{if } y_i^* > 0 \\ 1 - \Phi\left(\frac{\mu}{\sigma}\right) & y_i^* \leq 0 \end{cases}$$

Moments of the censored variable

$$E(y) = \Phi a + (1 - \Phi)(\mu + \sigma\lambda) \quad \text{and} \quad \text{var}(y) = \sigma^2(1 - \Phi)\left[\left((1 - \delta) + (\alpha - \lambda)^2\right)\Phi\right]$$

$$\delta = \lambda^2 - \lambda\alpha; \quad \lambda = \frac{\phi}{1 - \Phi}$$

$$E(y_i^* \leq 0) = \Phi(\alpha) = \Phi$$

Tobit Model

$$Y_i = X_i\beta + \varepsilon_i \quad Y_i = \begin{cases} Y_i^* & \text{if } Y_i^* > 0 \\ 0 & Y_i^* \leq 0 \end{cases}$$

The unknown parameters β and σ can be estimated consistently by the Maximum likelihood technique as:

$$\log L = \left[\sum_{Y_i^* > 0} -\frac{n}{2} \log 2\pi + \log \sigma^2 + \frac{(Y_i - \beta X_i)^2}{\sigma^2} \right] + \sum_{Y_i^* = 0} \left(1 - \Phi\left(\frac{X_i\beta}{\sigma}\right) \right)$$

Uncensored part
Censored part

Use the OLS estimates as the starting values.

The two step estimation procedure proceeds as following:

First construct the index variable $I_i = \begin{cases} Y_i^* & \text{if } Y_i^* > 0 \\ 0 & Y_i^* \leq 0 \end{cases}$

$\Pr(I_i = 1) = 1 - \Phi_i = p_i$ and $\Pr(I_i = 0) = \Phi_i = 1 - p_i$

Then the log likelihood

$$L = \prod_{I_i=1} P_i \prod_{I_i=0} (1 - P_i)$$

Using Probit and MLE find estimates of parameters β and σ and $\lambda \hat{\lambda}_i = \frac{\phi(X_i\hat{\beta}/\hat{\sigma})}{\Phi(X_i\hat{\beta}/\hat{\sigma})}$

Secondly apply OLS to

$$Y_i = X_i\beta + \sigma\lambda_i + w_i$$

Two limit Tobit

$$Y_i = X_i\beta + \varepsilon_i \quad Y_i = \begin{cases} Y_i^* & \text{if } L_1 < Y_i^* < L_2 \\ L_{1,i} & \text{if } Y_i^* < L_1 \\ L_{2,i} & \text{if } Y_i^* \geq L_2 \end{cases}$$

Formulate a maximum likelihood function

$$L = \prod_{Y_i=L_1} \Phi\left(\frac{L_{1,i} - X_i\beta}{\sigma}\right) \prod_{Y_i=Y_i^*} \Phi\left(\frac{Y_i - X_i\beta}{\sigma}\right) \prod_{Y_i=L_{12}} \Phi\left(\frac{L_{1,i} - X_i\beta}{\sigma}\right)$$

For instance think of a minimum, partial or maximum coverage of insurance.

Expected values

$$E(Y_i / X_i, L_1 < Y_i^* < L_2) = X_i\beta + \frac{\sigma(\Phi_{1,i} - \Phi_{2,i})}{\Phi_{2,i} - \Phi_{1,i}}$$

$$E(Y_i / X_i) = \Phi_{1,i}L_{1,i} + X_i\beta(\Phi_{2,i} - \Phi_{1,i}) + \sigma(\phi_{1,i} - \phi_{2,i}) + (1 - \Phi_{2,i})L_{2,i}$$

$$\Phi_{1,i} = \Phi\left(\frac{L_{1,i} - X_i\beta}{\sigma}\right); \quad \Phi_{2,i} = \Phi\left(\frac{L_{1,i} - X_i\beta}{\sigma}\right)$$

$$L = \prod_{n1} \Phi_{1,i} \prod_{n2} (\Phi_{2,i} - \Phi_{1,i}) \prod_{n3} (1 - \Phi_{2,i})$$

Estimation of Two Limit Tobit

$$L = \prod_{n1} \Phi_{1,i} \prod_{n2} (\Phi_{2,i} - \Phi_{1,i}) \prod_{n3} (1 - \Phi_{2,i})$$

$$\log L = N_2 \log \rho + N \log \sigma + \sum_{i=1}^N (Y_i - \beta X_i)^2 - \sum \log[\rho(1 - \rho)\Phi_i]$$

If ρ known estimate β_{ML} and σ_{ML} if not β_{ML} and σ_{ML} and test for $\rho = 1$

$$E(Y_i/X_i; I_i = 1) = X_i\beta + E(\varepsilon_i/I_i = 1) = X_i\beta + \sigma \frac{\phi_i}{\Phi_i}$$

$$E(Y_i/X_i; I_i = 0) = X_i\beta + \sigma \frac{\phi_i}{1 - \Phi_i}$$

$$E(Y_i/X_i) = X_i\beta = \frac{\sigma\rho_1\Phi_i - \rho_2\Phi_i}{\rho_1\Phi_i - \rho_2\Phi_i}$$

Heckman's Model of Sample Selection

Inference population when the sample is non-random and some observations are omitted causes a sample selection bias.

Heckman's procedure is to microeconometrics as is the unit root for time series data.

When the sample selection is not corrected inference drawn from the regression analysis is not efficient or robust.

$$Y_{1,i} = X_{1,i}\beta + \varepsilon_{1,i} \quad \text{if } Y_{1,i} > Y_{2,i}$$

$$Y_{2,i} = X_{2,i}\beta + \varepsilon_{2,i}$$

Both $Y_{1,i}$ and $Y_{2,i}$ are stochastic. For instance, if $Y_{1,i}$ is market wage and $Y_{2,i}$ is the reservation wage; an individual works only when $Y_{1,i} > Y_{2,i}$.

$$\begin{pmatrix} \varepsilon_{1,i} \\ \varepsilon_{2,i} \end{pmatrix} \sim IN(\underline{0}, \Sigma) \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{bmatrix}$$

At least one more variable in $X_{2,i}$ than in $X_{1,i}$; if $\sigma_{1,2} = 0$ it is a regular Tobit.

But the sample selection problem arise when $\sigma_{1,2} \neq 0$.

$$Y_{1,i} > Y_{2,i} \quad \text{implies } \varepsilon_{1,i} = Y_{1,i} - X_{1,i}\beta$$

$$\varepsilon_{2,i} < Y_{2,i} - X_{2,i}\beta$$

Joint density of $(\varepsilon_{1,i}, \varepsilon_{2,i}) = f(\varepsilon_{1,i}, \varepsilon_{2,i}) = g(\varepsilon_{1,i})h(\varepsilon_{2,i}/\varepsilon_{1,i})$

Likelihood Function and Correlated Errors in Heckman's Model

For some individuals you do not observe $Y_{1,i}$ because $Y_{1,i} < Y_{2,i}$.

$$Y_{1,i} < Y_{2,i} = \varepsilon_{1,i} - \varepsilon_{2,i} < X_{2,i}\beta - X_{1,i}\beta$$

$$(\varepsilon_{1,i}, \varepsilon_{2,i}) = IN(0, \sigma^2); \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}$$

$$\log L = -N_1 \log \sigma^2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^N (Y_{1,i} - X_{1,i}\beta)^2 + \sum_{\text{Instruemnt}} \log \Phi_i(w) + \sum_0 \log \left[\frac{(X_{2,i}\beta_2 - X_{1,i}\beta_1)}{\sigma} \right]$$

Observed *Unobsorved*

$$w = \frac{Y_{2,i} - X_{2,i}\beta_2}{\sigma_{2,1}} - \frac{\sigma_{1,2}}{\sigma_1^2} (Y_{1,i} - X_{1,i}\beta_1)$$

$$W = \beta_0 + X_{1,i}\beta + \varepsilon_{1,i} \quad E(\varepsilon_{2,i}/H > 0) = E(\varepsilon_{2,i}/V > -RD) = E(\varepsilon_{2,i}/\varepsilon_{2,i} - \varepsilon_{1,i} > \gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta) =$$

$$= \frac{\sigma_2^2 - 2\sigma_{1,2}}{\sigma} \frac{\phi(\Delta)}{1 - \Phi(\Delta)} = \frac{\alpha\phi(\Delta)}{1 - \Phi(\Delta)}$$

$$W_R = \gamma_0 + \gamma_1 H + \gamma_2 Z + \varepsilon_{2,i}$$

X and Z are exogenous variables and H and W are endogenous. If $H=0$ $W_R > W$.

Heckman's Lambda for Correction of Selection Bias

$$H = \frac{\beta_0 + X_{1,i}\beta - \gamma_0 - \gamma_2 Z}{\gamma_1} + \frac{\varepsilon_{2,i} - \varepsilon_{1,i}}{\gamma_1} = X\delta + V$$

$$W = \left. \begin{array}{l} \beta_0 + X_{1,i}\beta + \varepsilon_{1,i} \\ H = X\delta + V \end{array} \right\} \text{ if } H > 0$$

$$W = H = 0 \quad \text{Otherwise}$$

$$\Pr(H = 0) = \Pr(V \leq -R\delta) = \Pr\left(\frac{\varepsilon_{2,i} - \varepsilon_{1,i}}{\gamma_1} \leq \frac{\gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta}{\gamma_1}\right)$$

$$\Pr(\varepsilon_{2,i} - \varepsilon_{1,i} \leq (\gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta)) = \Phi(\Delta)$$

$$\Delta = \frac{\gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta}{\sigma} \quad ; \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}$$

$$L = \prod_{H>0} WH \prod_{H=0} (\Delta)$$

$$E(W/H > 0) = \beta_0 + X_{1,i}\beta + (\varepsilon_{1,i}/H > 0) = \beta_0 + X_{1,i}\beta + \frac{\alpha\phi(\Delta)}{1 - \Phi(\Delta)} = \beta_0 + X_{1,i}\beta + \alpha\lambda$$

$$E(W/H > 0) = \beta_0 + X_{1,i}\beta + \alpha\lambda = \beta_0 + X_{1,i}\beta + \sigma_{1,2} \frac{\phi\left(\frac{\gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta}{\sigma}\right)}{1 - \Phi\left(\frac{\gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta}{\sigma}\right)}$$

Thus the sample selection bias is due to the **Heckman's Lambda** term.

Estimation of Heckman's Model (look at Heckit Procedure in Limdep)

To estimate $E(W/H > 0) = \beta_0 + X_{1,i}\beta + \alpha\lambda$

Use probit to estimate $\hat{\lambda} \quad I_i = \begin{cases} 1 & \text{if working} \\ 0 & \text{Otherwise} \end{cases}$

and replace it in $W_i = \beta_0 + X_{1,i}\beta + \alpha\hat{\lambda} + \eta_i$

$$\Pr(I_i = 1) = \Pr(H > 0) = 1 - \Phi(\Delta)$$

$$\Pr(V > -RD) = \Pr(\varepsilon_{2,i} - \varepsilon_{1,i} > \gamma_0 + \gamma_2 Z - \beta_0 - X_{1,i}\beta) \quad \Pr(I_i = 0) = \Phi(\Delta)$$

Heckman's Tobit then:

$$L = \prod_{H>0} (1 - \Phi(\Delta)) \prod_{H \leq 0} \Phi(\Delta) \text{ by MLE get } \Delta \text{ and } \hat{\lambda} = \frac{\phi(\hat{\Delta})}{1 - \Phi(\hat{\Delta})}$$

$$H = \frac{\beta_0 + X_{1,i}\beta - \gamma_0 - \gamma_2 Z}{\gamma_1} + \frac{\varepsilon_{2,i} - \varepsilon_{1,i}}{\gamma_1} = \hat{\Delta}$$

Apply OLS to $W_i = \beta_0 + X_{1,i}\beta + \alpha\hat{\lambda} + \eta_i$

Regime Switching Model

Regime 1: $\theta_t = X_{1,i}\beta + u_{1,i}$ iff $X_{1,i}\beta + u_{1,i} < X_{2,i}\beta + u_{2,i}$

Regime 2: $\theta_t = X_{2,i}\beta + u_{2,i}$ iff $X_{1,i}\beta + u_{1,i} > X_{2,i}\beta + u_{2,i}$

Criterion $Z\gamma \geq u_t$ in regimes and if $Z\gamma < u_t$ in regime 2.

$$Z = \frac{X_{1,i}\beta - X_{2,i}\beta\gamma_0 - \gamma_2 Z}{\sigma}; \quad u = \frac{u_{2,i} - u_{1,i}}{\sigma}; \quad \sigma_1^2 - \sigma_2^2 = \text{var}(u_{2,i} - u_{1,i})$$

$$\begin{pmatrix} u_{1,i} \\ u_{2,i} \\ \varepsilon \end{pmatrix} \sim IN(\underline{0}, \Sigma) \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,\varepsilon} \\ & \sigma_2^2 & \sigma_{2,\varepsilon} \\ & & \sigma_\varepsilon^2 \end{bmatrix}$$

$$\text{Regime 1: } \theta_t = X_{1,i}\beta - \sigma \frac{\phi(Z\gamma)}{\Phi(Z\gamma)} + u_{1,i}$$

$$\text{Regime 2: } \theta_t = X_{2,i}\beta + \sigma \frac{\phi(Z\gamma)}{1 - \Phi(Z\gamma)} + u_{2,i}$$

Calculate $\frac{\phi}{\Phi}$ using Probit; where $L = \prod_{R2} \Pr(Z\gamma \geq 1) \prod_{R1} (Z\gamma < 0)$

Then apply OLS to estimate $\hat{\beta}_1 \hat{\beta}_2 \sigma_1^2 \sigma_{1,2} \sigma_{1,\varepsilon}$

Disequilibrium Models

Demand : $D_t = \alpha_1 P_t + X_{1,t} \beta_1 + u_{1,t}$

Supply: $S_t = \alpha_2 P_t + X_{2,t} \beta_2 + u_{2,t}$

Equilibrium: $Q_t = D_t = S_t$

Disequilibrium : $Q_t = \text{Min}(D_t, S_t) = \begin{cases} D_t > S_t & Q_t \text{ on } S \\ D_t < S_t & Q_t \text{ on } D \end{cases}$

Let λ_t be probability of observing D; $\Pr(D_t < S_t) = \Pr(u_{1,t} - u_{2,t} < X_{2,t} \beta_2 - X_{1,t} \beta_2)$ and $(1 - \lambda_t)$ be the probability of observing $\Pr(S_t < D_t)$. Then the joint density

$$g(D, S)$$

$$h(Q_t) = \lambda_t h(Q_t / Q_t = D_t) + (1 - \lambda_t) h(Q_t / Q_t = S_t) = \int_{Q_t}^{\infty} g(Q_t, S_t) dS_t + \int_{Q_t}^{\infty} g(Q_t, D_t) dD_t$$

Fixed Effect Models for Analysis of Panel Data

$$Y = D\alpha + X\beta + \varepsilon$$

This can be easily estimated by the OLS when the number of cross section units are small. Many panel data studies have much larger observations. It results in over parameterisation and loss of degree of freedom. For this the model is transformed by a projection matrix

$$M_d = I - D(D'D)^{-1}D'$$

$$M_d Y = M_d D\alpha + M_d X\beta + M_d \varepsilon$$

$$b = [X' M_d X]^{-1} [X' M_d Y]$$

$$M_d = \begin{bmatrix} M^0 & 0 & 0 & \dots & 0 \\ 0 & M^0 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & M^0 \end{bmatrix} \quad \text{where } M^0 = I_T - \frac{1}{T} ii'$$

Multiplying any variable by M^0 is equivalent taking deviation from the mean ie

$$M^0 X_i = X_i - \bar{X}_i$$

$$Y = D\alpha + X\beta + \varepsilon$$

$$a = (D'D)^{-1} D(Y - Xb)$$

$$\text{var}(b) = s^2 [X' M_d X]^{-1} \quad \text{and} \quad s^2 = \frac{\sum_t \sum_i (y_{it} - a_i - x_{i,t} b)}{nT - n - k}$$

Determinants of Economic growth in the US, UK, Germany, France and Japan: Fixed Effect Panel
Data Model (1960-2000)

| | Coefficient | Std.Error | t-value | t-prob |
|------------------------------|-------------|-----------|----------|--------|
| Investment ratio (I/Y) | 0.0007 | 0.0003 | 2.1500 | 0.0330 |
| Real interest rate | 0.0003 | 0.0006 | 0.4510 | 0.6530 |
| Population growth rate | 1.0218 | 0.0025 | 405.0000 | 0.0000 |
| Saving ratio (S/Y) | 0.0001 | 0.0006 | 0.0973 | 0.9230 |
| Per-capita income | 1.0069 | 0.0017 | 594.0000 | 0.0000 |
| Trade balance ratio (X-M)/Y) | -0.0001 | 0.0002 | -0.7370 | 0.4630 |
| Constant | -0.0221 | 0.0090 | -2.4400 | 0.0160 |
| UK | -0.0014 | 0.0072 | -0.1970 | 0.8440 |
| US | -0.0039 | 0.0054 | -0.7180 | 0.4740 |
| Germany | -0.0029 | 0.0039 | -0.7280 | 0.4680 |
| France | -0.0107 | 0.0105 | -1.0200 | 0.3070 |

sigma 0.009673186 sigma^2 9.357053e-005
 R^2 0.9999804; RSS 0.012538450674 TSS 639.4401902
 no. of observations 145 no. of parameters 11; individual: 4
 number of individuals 5 (derived from year); longest time series 31 [1970 - 2000]
 shortest time series 23 (unbalanced panel)

Determinants of Economic growth in the US, UK, Germany, France and Japan: Within Group Panel Data Model (1960-2000)

| | Coefficient | Std.Error | t-value | t-prob |
|------------------------------|-------------|-----------|----------|--------|
| Investment ratio (I/Y) | 0.0007 | 0.0003 | 2.1500 | 0.0330 |
| Real interest rate | 0.0003 | 0.0006 | 0.4510 | 0.6530 |
| Population growth rate | 1.0218 | 0.0025 | 405.0000 | 0.0000 |
| Saving ratio (S/Y) | 0.0001 | 0.0006 | 0.0973 | 0.9230 |
| Per-capita income | 1.0069 | 0.0017 | 594.0000 | 0.0000 |
| Trade balance ratio (X-M)/Y) | -0.0001 | 0.0002 | -0.7370 | 0.4630 |

sigma 0.009673186 sigma^2 9.357053e-005 R^2 0.9999789
 RSS 0.012538450674 TSS 594.73136048 no. of observations 145
 no. of parameters 11; number of individuals 5 (derived from year);
 longest time series 31 [1970 - 2000]; shortest time series 23 (unbalanced panel)

Random Effect Models for Panel Data Analysis

$y_{i,t} = \mu + \mathbf{x}_{i,t}\boldsymbol{\beta} + \alpha_i + \varepsilon_{i,t}$; $\alpha_i, \sim IID(0, \sigma_\alpha^2)$ are individual specific random errors
and $\varepsilon_{i,t} \sim IID(0, \sigma_\varepsilon^2)$ are remaining random errors.

$$\alpha_i \mathbf{l}_T + \varepsilon_i \quad \mathbf{l}_T = (1, 1, \dots, 1)'$$

$$\text{Var}(\alpha_i \mathbf{l}_T + \varepsilon_i) = \boldsymbol{\Omega} = \sigma_\alpha^2 \mathbf{l}_T \mathbf{l}_T' + \sigma_\varepsilon^2 I_T$$

Errors are correlated, therefore this requires estimation by the GLS estimator
Transform the model by pre-multiplying by $\boldsymbol{\Omega}^{-1}$ where

$$\boldsymbol{\Omega}^{-1} = \sigma_\varepsilon^2 \left[I_T - \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2 + T\sigma_\alpha^2} \mathbf{l}_T \mathbf{l}_T' \right]$$

$$\boldsymbol{\beta}_{GLS} = \left(\sum_t \sum_i (\mathbf{x}_{i,t} - \bar{\mathbf{x}}_i)(\mathbf{x}_{i,t} - \bar{\mathbf{x}}_i)' + \psi T \sum_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \right)^{-1}$$

$$\left(\sum_t \sum_i (\mathbf{x}_{i,t} - \bar{\mathbf{x}}_i)(y_{i,t} - \bar{y}_i) + \psi T \sum_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{y}_i - \bar{y})' \right)$$

Random Effect Models for Panel Data Analysis

$$\Omega = \begin{bmatrix} \sigma_\alpha^2 + \sigma_\varepsilon^2 & \sigma_\alpha^2 & \sigma_\alpha^2 & \cdot & \cdot & \sigma_\alpha^2 \\ 0 & \sigma_\alpha^2 + \sigma_\varepsilon^2 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \sigma_\alpha^2 & \cdot & \cdot & \sigma_\alpha^2 + \sigma_\varepsilon^2 \end{bmatrix}$$

$$\Omega^{-\frac{1}{2}} = \frac{1}{\sigma_\varepsilon} \left[I_T - 1 - \frac{\sigma_\varepsilon}{\sqrt{T\sigma_\alpha^2 + \sigma_\varepsilon^2}} \right]$$

$$\beta_{GLS} = \sum_i (X' \Omega^{-1} X)^{-1} \sum_i (X' \Omega^{-1} Y)$$

Dynamic Panel data models

$$y_{i,t} = \gamma y_{i,t-1} + \beta + \alpha_i + \varepsilon_{i,t} \quad \text{with } \gamma < 1$$

$$\gamma_{FE} = \frac{\sum_t \sum_i (y_{i,t} - \bar{y}_i)(y_{i,t} - \bar{y}_{i,t-1})}{\sum_t \sum_i (y_{i,t-1} - \bar{y}_{i,-1})^2} \quad ; \quad \bar{y}_i = T^{-1} \sum_i y_{i,t} \quad \text{and} \quad \bar{y}_{i,-1} = T^{-1} \sum_i y_{i,t-1}$$

This is not asymptotically unbiased estimator:

$$\gamma_{FE} = \gamma + \frac{(1/NT) \sum_t \sum_i (\varepsilon_{i,t} - \bar{\varepsilon}_i)(y_{i,t} - \bar{y}_{i,t-1})}{(1/NT) \sum_t \sum_i (y_{i,t-1} - \bar{y}_{i,-1})^2}$$

$$P \lim_{N \rightarrow \infty} (1/NT) \sum_t \sum_i (\varepsilon_{i,t} - \bar{\varepsilon}_i)(y_{i,t} - \bar{y}_{i,t-1}) = -\frac{\sigma_\varepsilon^2}{T^2} \frac{(T-1) - T\gamma + \gamma^T}{(1-\gamma)^2} \neq 0$$

Dynamic Panel data models

Instrumental variable methods have been suggested to solve this inconsistency

$$\hat{\gamma}_{IV} = \frac{\sum_t \sum_i y_{i,t-2} (y_{i,t-1} - \bar{y}_{i,t-2})}{\sum_t \sum_i y_{i,t-2} (y_{i,t-1} - y_{i,t-2})} \quad \text{and}$$

where y_{i-2} is used as instrument of $(y_{i,t-1} - y_{i-2})$

It is asymptotically

$$P \lim_{N \rightarrow \infty} (1/NT) \sum_t \sum_i (\varepsilon_{i,t} - \bar{\varepsilon}_i) y_{i,t-2} = 0$$

Moment conditions with vector of transformed error terms

$$\Delta \varepsilon_i = \begin{pmatrix} \varepsilon_{i,2} - \varepsilon_{i,1} \\ \varepsilon_{i,3} - \varepsilon_{i,2} \\ \cdot \\ - \\ \varepsilon_{i,T} - \varepsilon_{i,T-1} \end{pmatrix}$$

Dynamic Panel data models

$$Z_i = \begin{bmatrix} [y_{i0}] & 0 & 0 & \dots & 0 \\ 0 & [y_{i0}, y_{i1}] & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & [y_{i0}, y_{i,T-2}] \end{bmatrix}$$

$$E\{Z_i' \Delta \varepsilon_i\} = 0$$

Or for moment estimator write the transformed errors as

$$E\{Z_i' (\Delta y_{i,t} - \gamma \Delta y_{i,t-1})\} = 0$$

$$\min_{\gamma} \left(\frac{1}{N} \sum_{i=1}^N Z_i' (\Delta y_{i,t} - \gamma \Delta y_{i,t-1}) \right)' W_N \left(\sum_{i=1}^N Z_i' (\Delta y_{i,t} - \gamma \Delta y_{i,t-1}) \right)$$

GMM method includes the most efficient instrument

$$\hat{\gamma}_{GMM} = \left(\left(\sum_{i=1}^N \Delta y_{i,t-1}' Z_i \right) W_N \left(\sum_{i=1}^N Z_i' \Delta y_{i,t-1} \right) \right)^{-1} \times \left(\sum_{i=1}^N \Delta y_{i,t-1}' Z_i \right) W_N \left(\sum_{i=1}^N Z_i' \Delta y_{i,t} \right)$$

The optimal weighting matrix W_N can guarantee the most efficient estimator, this should be inversely proportional to transformed covariance matrix.

$$\hat{W}_N^{opt} = \left(\frac{1}{N} \sum_{i=1}^N Z_i' \Delta \varepsilon_i \Delta \varepsilon_i' Z_i \right)^{-1}$$

Computer Demonstration: Generating Panel data in Excel, Growth batch, Unemployment batch,

Panel Data Estimation Example from the BHPS

Stata Analysis

```
. xtreg stsrc duration, fe
```

```
Fixed-effects (within) regression      Number of obs   =   35016  
Group variable (i): pid                Number of groups =    8995
```

```
R-sq:  within = 0.0045      Obs per group: min =    1  
        between = 0.1948      avg =    3.9  
        overall = 0.0285     max =    37
```

```
corr(u_i, Xb) = -0.3383      F(1,26020)      = 118.35  
                                Prob > F          = 0.0000
```

```
-----+-----  
stsrc |   Coef.  Std. Err.   t  P>|t|  [95% Conf. Interval]  
-----+-----  
duration | .0000141  1.29e-06  10.88  0.000  .0000115  .0000166  
_cons | 22.03168  .0001639    .  0.000  22.03136  22.032  
-----+-----  
sigma_u | .01359554  
sigma_e | .02161173
```

Panel Data Models with Limited Dependent Variables

$$y_{i,t}^* = x_{i,t}'\beta + \alpha_i + \varepsilon_{i,t}$$

$y_{i,t} = 1$ if $y_{i,t}^* > 0$ where $y_{i,t}^*$ is a latent variable; $y_{i,t} = 0$ otherwise.

Alternative names: dichotomous dependent variables, discrete dependent random variable, binary variable, either or choice variables

$$Y_i = \begin{cases} \beta_0 + \beta_1 X_t + u_t & \text{if event occurs} \\ 0 & \text{otherwise} \end{cases}$$

- the labour force participation (1 if a person participates in the labour force, 0 otherwise)
- yes or no vote in particular issue
- to marry or not to marry
- to study further or to start a job
- to buy or not to buy a particular stock
- choice of transportation mode to work (1 if a person drives to work, 0 otherwise)
- Union membership (1 if one is a member of the union, 0 otherwise)
- Owning a house (1 if one owns 0 otherwise)
- Multinomial choices: work as a teacher, or as a clerk, or as a self employed or professional or as a factory worker
- Multinomial ordered choices: strongly agree, agree, neutral, disagree

Probit and Logit: Introduction

$$\Pr(Y_i = 1) = \Pr(Z_i^* \leq Z_i) = F(Z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_i} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta_1 + \beta_2 X} e^{-\frac{t^2}{2}} dt \quad (13)$$

Here t is standardised normal variable, $t \sim N(0,1)$

Steps for a probit model

probability depends upon unobserved utility index Z_i which depends upon observable variables such as income. There is a thresh-hold of this index when after which family starts owning a house, $Z_i \geq Z_i^*$.

Logit model

variable Y_i which takes value 1 ($Y_i = 1$) if a student gets a first class mark, value 0 ($Y_i = 0$) otherwise. Probability of getting a first class mark in an exam is a

function of student effort index denoted by Z_i . $P_i = \frac{1}{1 + e^{-Z_i}}$; where

$$Z_i = \beta_1 + \beta_2 X_i + u_i$$

An example of a logit model: what determines that a student gets the first class degree?

$$Z_i = \beta_1 + \beta_2 H + \beta_3 E + \beta_4 A + \beta_5 P + e_t$$

H = hours of study, E= exercises, A = attendance in lectures and classes; P = papers written for assignment.

Ratio of odds: $\left[\frac{P_i}{1 - P_i} = \frac{1 + e^{Z_i}}{1 + e^{-Z_i}} = e^{Z_i} \right]$; taking log of the odds $\ln\left(\frac{P_i}{1 - P_i}\right) = Z_i$

Features of a logit model

- a. probability goes from 0 to 1 as the index variable Z_i goes from $-\infty$ to $+\infty$.
Probability lies between 0 and 1.
- b. Log of the odds is linear in x, characteristic variables but probabilities themselves are not linear but non linear function of the parameters. Probabilities are estimated using the maximum likelihood method.
- c. Any explanatory variable that determines the value of Z_i , measures how the log of odds of an event (i.e. owning a house) changes as a result of change in explanatory variable such as income.
- d. We can calculate P_i for given estimates of $\hat{\beta}_1$ and $\hat{\beta}_2$ or all other $\hat{\beta}_i$.
- e. Limiting case when $P_i=1$; $\ln\left(\frac{P_i}{1-P_i}\right)$ or when $P_i=0$ $\ln\left(\frac{0}{1-1}\right)$; OLS cannot be applied in such case but the maximum likelihood method may be used to estimate the parameters.

Tobit Model

It is an extension of the probit model, named after Tobin.

We observe variables if the event occurs: ie if some one buys a house.

We do not observe explanatory variables for people who have not bought a house.

The observed sample is censored, contains observations for only those who buy the house.

$$Y_i = \begin{cases} \beta_0 + \beta_1 X_t + u_t & \text{if event occurs} \\ 0 & \text{otherwise} \end{cases}$$

Y_t is equal to $\beta_0 + \beta_1 X_t + u_t$ if the event is observed equal to zero if the event is not observed.

It is unscientific to estimate probability only with observed sample without worrying about the remaining observations in the truncated distribution.

The Tobit model tries to correct this bias.

Impacts of Explanatory Variables in Probability of Certain Event

The effect of observed variables on probability

$$\frac{\partial P_i}{\partial x_{i,j}} = \begin{cases} \beta_j & \text{for the linear probability model} \\ \beta_j P_j (1 - P_j) & \text{for the logit probability model} \\ \beta_j \phi(Z_i) & \text{for the probit probability model} \end{cases}$$

where $Z_i = \beta_0 + \sum_{i=1}^k \beta_i x_{i,j}$ **and** $\phi(\cdot)$ **is the standard normal density fu**

Panel models of limited dependent variables

$$y_{i,t}^* = x'_{i,t} \beta + \alpha_i + \varepsilon_{i,t}$$

$y_{i,t} = 1$ if $y_{i,t}^* > 0$ where $y_{i,t}^*$ is a latent variable; $y_{i,t} = 0$ otherwise.

$$\text{Log}L(\beta, \alpha_1, \dots, \alpha_N) = \sum_{i,t} y_{i,t} \log F(\alpha_i + x'_{i,t} \beta) + \sum_{i,t} (1 - y_{i,t}) (1 - \log F(\alpha_i + x'_{i,t} \beta))$$

Random Effect Probit model

$$y_{i,t}^* = x'_{i,t} \beta + \alpha_i + \varepsilon_{i,t}$$

$$y_{i,t} = 1 \text{ if } y_{i,t}^* > 0$$

$$y_{i,t} = 0 \text{ if } y_{i,t}^* \leq 0$$

$$\begin{aligned} f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) &= \int_{-\infty}^{\infty} f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) f(\alpha_i) d\alpha_i \\ &= \int_{-\infty}^{\infty} \left[\prod_i f(y_{i,t} / x_{i,t}, \beta) \right] f(\alpha_i) d\alpha_i \end{aligned}$$

$$\begin{aligned} f(y_{i,t} / x_{i,t}, \beta) &= \Phi \left(\frac{x'_{i,t} \beta + \alpha_i}{\sqrt{1 - \sigma_u^2}} \right) \text{ if } y_{i,t} = 1 \\ \dots\dots\dots &= 1 - \Phi \left(\frac{x'_{i,t} \beta + \alpha_i}{\sqrt{1 - \sigma_u^2}} \right) \text{ if } y_{i,t} = 0 \end{aligned}$$

Tobit Panel model

$$y_{i,t}^* = x_{i,t}'\beta + \alpha_i + \varepsilon_{i,t}$$

$$y_{i,t} = 1 \text{ if } y_{i,t}^* > 0$$

$$y_{i,t} = 0 \text{ if } y_{i,t}^* \leq 0$$

$$f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) = \int_{-\infty}^{\infty} f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) f(\alpha_i) d\alpha_i$$

$$f(y_{i,t} / x_{i,t}, \beta) = \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} \exp\left(-\frac{1}{2} \frac{(y_{i,t}^* = x_{i,t}'\beta + \alpha_i + \varepsilon_{i,t})^2}{\sigma_\varepsilon^2}\right) \text{ if } y_{i,t} = 1$$

$$\dots = 1 - \Phi\left(\frac{x_{i,t}'\beta + \alpha_i}{\sigma_\varepsilon}\right) \text{ if } y_{i,t} = 0$$

Dynamics and initial conditions

$$y_{i,t}^* = x_{i,t}'\beta + \gamma_{i,t-1} + \alpha_i + \varepsilon_{i,t}$$

$$f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) = \int_{-\infty}^{\infty} f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) f(\alpha_i) d\alpha_i$$

$$= \int_{-\infty}^{\infty} \left[\prod_i f(y_{i,t} / y_{i,t-1}, \alpha_i, \beta) \right] f(y_{i,t} / x_{i,t}, \alpha_i, \beta) f(\alpha_i) d\alpha_i$$

$$f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) = \int_{-\infty}^{\infty} f(y_{i,t}, \dots, y_{i,T} / x_{i,t}, \dots, x_{i,T}, \beta) f(\alpha_i) d\alpha_i$$

$$f(y_{i,t} / y_{i,t-1}, \alpha_i, \beta) = \Phi \left(\frac{(x_{i,t}'\beta + \gamma_{i,t-1} + \alpha_i)}{\sqrt{1 - \sigma_\alpha^2}} \right) \quad \text{if } y_{i,t} = 1$$

$$\dots \dots \dots = 1 - \Phi \left(\frac{(x_{i,t}'\beta + \gamma_{i,t-1} + \alpha_i)}{\sqrt{1 - \sigma_\alpha^2}} \right) \quad \text{if } y_{i,t} = 0$$

Table 3. The wage model estimates

Dependent variable - log gross hourly earnings (*derived variable*)
 Number of observations - 5108

| Variable | Coefficient | t-stat. | Mean |
|--|-------------|----------|----------|
| Intercept | 0.042865 | 0.2773 | |
| Years of schooling (<i>derived variable</i>) | 0.041662 | 14.2672 | 12.136 |
| Age at Date of Interview | 0.059554 | 21.3142 | 36.662 |
| Age squared | -0.000650 | -18.3420 | 1492.846 |
| Vocational qualifications | 0.042175 | 3.3939 | 0.389 |
| Sex | 0.223546 | 18.1097 | 0.500 |
| English 2 nd language | -0.222350 | -2.7740 | 0.025 |
| <i>RG Social Class: most recent job</i> | | | |
| Professional occupation | 0.162362 | 0.7498 | 0.073 |
| Managerial and technical | 0.073698 | 0.3422 | 0.304 |
| Skilled non-manual | -0.210799 | -0.9770 | 0.288 |
| Skilled manual | -0.264472 | -1.2254 | 0.205 |
| Partly skilled occupation | -0.360438 | -1.6732 | 0.177 |
| Unskilled occupation | -0.424298 | -1.9709 | 0.069 |
| Armed forces | 0.113362 | 0.4318 | 0.020 |
| <i>Region / Metropolitan Area Dummies</i> | | | |
| Inner London | 0.289657 | 3.4308 | 0.048 |
| Outer London | 0.252697 | 3.1381 | 0.080 |
| R. of South East | 0.155620 | 1.9941 | 0.219 |
| South West | 0.083689 | 1.0566 | 0.112 |
| East Anglia | 0.046243 | 0.5614 | 0.058 |
| East Midlands | 0.013843 | 0.1746 | 0.110 |
| West Midlands Conurb. | 0.061383 | 0.7387 | 0.054 |
| R. of West Midlands | 0.045400 | 0.5634 | 0.078 |
| Greater Manchester | 0.151121 | 1.8403 | 0.060 |
| Merseyside | 0.153563 | 1.7593 | 0.039 |
| R. of North West | 0.090306 | 1.1025 | 0.062 |
| South Yorkshire | 0.070065 | 0.8220 | 0.044 |
| West Yorkshire | 0.050060 | 0.5990 | 0.051 |
| R. of Yorks & Humber | 0.032916 | 0.3931 | 0.050 |
| Tyne & Wear | 0.051728 | 0.6001 | 0.042 |
| R. of North | 0.035033 | 0.4255 | 0.059 |
| Wales | 0.051493 | 0.6336 | 0.068 |
| Scotland | 0.106978 | 1.3404 | 0.094 |
| Northern Ireland | 0.024426 | 0.2779 | 0.038 |
| R ² = 0.92273928 Adjusted R ² = 0.92225211 | | | |

Note: Mean of dependent variable is 1.8226.

The BHPS

The British Household Panel Survey (BHPS) is a longitudinal data set. It started in 1991 with approximately 5500 households and more than 10,300 individuals drawn from 250 different areas of Great Britain.

There have been eight waves of BHPS until so far. It includes variables describing the economic, social and regional background of individuals and their labour market status and earnings over years.

The BHPS compares to the PSID data set in the United States, which started much earlier in 1968 with 4800 families and about 31,000 individuals.

The British Household Panel Survey is a multi-purpose study whose unique value resides in the fact that (1) it follows the same representative sample of individuals - the panel - over a period of years; (2) it is household-based, interviewing every adult member of sampled households; (3) it contains sufficient cases for meaningful analysis of certain groups such as the elderly or one parent families; (4) it allows for linkage of data both from other surveys and from local area statistics. The Wave 1 panel consists of some 5,500 households and 10,300 individuals drawn from 250 different areas of Great Britain. Continuing representativeness of the survey is ensured (and loss of panel members minimised) through (a) following panel members wherever they move in the UK; (b) including in the panel the new members of households formed by original panel members; (c) efficient fieldwork practices; (d) training videos for the interviewers; and by (e) regular contact with panel members through special reports and letters.” Source: <http://www.iser.essex.ac.uk/bhps/index.php#topcent>

Table 1

Descriptive Statistics

| | Minimum | Maximum | Mean | Std. Deviation | Skewness | | Kurtosis | |
|-----------|-----------|-----------|-----------|----------------|-----------|------------|-----------|------------|
| | Statistic | Statistic | Statistic | Statistic | Statistic | Std. Error | Statistic | Std. Error |
| WAGE | .40 | 64.67 | 7.3291 | 3.9415 | 2.421 | .024 | 14.977 | .047 |
| Real wage | .38 | 64.67 | 6.8506 | 3.6330 | 2.470 | .024 | 16.450 | .047 |
| AGE | 16.00 | 72.00 | 38.8614 | 10.0534 | .133 | .024 | -.580 | .047 |
| Schooling | .00 | 17.00 | 12.4292 | 3.7152 | -1.324 | .024 | 1.117 | .047 |
| <9 | .00 | 1.00 | .1357 | .3425 | 2.127 | .024 | 2.526 | .047 |
| >500 | .00 | 1.00 | .1887 | .3913 | 1.592 | .024 | .534 | .047 |
| PRO | .00 | 1.00 | 5.514E-02 | .2283 | 3.898 | .024 | 13.199 | .047 |
| MAN | .00 | 1.00 | .3019 | .4591 | .863 | .024 | -1.255 | .047 |
| SNM | .00 | 1.00 | .2485 | .4322 | 1.164 | .024 | -.645 | .047 |
| SM | .00 | 1.00 | .1831 | .3868 | 1.639 | .024 | .686 | .047 |
| PSO | .00 | 1.00 | .1267 | .3326 | 2.245 | .024 | 3.042 | .047 |
| UO | .00 | 1.00 | 4.053E-02 | .1972 | 4.661 | .024 | 19.728 | .047 |
| ARMY | .00 | 1.00 | 1.925E-02 | .1374 | 7.000 | .024 | 47.003 | .047 |

Table 2

| Social Group | Average real wage rate |
|---|-------------------------------|
| Managers and Professionals | 8.697048 |
| Skilled non-manual workers | 5.853007 |
| Skilled manual and partly skilled workers | 5.634582 |
| Unskilled workers | 5.077317 |

Determinants of log real wages in the BHPS

| | Wave specific Model | | Time trend Model | | Simple Model | |
|------------|---------------------|----------|------------------|----------|--------------|----------|
| | Coefficients | t-stat | Coefficients | t-stat | Coefficients | t-stat |
| (Constant) | -0.14747 | -2.59251 | -0.06943 | -1.21463 | 0.077762 | 1.398395 |
| Schooling | 0.02773 | 24.69335 | 0.024771 | 22.89871 | 0.02236 | 20.63917 |
| AGE | 0.061577 | 25.59521 | 0.061922 | 25.59911 | 0.064557 | 26.37733 |
| AGE SQ | -0.00071 | -23.6162 | -0.00072 | -23.7254 | -0.00073 | -23.9861 |
| REGION | 0.108861 | 13.76351 | 0.110162 | 13.85307 | 0.108349 | 13.43424 |
| SEX | 0.289535 | 38.1348 | 0.290434 | 38.0495 | 0.291724 | 37.68031 |
| <9 | -0.22086 | -20.5803 | -0.21948 | -20.3914 | -0.21897 | -20.0628 |
| >500 | 0.09911 | 9.982215 | 0.093538 | 9.430559 | 0.094012 | 9.346495 |
| PRO | 0.27677 | 8.92049 | 0.22158 | 7.804683 | 0.180633 | 6.313034 |
| MAN | 0.242631 | 8.659276 | 0.185404 | 7.434985 | 0.143636 | 5.730343 |
| SNM | 0.011577 | 0.409853 | -0.04718 | -1.85722 | -0.09587 | -3.7605 |
| SM | -0.07086 | -2.49312 | -0.13256 | -5.14719 | -0.18186 | -7.0363 |
| PSO | -0.13489 | -4.67352 | -0.19871 | -7.53563 | -0.24978 | -9.44191 |
| UO | -0.15041 | -4.77689 | -0.20085 | -6.64589 | -0.24771 | -8.12958 |
| ARMY | -0.0054 | -0.15241 | -0.01209 | -0.33963 | -0.01323 | -0.36642 |
| W2 | 0.129078 | 8.836155 | - | - | - | - |
| W3 | 0.016599 | 1.164677 | - | - | - | - |
| W4 | 0.133982 | 9.182063 | - | - | - | - |
| W5 | 0.150253 | 10.28989 | - | - | - | - |
| W6 | 0.234902 | 14.26546 | - | - | - | - |
| W7 | 0.188706 | 12.92747 | - | - | - | - |
| W8 | 0.22254 | 15.22138 | - | - | - | - |
| TIME | - | - | 0.039577 | 5.361771 | - | - |
| TIME SQ | - | - | -0.00119 | -1.50042 | - | - |
| R squared | | 0.417 | | 0.416 | | 0.399 |

Variation in real wages by the size of the company

| | Big companies | | Small Business | |
|------------|---------------|----------|----------------|----------|
| | Coefficients | t-stat | Coefficients | t-stat |
| (Constant) | 0.29994 | 1.099604 | -0.0457 | -0.33149 |
| Schooling | 0.022003 | 8.608514 | 0.024922 | 7.732276 |
| AGE | 0.060397 | 9.556513 | 0.05835 | 9.261173 |
| AGE SQ | -0.00072 | -8.85622 | -0.00069 | -8.9733 |
| REGION | 0.129167 | 7.660593 | 0.134833 | 5.584151 |
| SEX | 0.255387 | 15.62679 | 0.308492 | 12.72529 |
| PRO | 0.120275 | 0.491738 | -0.08024 | -1.61605 |
| MAN | 0.03926 | 0.16106 | - | - |
| SNM | -0.20562 | -0.84153 | -0.21833 | -7.06839 |
| SM | -0.22536 | -0.92101 | -0.38824 | -10.9582 |
| PSO | -0.28734 | -1.17199 | -0.36387 | -8.91642 |
| UO | -0.58074 | -2.32851 | -0.42895 | -7.62572 |
| W2 | 0.125986 | 3.872638 | 0.220977 | 5.109043 |
| W3 | 0.029339 | 0.940177 | 0.100242 | 2.295374 |
| W4 | 0.129666 | 4.149246 | 0.187425 | 4.15907 |
| W5 | 0.130334 | 4.161889 | 0.242116 | 5.341385 |
| W6 | 0.026827 | 0.109107 | 0.09715 | 2.083886 |
| W7 | 0.145871 | 4.711325 | 0.274512 | 6.472693 |
| W8 | 0.201444 | 6.403804 | 0.300538 | 6.962366 |
| R squared | | 0.401 | | 0.369 |

Table 5 Variation in wages across professional classes

| Managers and Professionals | | Skilled non-manual | | Skilled manual and partly skilled | | Unskilled Occupation | | |
|----------------------------|----------|--------------------|----------|-----------------------------------|----------|----------------------|----------|-------|
| Coeff. | t-stat | Coeff. | t-stat | Coeff. | t-stat | Coeff. | t-stat | |
| -0.24238 | -2.30467 | -0.279 | -3.21818 | 0.079343 | 0.997643 | 0.265824 | 1.2131 | |
| 0.033991 | 14.63375 | 0.025129 | 9.700202 | 0.020295 | 12.37499 | 0.01783 | 4.236874 | |
| 0.07674 | 15.00896 | 0.072049 | 17.12924 | 0.045051 | 11.8383 | 0.03088 | 2.958091 | |
| -0.00089 | -14.2944 | -0.00085 | -15.8118 | -0.0005 | -10.5144 | -0.00032 | -2.54643 | |
| 0.155313 | 12.07284 | 0.08104 | 5.171669 | 0.074779 | 5.100802 | 0.106354 | 3.106725 | |
| 0.205673 | 16.6282 | 0.359635 | 22.84687 | 0.368248 | 28.36602 | 0.275157 | 8.168141 | |
| -0.19619 | -9.80429 | -0.15601 | -7.43583 | -0.19338 | -10.01 | -0.06789 | -1.47323 | |
| 0.081668 | 5.47145 | 0.083755 | 4.03365 | 0.159345 | 8.910793 | 0.016583 | 0.314742 | |
| 0.131612 | 5.324048 | 0.115727 | 4.170396 | 0.107282 | 4.412229 | 0.1093 | 1.578589 | |
| 0.02364 | 0.981268 | -0.00268 | -0.09843 | 0.028341 | 1.212702 | 0.027789 | 0.42512 | |
| 0.122919 | 5.054341 | 0.117295 | 4.189873 | 0.138125 | 5.634414 | 0.097687 | 1.419719 | |
| 0.135043 | 5.596049 | 0.154982 | 5.445144 | 0.143342 | 5.879854 | 0.127856 | 1.803636 | |
| 0.034108 | 0.930621 | 0.113121 | 3.3613 | 0.350556 | 13.56767 | 0.58311 | 10.77931 | |
| 0.186893 | 7.839752 | 0.181157 | 6.241838 | 0.179 | 7.3533 | 0.064058 | 0.869105 | |
| 0.2438 | 10.16457 | 0.195785 | 6.728269 | 0.209928 | 8.637929 | 0.09961 | 1.43655 | |
| | | 0.302 | | 0.307 | | 0.331 | | 0.565 |

Regional variation in log real wages

| | South East | | Rest of the UK | |
|------------|--------------|----------|----------------|----------|
| | Coefficients | t-stat | Coefficients | t-stat |
| (Constant) | 0.070817 | 0.628037 | -0.18588 | -2.83481 |
| Schooling | 0.03283 | 14.53105 | 0.025668 | 19.87559 |
| AGE | 0.055873 | 12.01928 | 0.063831 | 22.75449 |
| AGE SQ | -0.00064 | -11.1071 | -0.00074 | -20.9837 |
| SEX | 0.253966 | 17.23961 | 0.300643 | 33.8597 |
| <9 | -0.19384 | -9.2052 | -0.22896 | -18.4183 |
| >500 | 0.100411 | 5.300475 | 0.09737 | 8.371546 |
| PRO | 0.289589 | 4.955016 | 0.261049 | 7.092697 |
| MAN | 0.228659 | 4.190061 | 0.246252 | 7.573783 |
| SNM | -0.05313 | -0.9588 | 0.035501 | 1.08647 |
| SM | -0.12119 | -2.14751 | -0.05636 | -1.71977 |
| PSO | -0.19596 | -3.39382 | -0.1169 | -3.52365 |
| UO | -0.17541 | -2.82167 | -0.13962 | -3.84105 |
| ARMY | -0.03561 | -0.51443 | 0.003636 | 0.088666 |
| W2 | 0.127825 | 4.512517 | 0.128312 | 7.55933 |
| W3 | -0.01947 | -0.70262 | 0.031207 | 1.887415 |
| W4 | 0.10896 | 3.848974 | 0.142439 | 8.401605 |
| W5 | 0.133526 | 4.699018 | 0.155435 | 9.172799 |
| W6 | 0.244195 | 7.584801 | 0.230597 | 12.08414 |
| W7 | 0.163457 | 5.720786 | 0.196882 | 11.64921 |
| W8 | 0.206867 | 7.245746 | 0.227713 | 13.44049 |
| R squared | 0.43 | | 0.401 | |

Using Stata PcGive, Limdep, Excel for Panel Data Analysis

Create a Panel id and time variable along with dependent and independent variables. Save your data in the CSV format; import of STATA; do preliminary statistics; Use data editor to check nature of data; select statistics/multivariate linear models/fixed effect random effect models, choose dependent and independent variables and study the results.

Estimating Panel data model in pcGive: Select panel data package, open data file, data need to be in the stacked format, determine dependent and independent variables, give time variables, estimate dynamic and static panels. Alternatively write batch programmes in PcGive as given in the next pages and do all calculations at one time.

Limdep is very good for Panel data model, particularly for limited dependent variable models; It has its own commands and routines which are not necessarily that difficult. In Shazam Pale can be computed as a system or SUR method.

Panel Unit Root Tests

Increasingly recent studies have looked into nonstationarity and heterogeneity issues in panel data model. Levin and Lin (1992)

$$\Delta y_{i,t} = a_i = \rho y_{i,t-1} + \sum_{k=1}^n \phi_k \Delta y_{i,t-k} + \delta_i t + \theta_t + u_{i,t}$$

$$H_0 : \rho = 0 \quad \text{against} \quad H_A : \rho < 0$$

Levin, A., C. Lin and C. Chu (2002): "Unit Root Tests in Panel Data: Asymptotic and finite sample properties", Journal of Econometrics, 108, p.12-24.

IM, Pesharan and Shin (1997)

Im, K.S., M. Pesaran and Y. Shin (2003): "Testing for Unit Roots in Heterogeneous Panels", Journal of Econometrics, 115, p.53-74.

$$\Delta y_{i,t} = a_i = \rho y_{i,t-1} + \sum_{k=1}^n \phi_k \Delta y_{i,t-k} + \delta_i t + \theta_t + u_{i,t}$$

$$\text{Heterogeneity in unit roots: } H_0 : \rho_i = 0 \quad \text{against} \quad H_A : \rho_i < 0$$

Maddala and Wu (1999) tests for unbalanced panel

Maddala, G.S. and S. Wu (1999): "A comparative study of unit root test with panel data and a new simple test", Oxford Bulletin of Economics and Statistics, 61, p.631-652.

$$\Pi = -2 \sum_{i=1}^N \ln \pi_i \quad \text{with } \chi^2 \text{ distribution}$$

where π_i is the probability limit of ADF test

Panel Unit Root Test

Increasingly recent studies have looked into nonstationarity and heterogeneity issues in panel data model. Levin and Lin (1992)

$$\Delta y_{i,t} = a_i = \rho y_{i,t-1} + \sum_{k=1}^n \phi_k \Delta y_{i,t-k} + \delta_i t + \theta_t + u_{i,t}$$

$$H_0 : \rho = 0 \quad \text{against} \quad H_A : \rho < 0$$

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$$\Delta y_{i,t} = a_i = \rho y_{i,t-1} + \sum_{k=1}^n \phi_k \Delta y_{i,t-k} + \delta_i t + \theta_t + u_{i,t}$$

$$\text{Heterogeneity in unit roots: } H_0 : \rho_i = 0 \quad \text{against} \quad H_A : \rho_i < 0$$

Panel Unit Root Test

Maddala and Wu (1999) tests for unbalanced panel

Maddala, G.S. and S. Wu (1999): “A comparative study of unit root test with panel data and a new simple test”, Oxford Bulletin of Economics and Statistics, 61, p.631-652.

$$\Pi = -2 \sum_{i=1}^N \ln \pi_i \text{ with } \chi^2 \text{ distribution}$$

where π_i is the probability limit of ADF test

KSSS test

$$KPSS = \sum_{t=1}^T S_t^2 / \hat{\sigma}^2 \text{ where } S_t = \sum_{s=1}^t e_s \text{ is the partial sum of errors in a regression of Y}$$

on an intercept and time trend. In contrast to the unit root test this test assumes that Y series are stationary and alternative hypothesis is nonstationarity.

Kwaitkowski, D. P.C. Phillips, P. Schmidt and Y. Shin (1992): “Testing the null hypothesis of Stationarity against the alternative of a unit root”, Journal of Econometrics, 54, p.159-178.

Panel Cointegration Test

See also Asteriou (2006) Applied Econometrics, Palgrave and Verbeek (2004) A guide to Modern Econometrics, Wiley.

Kao test

Kao, C. (1999): “Spurious Regression and residual-based Tests for Cointegration in Panel Data”, Journal of Econometrics, 90, p.1-44.

Start with $Y_{i,t} = a_i + \beta X_{i,t} + e_{i,t}$

Residual based cointegration $\hat{e}_{i,t} = \rho \hat{e}_{i,t-1} + v_{i,t}$

Estimate
$$\hat{\rho} = \frac{\sum_t \sum_i \hat{e}_{i,t}^2 \hat{e}_{i,t-1}^2}{\sum_t \sum_i \hat{e}_{i,t}^2} \text{ and}$$

related t statistics
$$t_p = \frac{(\hat{\rho} - 1) \sqrt{\sum_t \sum_i \hat{e}_{i,t}^2}}{1/(NT) \left(\sum_t \sum_i (\hat{e}_{i,t}^2 - \hat{\rho} \hat{e}_{i,t-1}^2)^2 \right)}$$

Panel Cointegration Test of Pedroni

$$Y_{i,t} = a_i + \delta_t + \sum_{m=1}^M \beta_{mi} X_{mi,t} + e_{i,t}$$

Within group tests

$$\text{Panel } v \text{ statistic } T^2 N^3 Z_{vNT} = \frac{T^2 N^{3/2}}{\left(\sum_t \sum_i L_{11}^{-2} e_{i,t}^2 \right)}$$

$$\text{Panel } \rho \text{ statistic } T\sqrt{N} Z_{\rho NT} = \frac{T\sqrt{N} \left(\sum_t \sum_i L_{11}^{-2} \hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right)}{\left(\sum_t \sum_i L_{11}^{-2} e_{i,t}^2 \right)}$$

$$\text{Panel } t \text{ statistic } Z_{tNT} = \sqrt{\sigma_{NT}^2 \left(\sum_t \sum_i L_{11}^{-2} e_{i,t-1}^2 \right)} \left(\sum_t \sum_i L_{11}^{-2} \hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right)$$

$$\text{Panel } t \text{ statistic (parametric)} Z_{tNT} = \sqrt{\sigma_{NT}^2 \left(\sum_t \sum_i L_{11}^{-2} e_{i,t-1}^2 \right)} \left(\sum_t \sum_i L_{11}^{-2} \hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right)$$

Between group tests

$$\text{Group } \rho \text{ statistic } T\sqrt{N} Z_{\rho NT} = T\sqrt{N} \frac{\sum_t \left(\hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right)}{\left(\sum_t \sum_i e_{i,t}^2 \right)}$$

$$\text{Group } t \text{ statistic } \sqrt{N} Z_{tNT-1} = \sqrt{N} \sum_{i=1}^N \sqrt{\sigma_i^2 \sum_t e_{it}^2} \left(\sum_t \left(\hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right) \right)$$

$$\text{Group } t \text{ statistic (parametric)} \sqrt{N} Z_{tNT-1} = \sqrt{N} \sum_{i=1}^N \sqrt{\sigma_i^2 \sum_t e_{it}^2} \left(\sum_t \left(\hat{e}_{i,t-1}^2 \Delta \hat{e}_{i,t-1}^2 - \hat{\lambda}_i \right) \right)$$

Panel Max and Trace Statistics

Larsson test

Based their test on Johannes' maximum likelihood procedure.

$$\Delta Y_{i,t} = \Pi_i y_{i,t-1} + \sum_{k=1}^n \Gamma_k \Delta Y_{i,t-k} + u_{i,t}$$

$$H_0 : \text{rank}(\Pi_i) = r_i \leq r \quad \text{for all } i \text{ from } 1 \text{ to } N.$$

$$H_A : \text{rank}(\Pi_i) = p \quad \text{for all } i \text{ from } 1 \text{ to } N.$$

The standard rank test statistics is defined in terms of average of the trace statistic for each cross section unit and mean and variance of trace statistics.

$$LR = \frac{\sqrt{N}(LR_{Nt} - E(Z_k))}{\sqrt{\text{Var}(Z_t)}}$$

Duration: A Random Variable

- How long a certain thing will last given that it has survived/existed for so long time?
- Duration of events is a random variable
- Length of duration of occurrence for period up to T period or survival after period T or what is probability of transition or the hazard rate between T and $T + \Delta$ period.
- Examples-
 - duration or probability of termination of strikes,
 - Unemployment Spells,
 - marriage,
 - disaster spells,
 - heart attacks or many other ill-spells,
 - likelihood of bankruptcy of a firm,
 - technological breakthrough,
 - probability of maintaining championship titles in sports).

Some Applications of Duration Analysis

- Manager of a company would be interested to know how long will a certain machine last given that it has been running so far.
- A life insurance company would be interested in probability of death of an individual with certain medical record or physical characteristic in the next $T + \Delta$ years given that the person has survived up to T years.
- When a central bank is likely to change the rate of interest given that the interest rate has remained at a certain rate up to so far?
- A union leader or the management negotiator will be interested about the probability of withdrawal of a strike given that the strike has continued up to T periods.

Densities of Duration and Survival

Cumulative Density of Duration

$$\Pr(t \leq T) = F(t) = \int_0^t f(t)dt$$

Survival Density

$$S(t) = 1 - F(t) = \Pr(t > T)$$

Hazard Rate

$$\lambda(t) = \lim_{\Delta \rightarrow 0} \frac{F(t + \Delta) - F(t)}{\Delta s(t)} = \frac{f(t)}{s(t)}$$

$$f(t) = s(t) \cdot \lambda(t)$$

Integrated Hazard and Exponential Hazard Functions

Hazard function is linked to the survival function as

$$\lambda(t) = \frac{\partial \log[1 - F(t)]}{\partial t} = \frac{-F'(t)}{1 - F(t)} = \frac{f(t)}{s(t)}$$

Integrated hazard function (the survival function)

$$\int_0^t \lambda(t) dt = -\log[1 - F(s)] + \log[1 - F(0)] = -\log[1 - F(s)]$$

$$F(s) = 1 - \exp\left(-\int_0^t \lambda(t) dt\right)$$

Modelling hazard: Exponential hazard function

$$\lambda(t, x_i) = \lambda_0(t) \exp(x_i \beta)$$

Duration Dependence

- Likelihood of how much hazard rate depends on the duration variable.
- Positive duration dependence if the longer the time spent in a given state, the higher the probability of leaving it soon.
- longer a light bulb works the higher the probability that it fails next period.
- Negative duration dependence implied longer the time spent in a given state, the lower the probability of leaving it soon.
- The longer the job search lasts, the less chance an unemployed person has finding a job.
- Absence of duration dependence is observed if the duration does not impact on the hazard rate, but this case is less appealing than the positive or negative duration dependence.

Duration Dependence

$$\lambda(t, x_i) = \lambda_0(t) \exp(x_i \beta)$$

Positive duration dependence and

$$\frac{\partial \lambda(t)}{\partial t} > 0$$

negative duration dependence

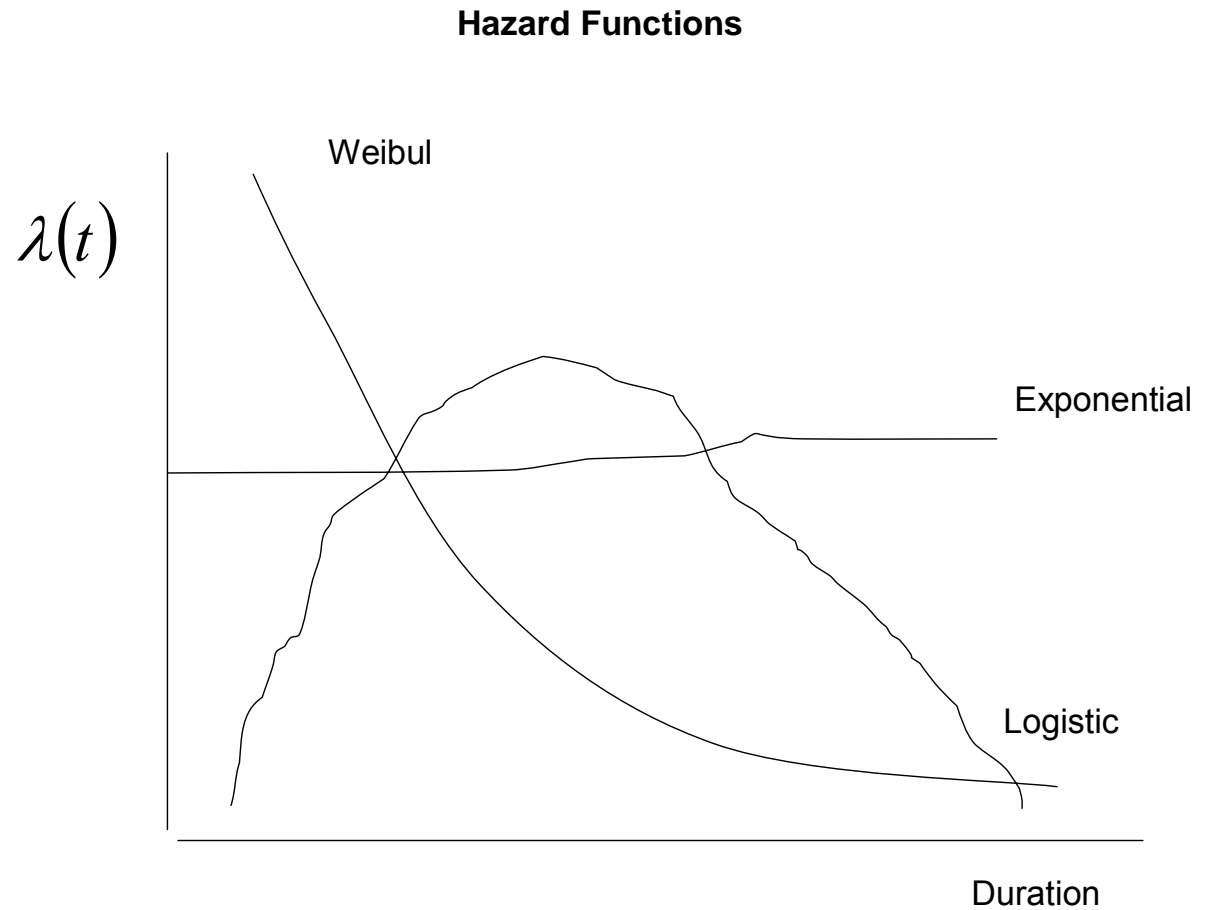
$$\frac{\partial \lambda(t)}{\partial t} < 0$$

no duration dependence

$$\lambda(t) = 0$$

Popular Distributions of Duration and Implied Hazard Functions

- Weibull
- Logistic
- Log Normal
- Gamma



Weibull Distribution of Duration and Implied Hazard Rate

$$\text{CDF: } F(t) = 1 - \exp(-\lambda \cdot t^\alpha)$$

where λ and α are nonnegative parameters;

$$\text{Density : } f(t) = \alpha\lambda \cdot t^{\alpha-1} \exp(-\lambda \cdot t^\alpha)$$

hazard function is

$$\lambda(t) = \frac{f(t)}{s(t)} = \frac{\alpha\lambda \cdot t^{\alpha-1} \exp(-\lambda \cdot t^\alpha)}{\exp(-\lambda \cdot t^\alpha)} = \alpha\lambda \cdot t^{\alpha-1}.$$

When $\alpha = 1$, the Weibull distribution reduces to the exponential distribution with $\lambda(t) = \lambda$

If $\alpha > 1$ the hazard is monotonically increasing, $\lambda(t) = \alpha\lambda \cdot t^{\alpha-1}$, which shows positive duration dependence.

If $\alpha < 1$, the hazard $\lambda(t) = \alpha\lambda \cdot t^{\alpha-1}$, is continuously decreasing.

Log-Normal Distribution of Duration

When $\log T$ follows a normal distribution with mean m and variance σ^2 it follows the normal distribution; its density is given by following function

$$f(t) = \frac{1}{\sigma \cdot t} \phi\left(\frac{\log T - m}{\sigma}\right)$$

and the survivor function is $S(t) = 1 - \Phi\left(\frac{\log T - m}{\sigma}\right)$ with Φ denoting the

CDF of a standard normal. The hazard function using $\lambda(t) = \frac{f(t)}{s(t)}$

$$\lambda(t) = \frac{f(t)}{s(t)} = \frac{1}{T} \frac{(1/\sigma)\phi\left(\frac{\log T - m}{\sigma}\right)}{1 - \Phi\left(\frac{\log T - m}{\sigma}\right)}$$

Log-Logistic Function of Hazard and Duration

Log logistic hazard function is $\lambda(t) = \frac{f(t)}{s(t)} = \frac{\gamma\alpha \cdot t^{\alpha-1}}{1 + \alpha \cdot t^\alpha}$ where the α and γ are positive parameters.

$$\int_0^\infty \lambda(st) ds = \int_0^\infty \frac{\gamma\alpha \cdot t^{\alpha-1}}{1 + \alpha \cdot t^\alpha} = \log(1 + \alpha \cdot t^\alpha) = -\left[\log(1 + \alpha \cdot t^\alpha)^{-1}\right]$$

Using $F(s) = 1 - \exp\left(-\int_0^t \lambda(t) dt\right)$ condition derived above

$$F(t) = 1 - (1 + \alpha \cdot t^\alpha)^{-1} \quad \text{for } t \geq 0$$

Differentiating with respect to t gives: $f(t) = \alpha\gamma \cdot t^\alpha (1 + \lambda \cdot t^\alpha)^{-2}$

GAMMA Distribution of Duration

$$f(t) = \left[a^\nu t^{\nu-1} \exp(-at) \right] / \Gamma(\nu) \quad \text{where } \Gamma(\nu) = \int_0^\infty \exp(-t) t^{\nu-1} dt$$

Weibull distribution is better to capture the duration variable and transition between states if the hazard is monotonically increasing or decreasing

Log normal distributions of durations give non-monotonic hazard functions; first the hazard rate increases with duration and then decreases. This type of analysis is good in modelling bankruptcy rates.

Summary of Duration Models and Hazard and Survival Functions

Exponential functions for survival.

$$S(t) = e^{-\gamma \cdot t} \quad \lambda(t) = \gamma \quad F(t) = 1 - e^{-\gamma \cdot t} \quad f(t) = \gamma e^{-\gamma \cdot t}$$

$$\text{Logistic} \quad S(t) = \frac{1}{1 + \alpha \gamma} \quad \Delta(t) = \ln(1 + \gamma t^\alpha). \quad f(t) = \frac{\gamma \alpha t^{\alpha-1}}{(1 + \gamma t^\alpha)^2} \quad \lambda(t) = \frac{\gamma \alpha t^{\alpha-1}}{1 + \gamma t^\alpha}$$

Weibull

$$S(t) = e^{-(\gamma t)^\alpha} \quad \lambda(t) = \gamma \quad F(t) = 1 - e^{-(\gamma t)^\alpha} \quad f(t) = \gamma \alpha (\gamma t)^{\alpha-1} - (1 - e^{-\gamma t})$$

Estimation of Duration Models

Parameters of above models $\theta = (\lambda, \gamma)$ can be estimated using the maximum likelihood function for uncensored and censored observations.

$$\ln L = \sum \ln f(t/\theta) + \sum \ln s(t/\theta)$$

Easily estimated by BHHH (Berdt-Hall-Hall-Hauseman (1974) estimator (See Greene (938-951))).

$$\ln L = \sum \ln \lambda(t/\theta) + \sum \ln s(t/\theta)$$

Empirical Estimation of Duration Models

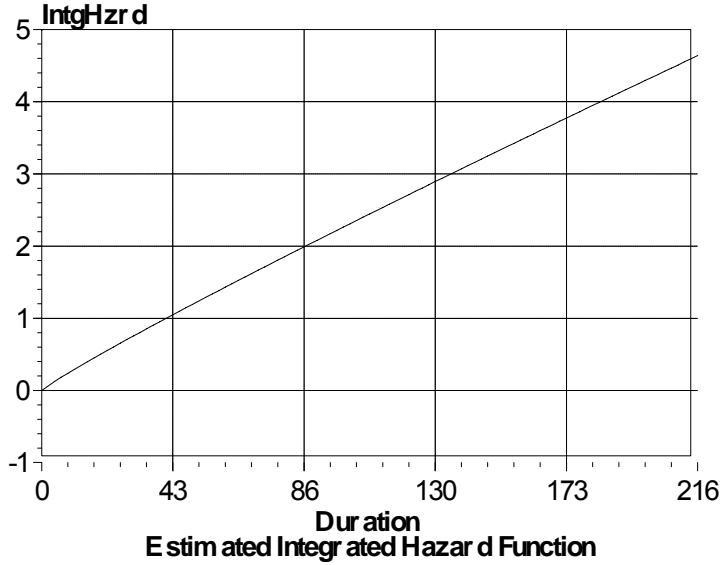
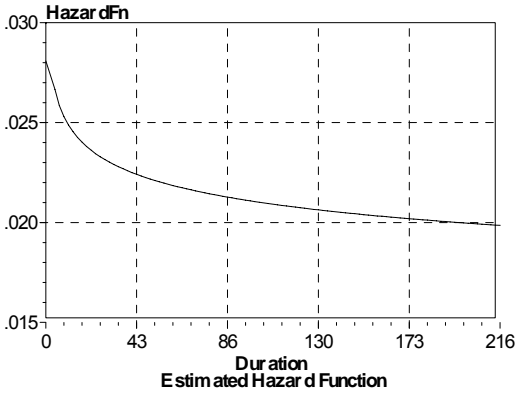
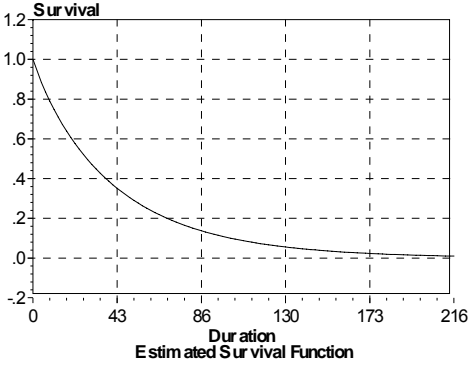
Empirical implementation (Greene (2000); Chapter 20; Using Limdep)

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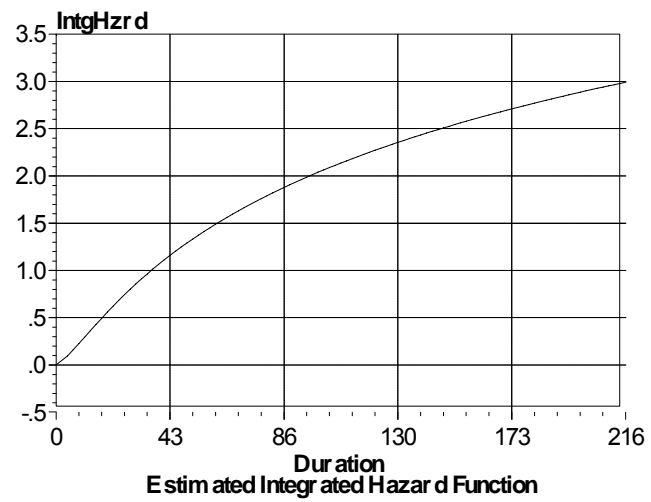
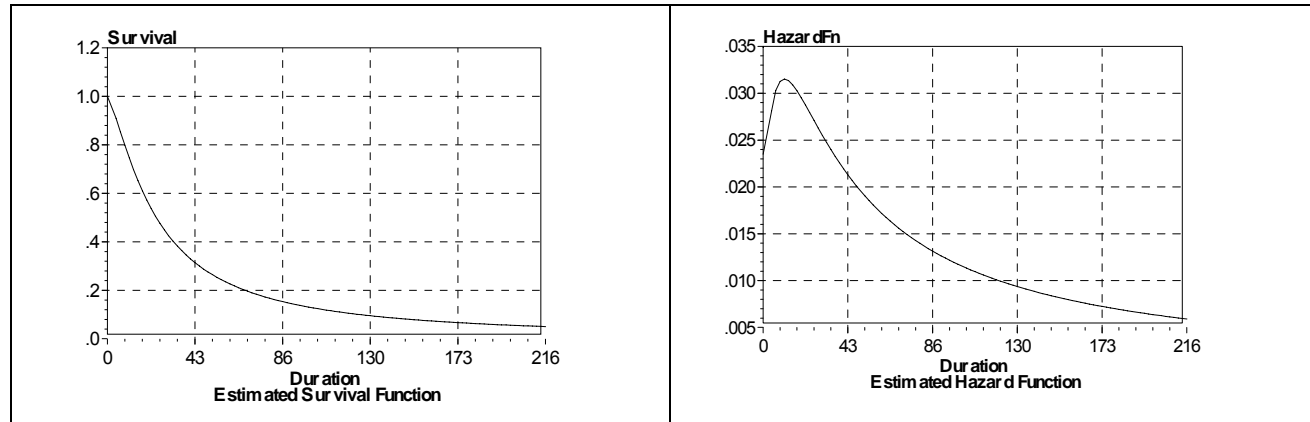
/*=====
Example 20.17. Log-Linear Survival Models for Strike Duration
*/=====
Read ; Nobs = 62 ; Nvar = 2 ; Names = T,Prod $
      T          Prod      T          Prod      T          Prod
7.00000      .0113800  9.00000      .0113800  13.0000      .0113800
14.0000      .0113800  26.0000      .0113800  29.0000      .0113800
52.0000      .0113800  130.000      .0113800  9.00000      .0229900
37.0000      .0229900  41.0000      .0229900  49.0000      .0229900
52.0000      .0229900  119.000      .0229900  3.00000      -.0395700
17.0000      -.0395700  19.0000      -.0395700  28.0000      -.0395700
72.0000      -.0395700  99.0000      -.0395700  104.000      -.0395700
114.000      -.0395700  152.000      -.0395700  153.000      -.0395700
216.000      -.0395700  15.0000      -.0546700  61.0000      -.0546700
98.0000      -.0546700  2.00000      .0053500  25.0000      .0053500
85.0000      .0053500  3.00000      .0742700  10.0000      .0742700
1.00000      .0645000  2.00000      .0645000  3.00000      .0645000
3.00000      .0645000  3.00000      .0645000  4.00000      .0645000
8.00000      .0645000  11.0000      .0645000  22.0000      .0645000
23.0000      .0645000  27.0000      .0645000  32.0000      .0645000
33.0000      .0645000  35.0000      .0645000  43.0000      .0645000
43.0000      .0645000  44.0000      .0645000  100.000      .0645000
5.00000      -.104430  49.0000      -.104430  2.00000      -.0070000
12.0000      -.0070000  12.0000      -.0070000  21.0000      -.0070000
21.0000      -.0070000  27.0000      -.0070000  38.0000      -.0070000
42.0000      -.0070000  117.000      -.0070000
?
? Four survival models for duration
?
Create ; logT = Log(T) $
Surv; Lhs=logT ; Rhs = One ; Model=Exponential ; Plot$
Surv; Lhs=logT ; Rhs = One ; Model=Weibull ; Plot$
Surv; Lhs=logT ; Rhs = One ; Model=Logistic ; Plot$
Surv; Lhs=logT ; Rhs = One ; Model=Normal ; Plot $

```

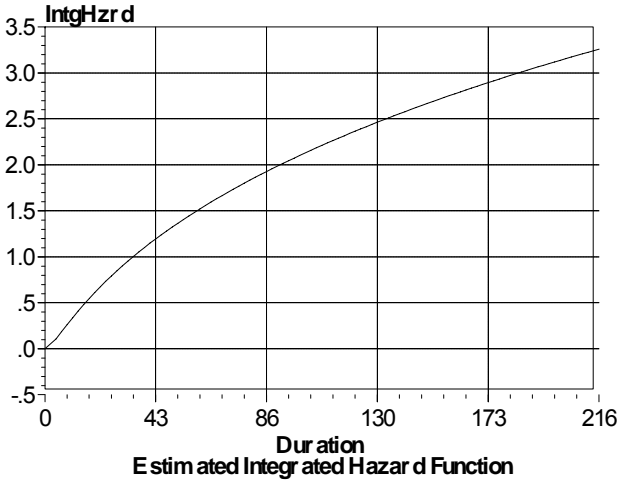
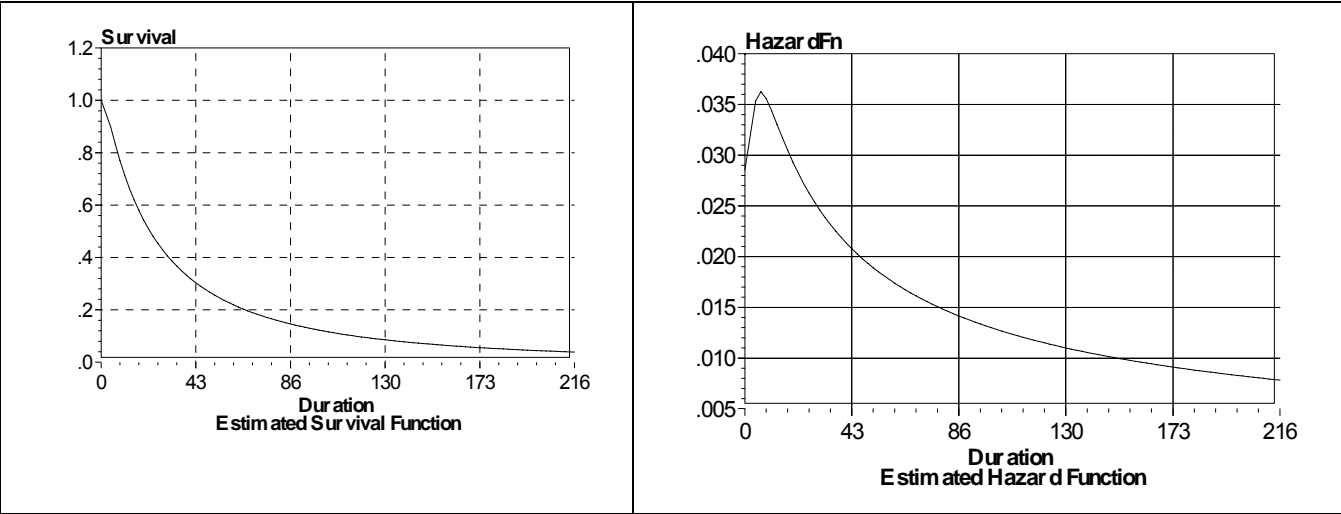

Weibull Hazard Model



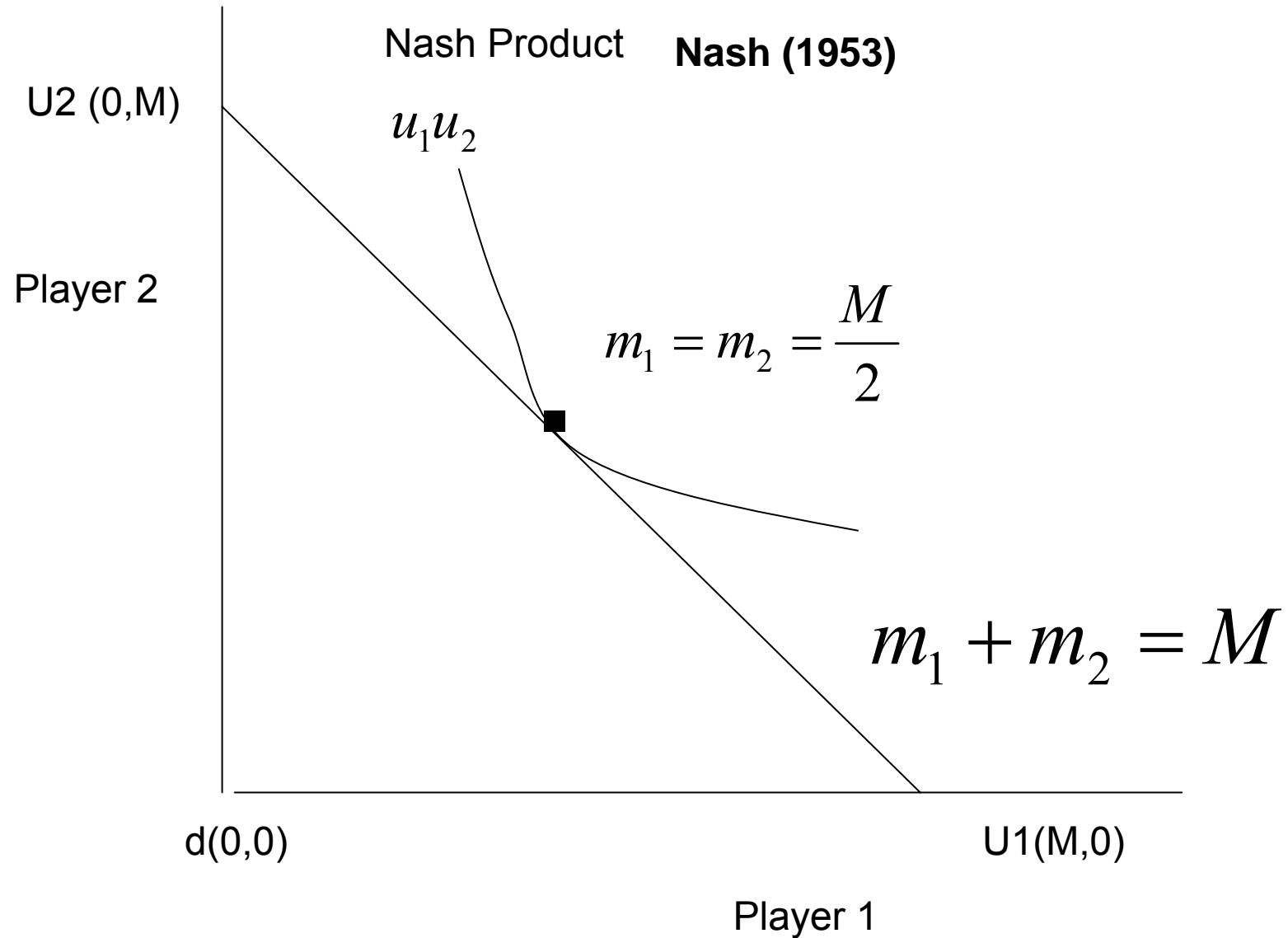
Logistic Hazard Models



Log Normal Hazard Model

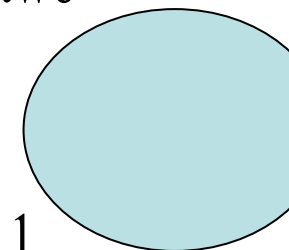


Nash Product: Utility Possibility Frontier



Bargaining on Splitting a Pie

The very common example for bargaining game is splitting a pie between two individuals.



The sum of the shares of the pie claimed by both cannot exceed more than 1, otherwise each will get zero.

If we denote these shares by θ_i and θ_j then $\theta_i + \theta_j \leq 1$ is required for a meaningful solution of the game where each get $\pi_i \geq 0$ and $\pi_j \geq 0$ payoff. When $\theta_i + \theta_j > 1$ then $\pi_i = 0$ and $\pi_j = 0$.

Standard technique to solve this problem is to use the concept of Nash Product .

Nash Product Solution of Splitting a Pie Game

$$\max U = (\theta_i - 0)(\theta_j - 0)$$

subject to

$$\theta_i + \theta_j \leq 1 \text{ or by non-satiation property } \theta_i + \theta_j = 1$$

Using a Lagrangian function

$$L(\theta_i, \theta_j, \lambda) = (\theta_i - 0)(\theta_j - 0) + \lambda[1 - \theta_i - \theta_j]$$

First order conditions

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_i} = \theta_j - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_j} = \theta_i - \lambda = 0$$

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \lambda} = 1 - \theta_i - \theta_j = 0$$

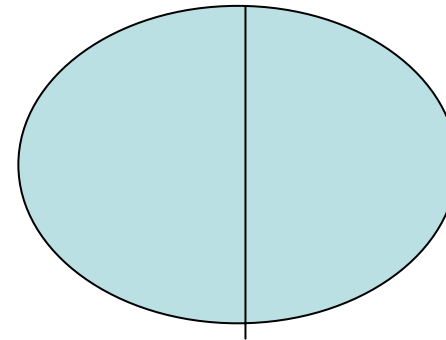
Nash Product Solution of Splitting a Pie Game

From the first two first order conditions

$$\theta_j - \lambda = \theta_i - \lambda \text{ implies}$$

$\theta_j = \theta_i$ and putting this into the third first order condition

$$\theta_j = \theta_i = \frac{1}{2}.$$



This is called focal point.

Thus Nash solution of this problem is to divide the pie symmetrically into two equal parts. Any other solution of this not stable.

Roy Gardner (2003) and Rasmusen (2007) have a number of interesting examples on bargaining game.

Nash Product Solution: A numerical Example

Suppose there is 1000 in the table to be split between two players.

What is the optimal solution from a symmetric bargaining game if the threat point is given by $d(0,0)$?

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda [1000 - u_1 - u_2]$$

Nash Product Solution A numerical Example

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

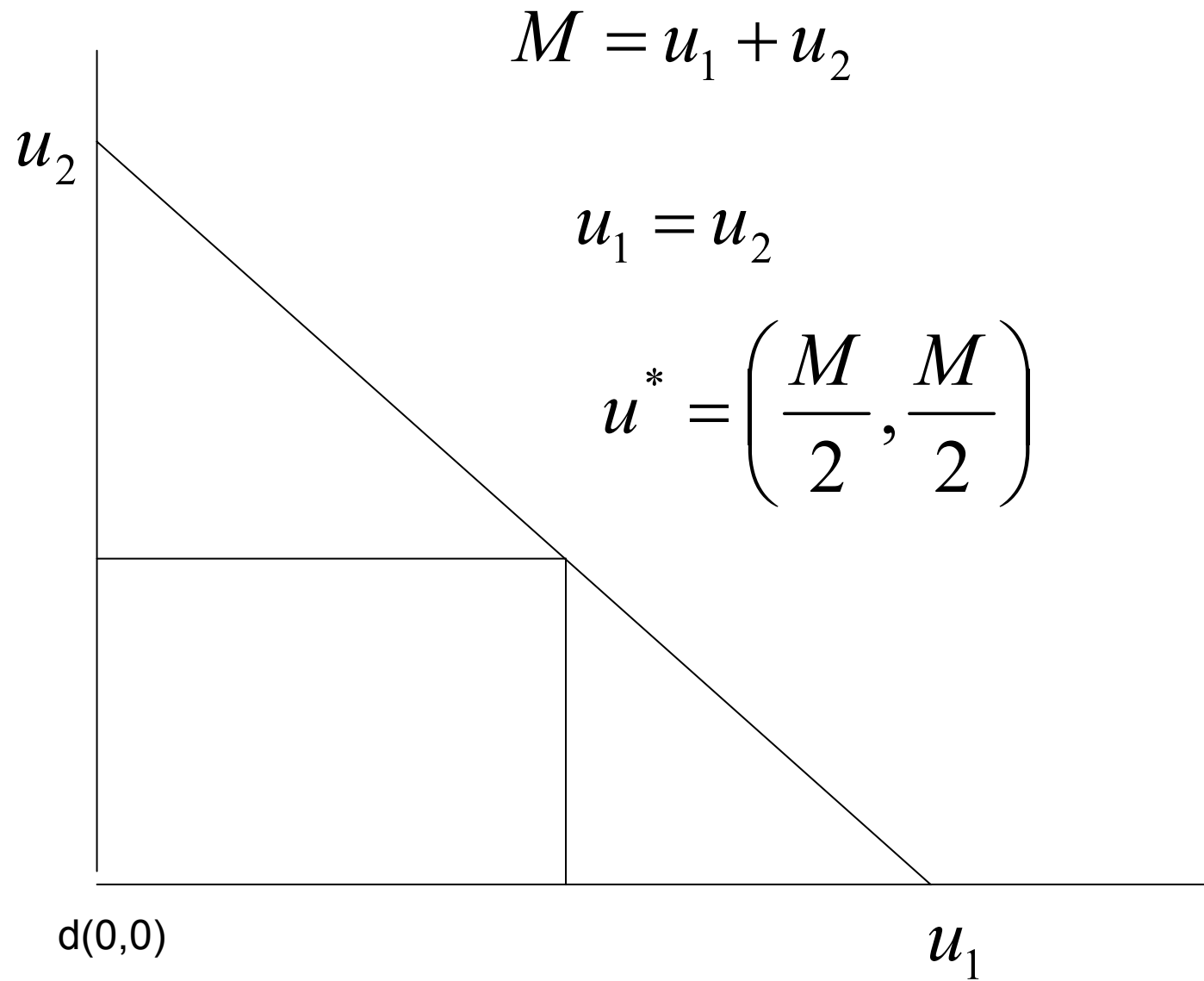
$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 1000 - u_1 - u_2 = 0$$

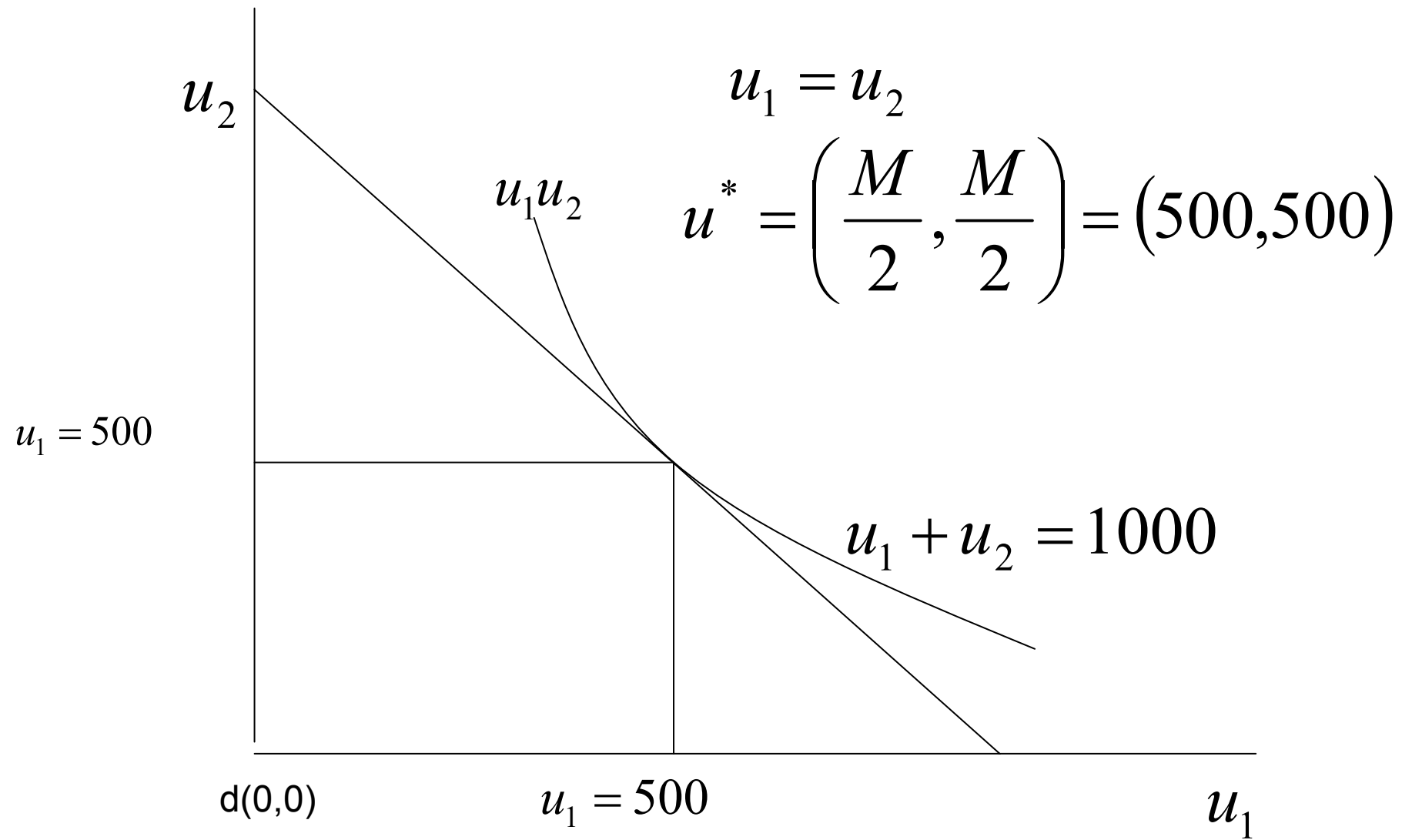
Thus $u_1 - \lambda = u_2 - \lambda$ implies $u_1 = u_2$ and putting this into the third

first order condition $u_1 = u_2 = \frac{1000}{2} = 500$.

Splitting M amount between two players



Symmetric Allocation of Amount 1000 Between Two Players



Linear Invariance in a Bargaining Game

$$L(u_1, u_2, \lambda) = (u_1 - d_1)(u_2 - d_2) + \lambda[1 - u_1 - u_2]$$

Suppose the player 1 has side payment $d_1 = 15000$

$$L(u_1, u_2, \lambda) = (u_1 - 15000)(u_2 - d_2) + \lambda[50000 - u_1 - u_2]$$

First order conditions of this maximization problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 15000 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 50000 - u_1 - u_2 = 0$$

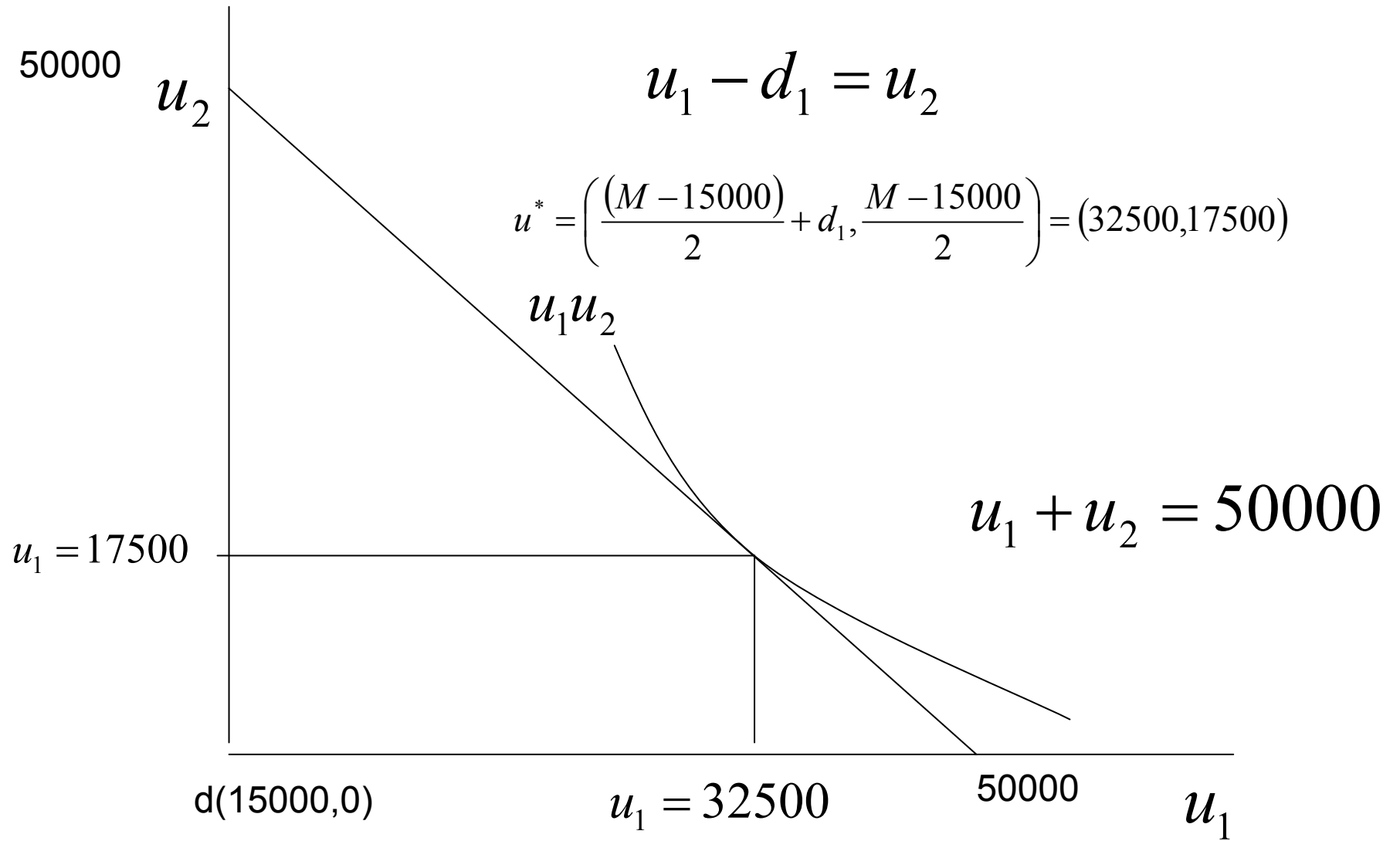
$u_1 - 15000 - \lambda = u_2 - \lambda$ implies $u_1 = 15000 + u_2$ and putting this into the third first order condition $u_2 + 15000 + u_2 = 50000$

$$2u_2 = 50000 - 15000$$

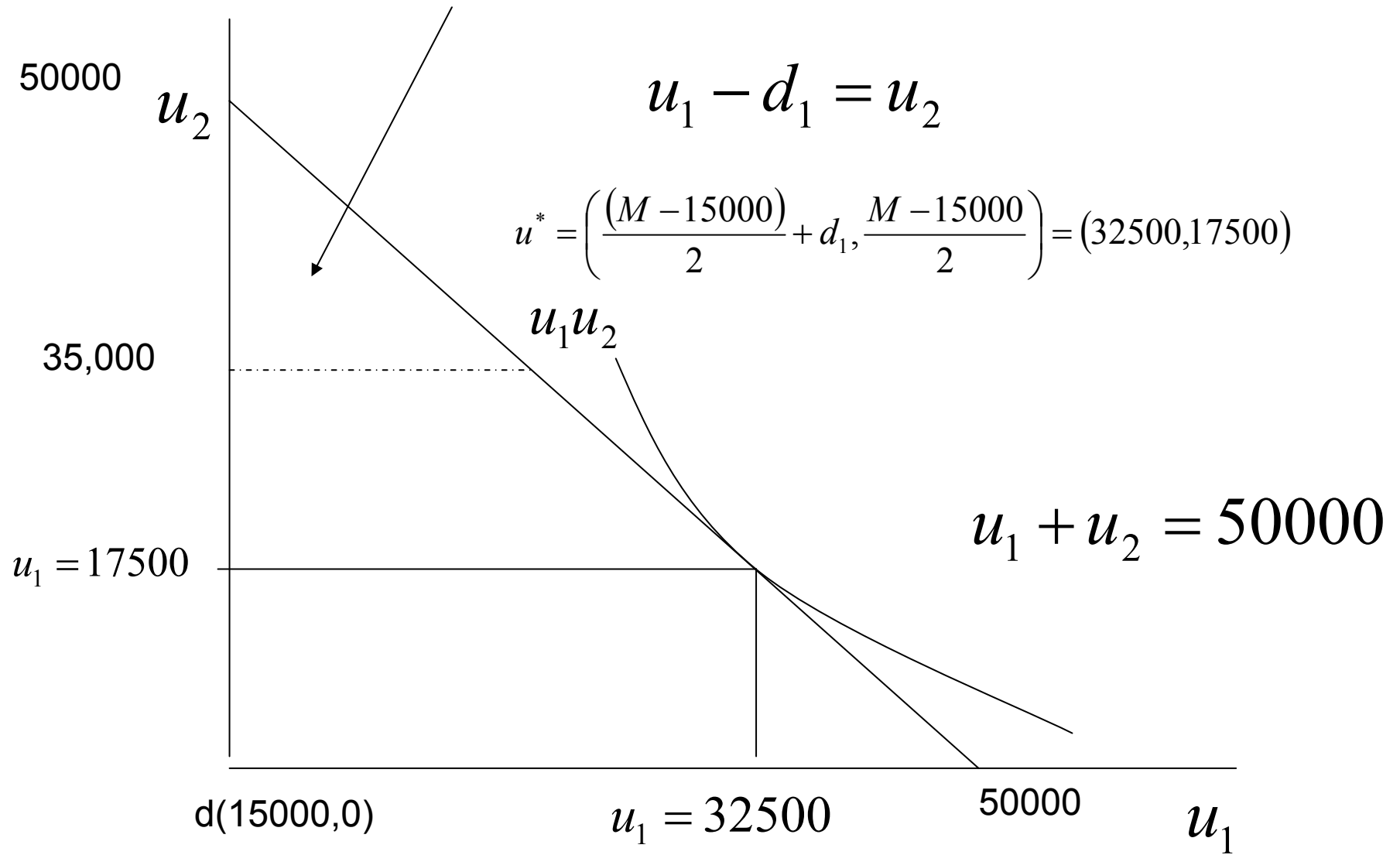
$$2u_2 = 35000 \quad u_2 = \frac{35000}{2} = 17500$$

$$u_1 = 15000 + 17500 = 32500$$

Symmetric Allocation of Amount 50,000 Between Two Players With Linear Invariance



Independence of Irrelevant Alternative (IIA)



Division of Gains between Risk Averse and Risk Neutral Players

A risk averse person loses in bargaining but the risk neutral person gains.

Suppose the utility functions of risk averse person is given by

$u_2 = (m_2)^{0.5}$ but the risk neutral person has a linear utility

$$u_1 = m_1.$$

$$m_1 + m_2 = M$$

$$u_1 + u_2^2 = 100$$

Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda [100 - u_1 - u_2^2]$$

Division of Gains between Risk Averse and Risk Neutral Players

First order conditions

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_1} = u_2 - \lambda = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial u_2} = u_1 - 2\lambda u_2 = 0$$

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \lambda} = 100 - u_1 - u_2^2 = 0$$

From the first two first order conditions $\frac{u_2}{u_1} = \frac{\lambda}{2\lambda u_2}$ implies $u_1 = 2u_2^2$

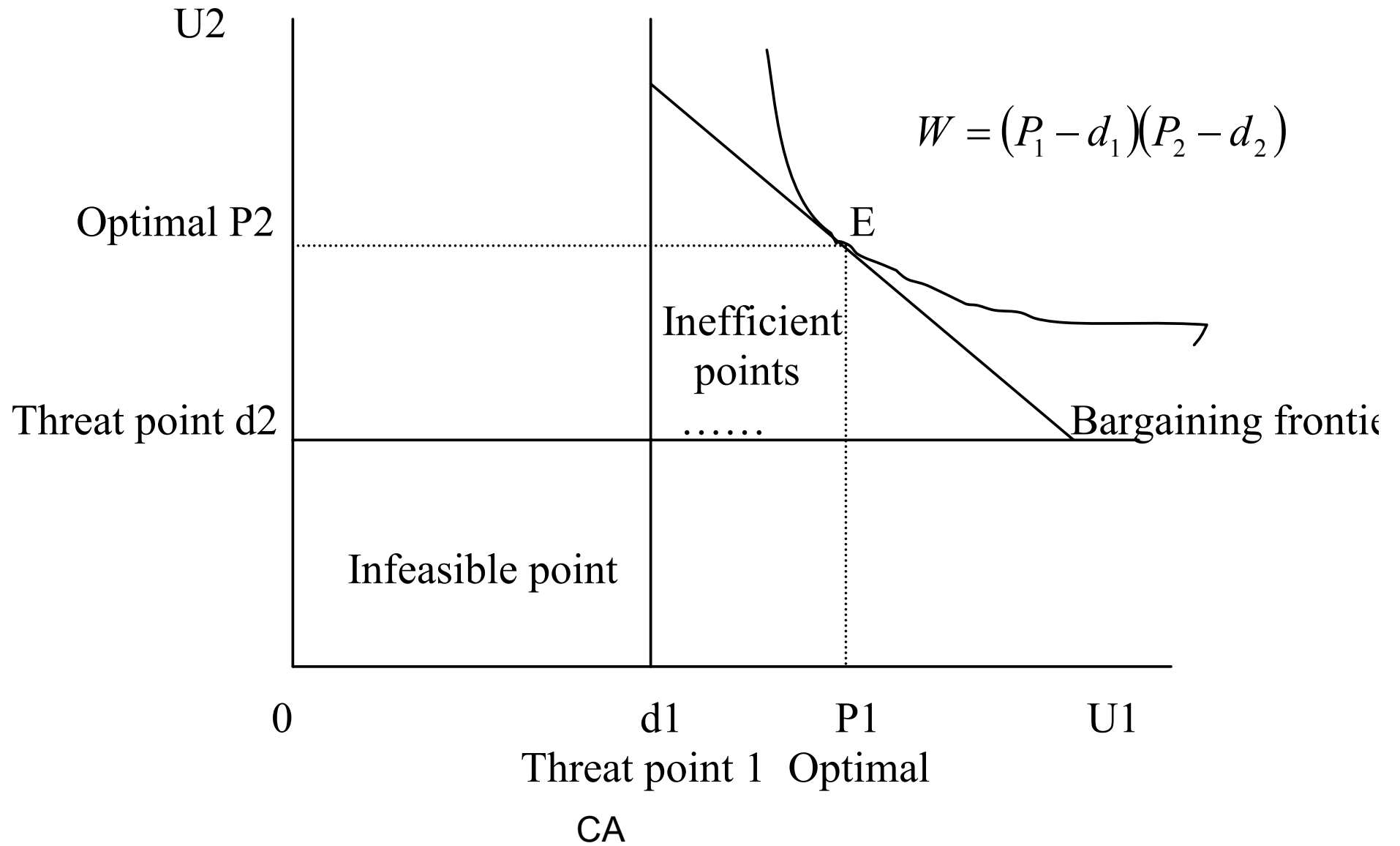
and putting this into the third first order condition $3u_2^2 = 100$.

$$u_2^2 = \frac{100}{3} = 33.33; u_2 = 5.77$$

$$u_1 = 2u_2^2 = 2(5.77)^2 = 66.6$$

$$u_1 + u_2^2 = 66.67 + 33.33 = 100$$

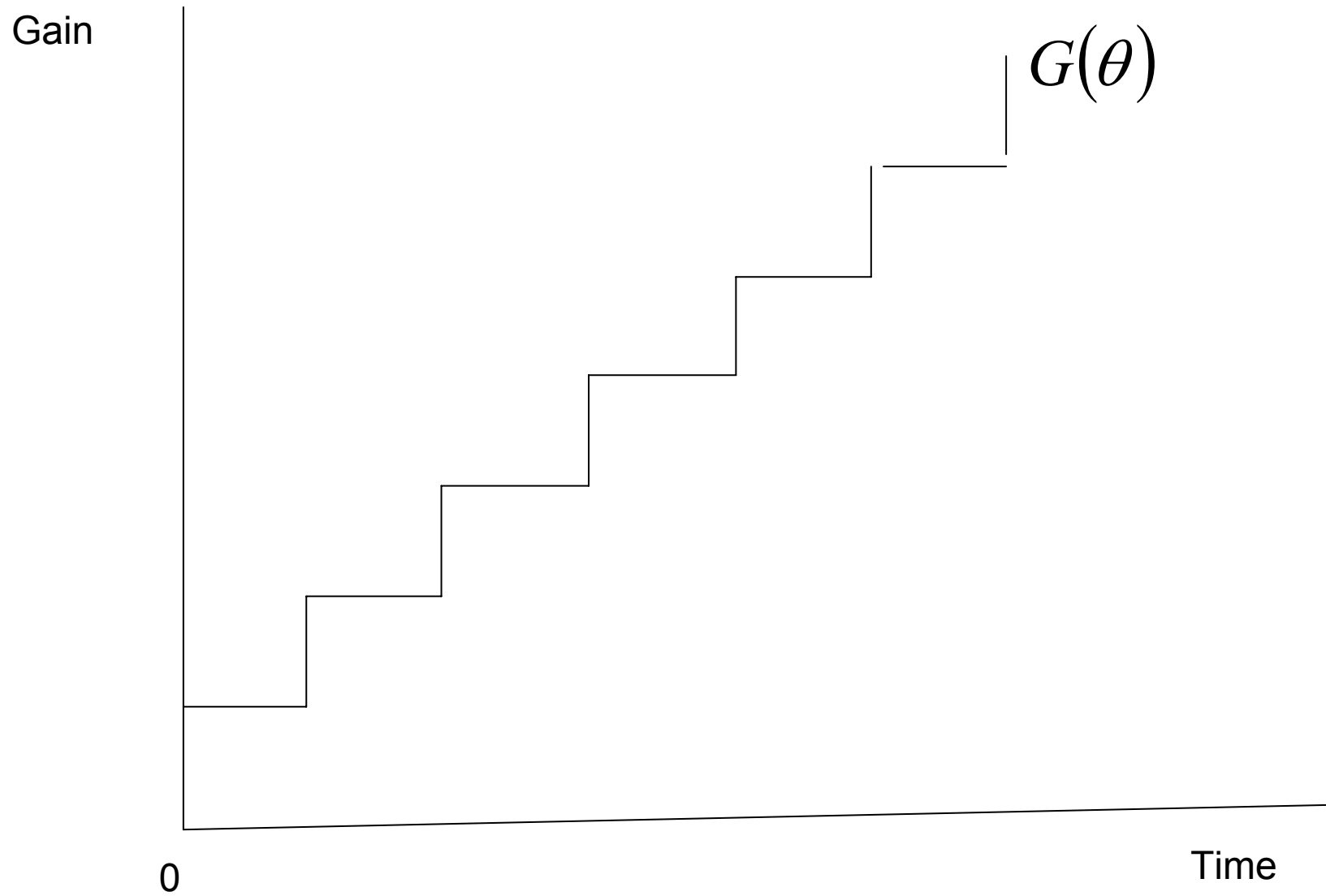
Efficient and Inefficient Bargaining Solutions



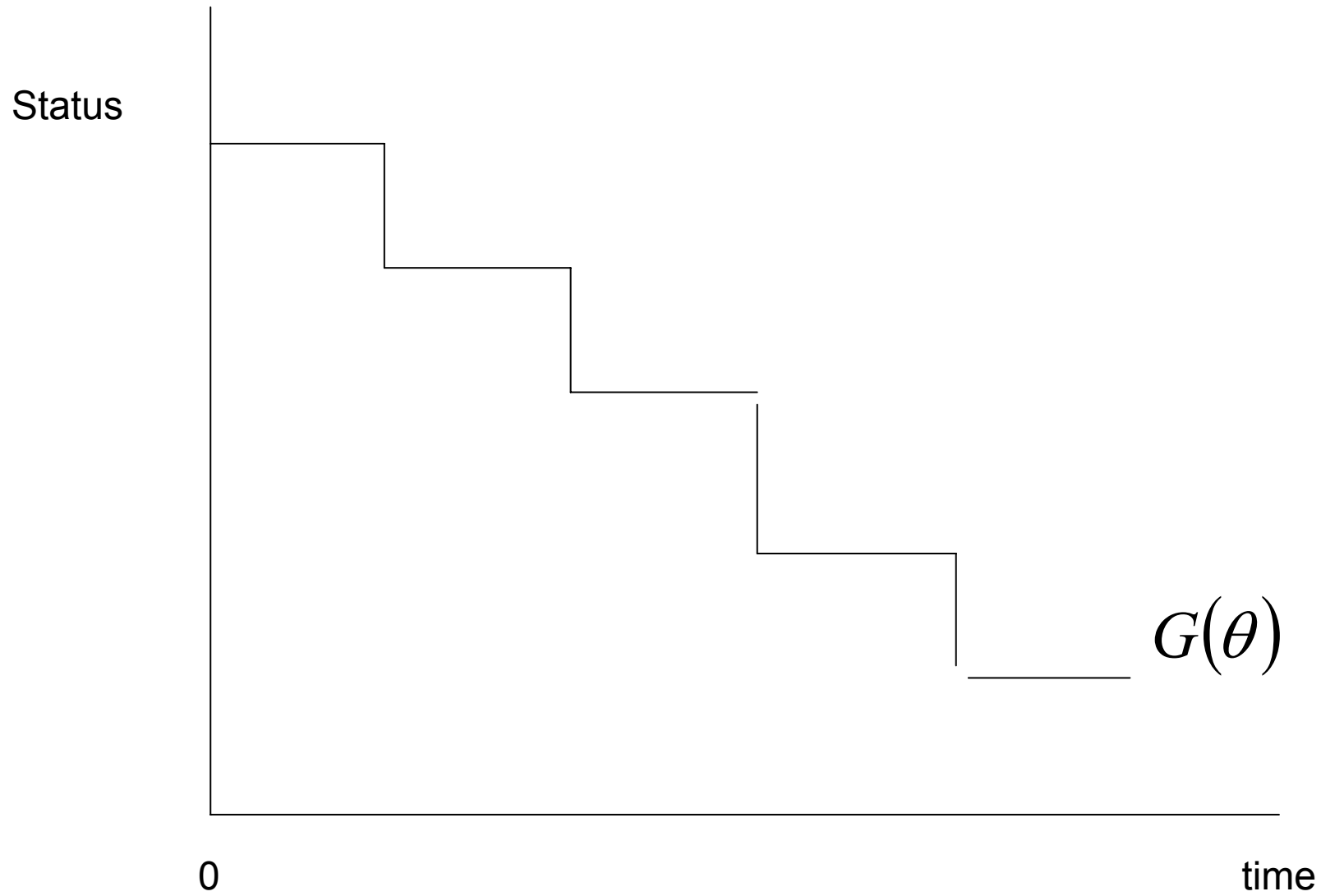
Coalition Formation and Cooperation and Core

- $2^N - 1$ rule for possible coalition
- Consider Four Players A,B,C,D
- A, B, C, D
- AB, AC, AD
- BC, BD, CD
- ABC, ACD, BCD
- ABCD
- $2^4 - 1 = 15$

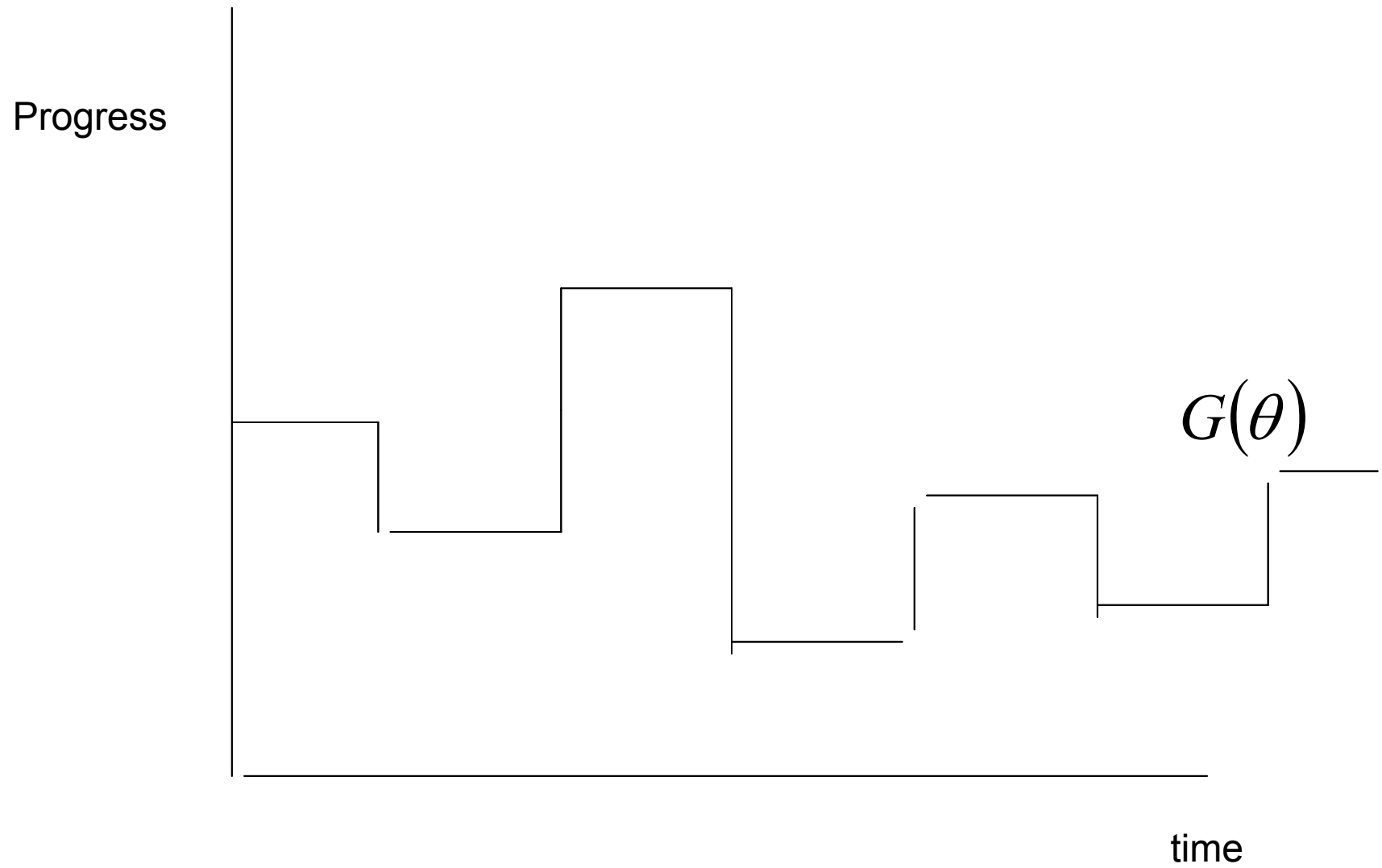
Results of a sequence of right actions following correct signals



Results of a sequence of wrong actions following reading signals incorrectly



Results of sequences of right and wrong actions reading signals correctly and incorrectly

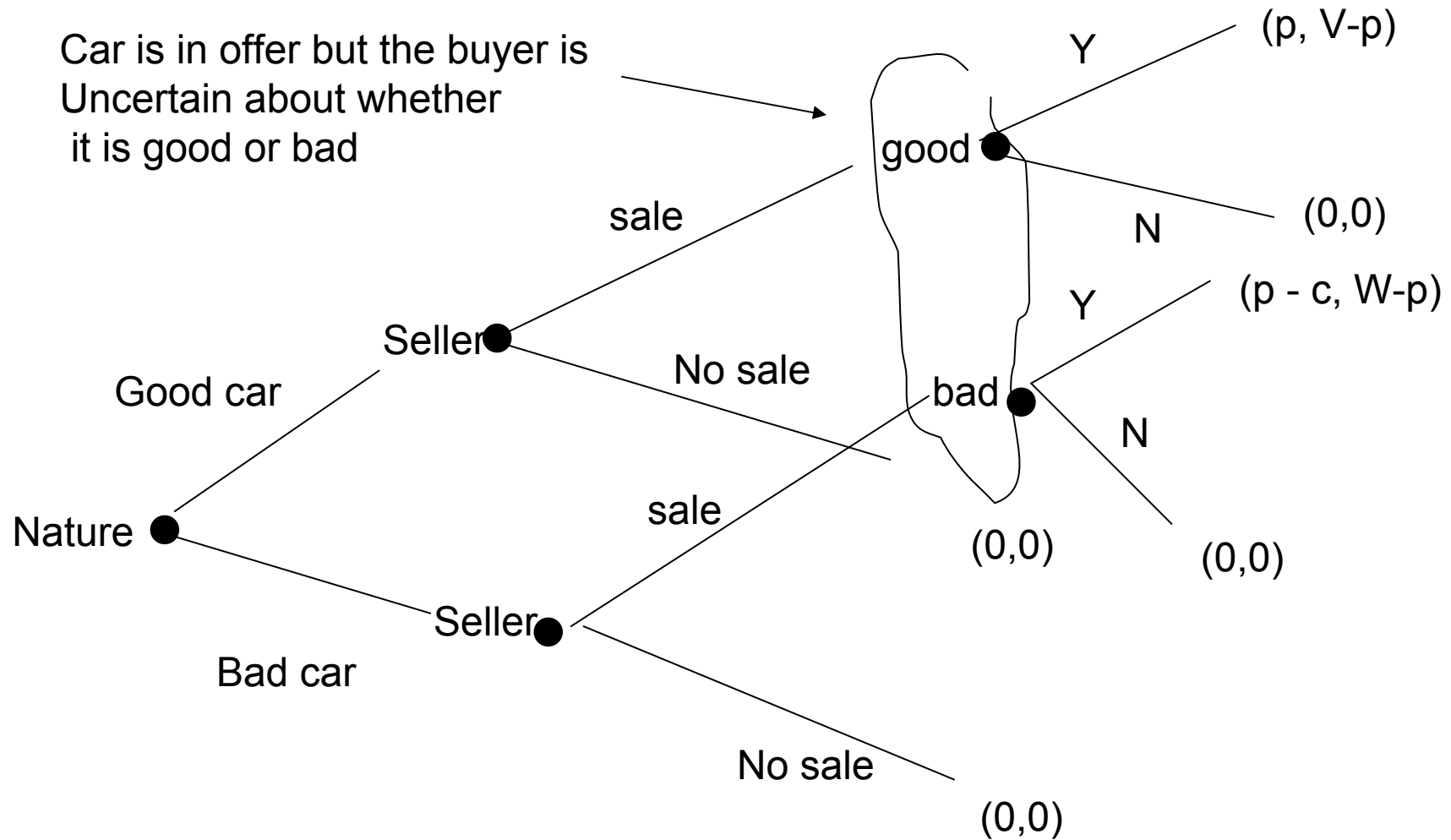


Market Situations in Signalling Game

- Complete market failure
 - pooling equilibrium (same price for good and bad cars; good cars disappear from the market)
- Complete market success
 - Separating equilibrium where players act as they should according to the signal (prices according to quality)
- Partial market success
 - (both good and bad cars are bought, some feel cheated)
- Near Market failure (mixed strategies)

Bayesian updating mechanism at work

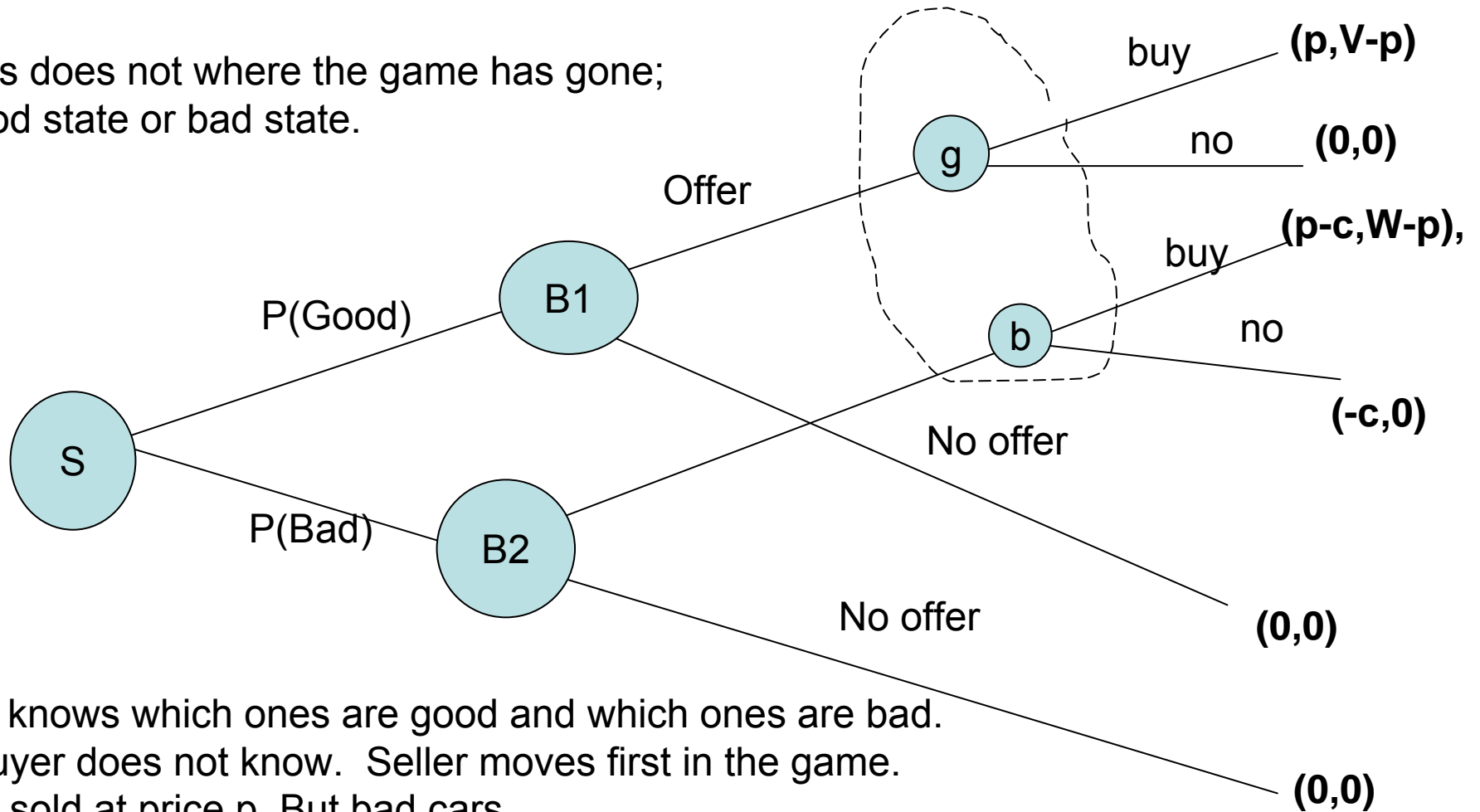
Signalling game in Markets for Used Car (Akerlof)



The buyer is uncertain about where the game is going- Need to resolve this- then the game can be solved by backward induction.

Informed Player Moves First in a Signalling Game (Akerlof)

Buyers does not where the game has gone;
At good state or bad state.



Seller knows which ones are good and which ones are bad.
But buyer does not know. Seller moves first in the game.
Car is sold at price p . But bad cars
need amount c to make repair and look just like a good one.
 V is value of good car is V and W for bad car for the buyer.

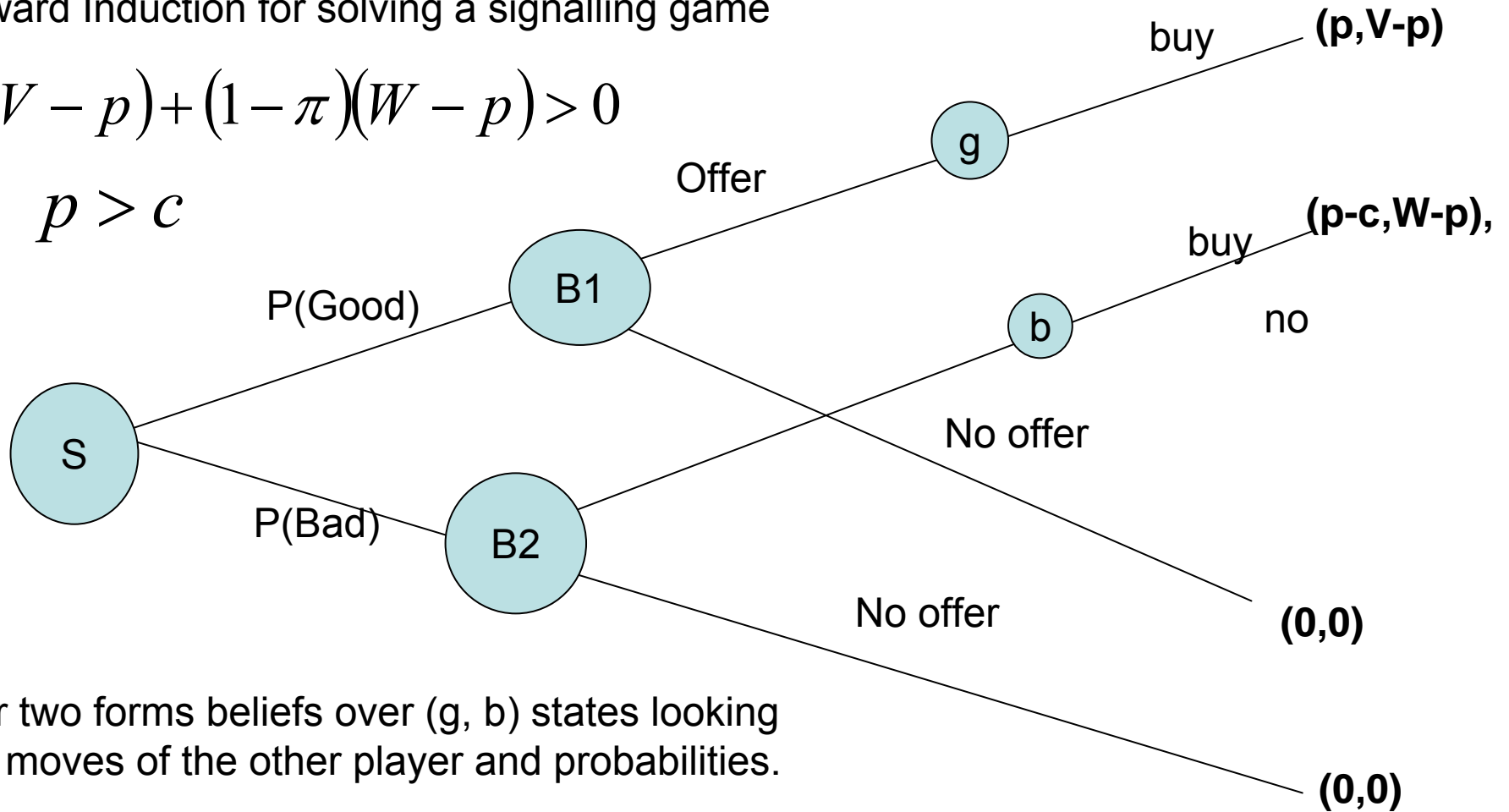
Partial Success of Market with asymmetric information.

Signals need to be credible

Backward Induction for solving a signalling game

$$\pi(V - p) + (1 - \pi)(W - p) > 0$$

$$p > c$$



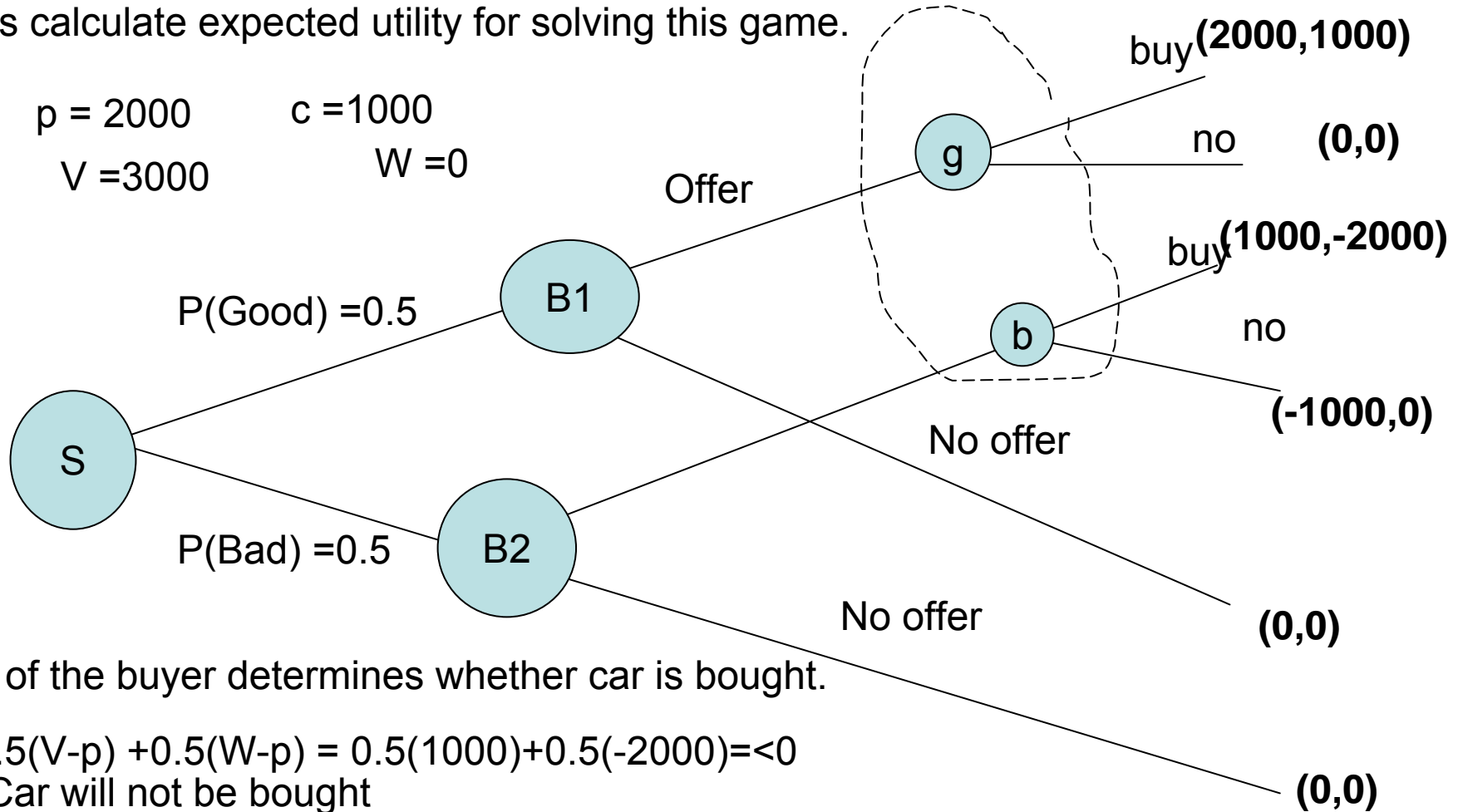
Player two forms beliefs over (g, b) states looking at the moves of the other player and probabilities.

Tries to find a sub-game perfect equilibrium maximising expected utility.

Both good and bad cars will be sold; Some buyers who get bad cars feel cheated ¹⁸¹

Mixed Strategy Sequential Equilibrium

Buyers calculate expected utility for solving this game.



Belief of the buyer determines whether car is bought.

$$0.5(V-p) + 0.5(W-p) = 0.5(1000) + 0.5(-2000) < 0$$

Car will not be bought

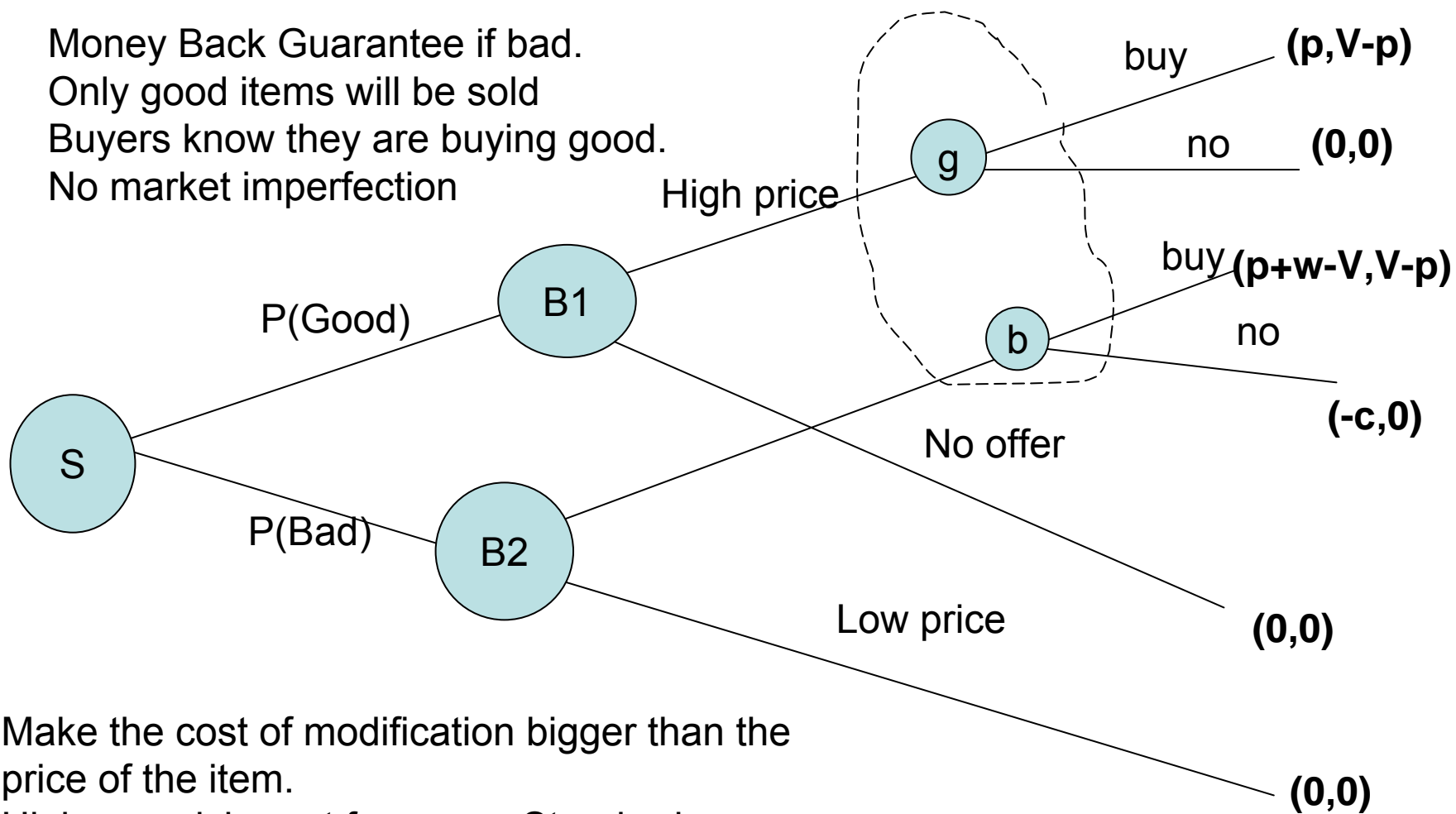
$$0.5(V-p) + 0.5(W-p) = 0.66(1000) + 0.33(-2000) = 0$$

$$EU(\text{good}) = 0.5(2000) + 0.5(0) = 1000 > 0$$

$$EU(\text{bad}) = 0.5(1000) + 0.5(-1000) = 0$$

Costly Commitment for Separating Equilibrium: Complete Market Success

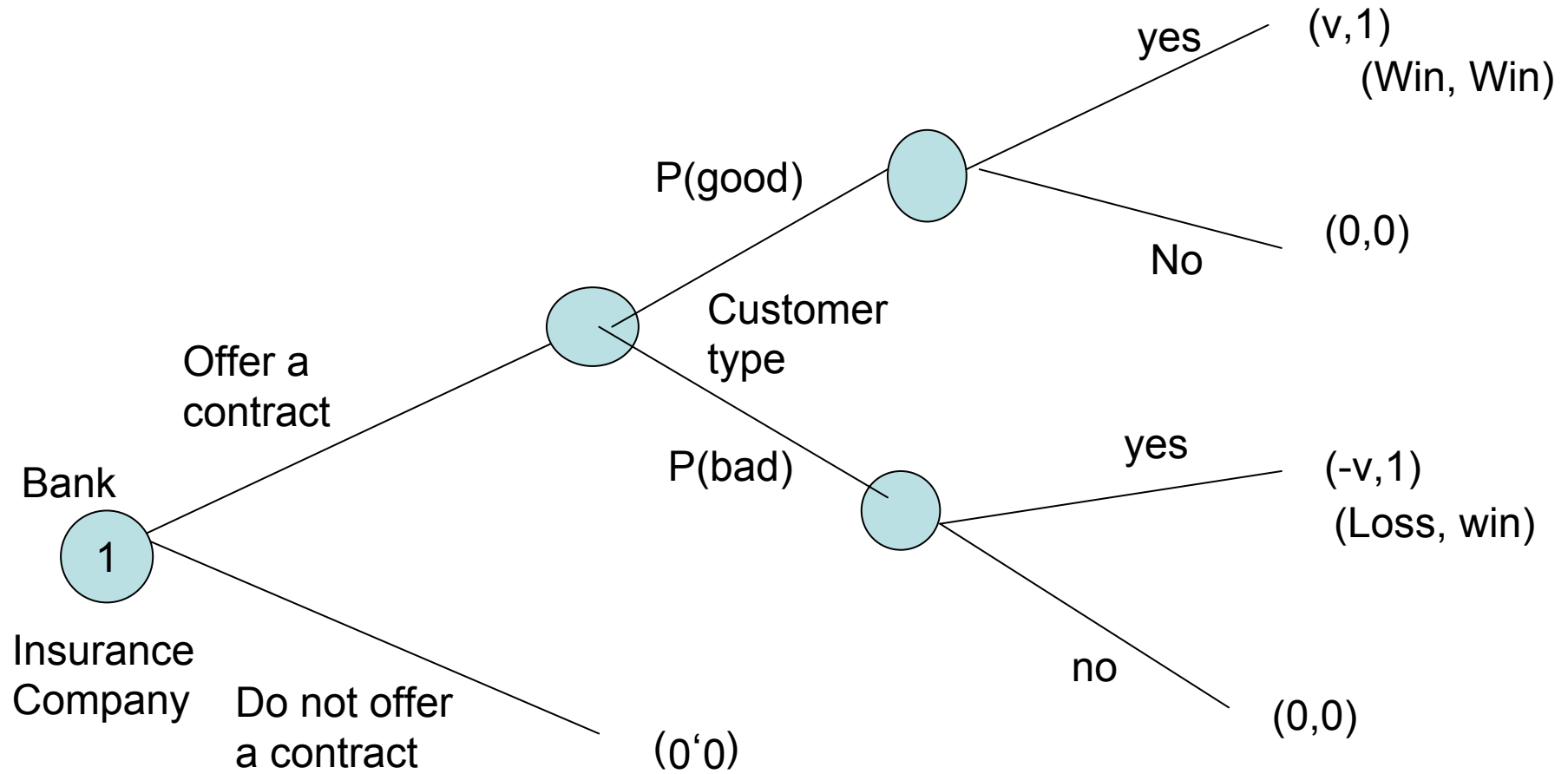
Money Back Guarantee if bad.
 Only good items will be sold
 Buyers know they are buying good.
 No market imperfection



Make the cost of modification bigger than the price of the item.
 Higher punishment for wrong Standards.

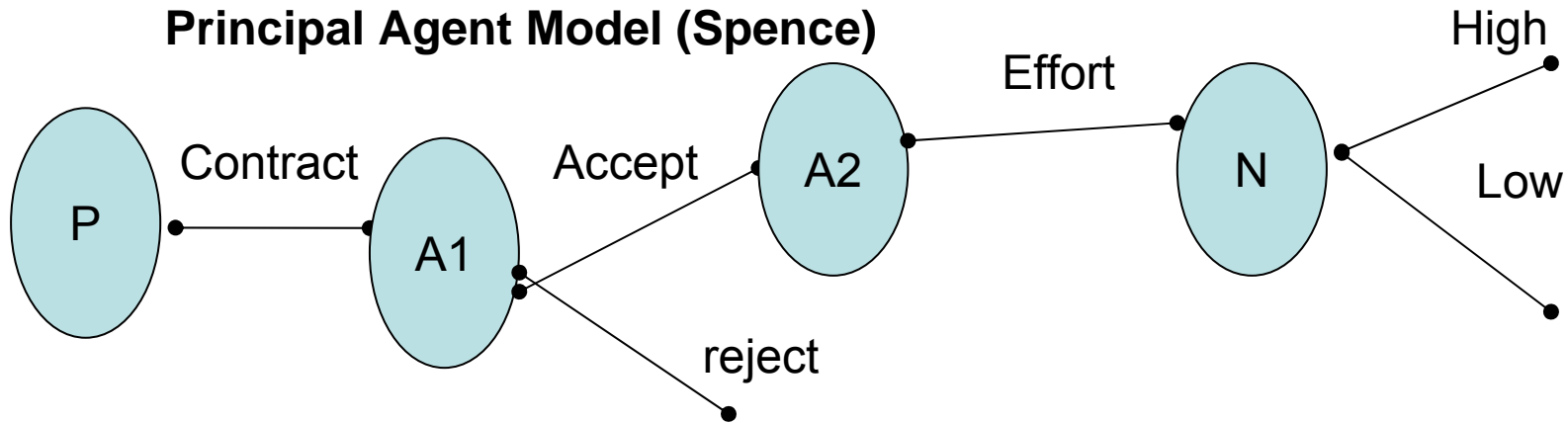
A Screening Game (Stiglitz)

Uninformed Players Moves First in

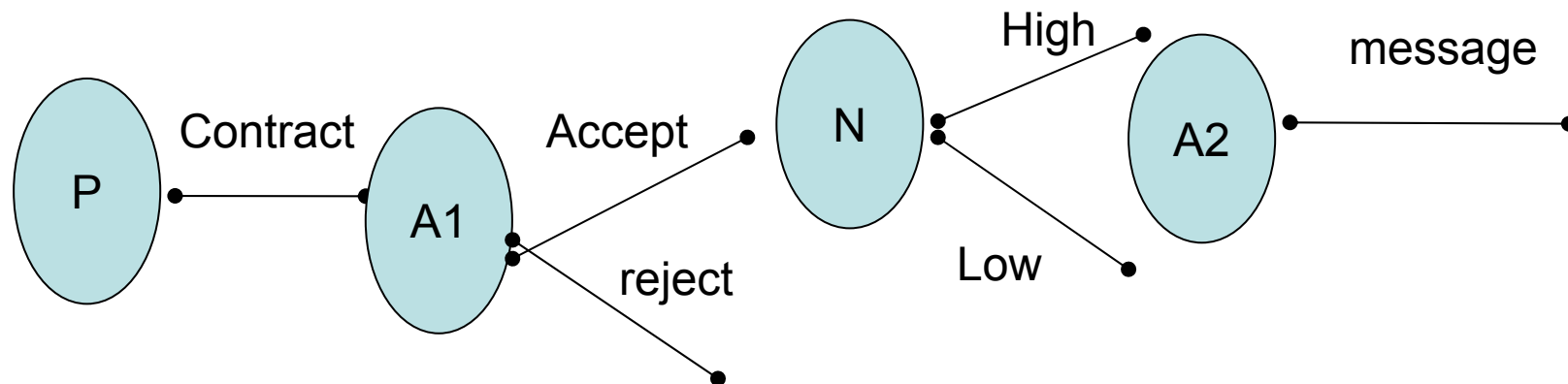


Credit Rationing or Insurance Market

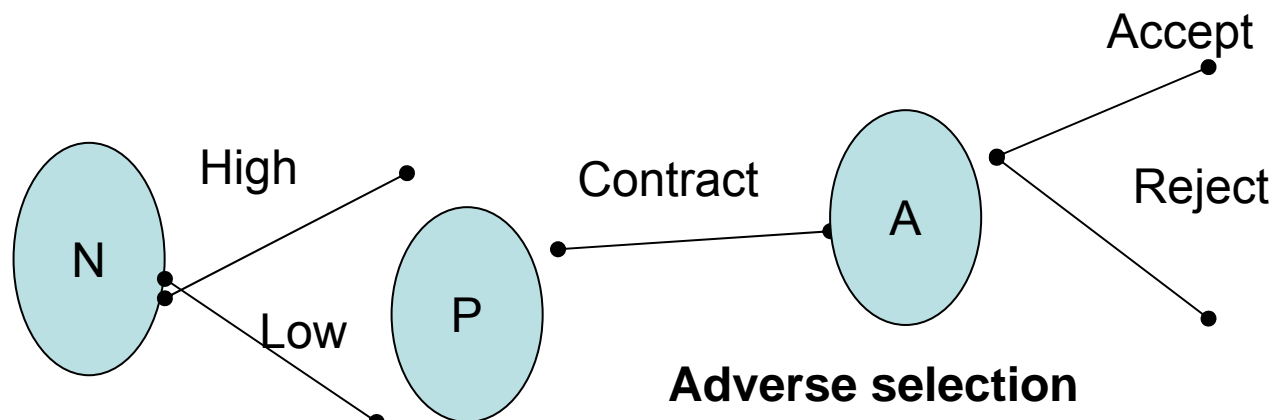
Principal Agent Model (Spence)



Moral Hazard game with Hidden action



Moral Hazard with Post contractual hidden knowledge

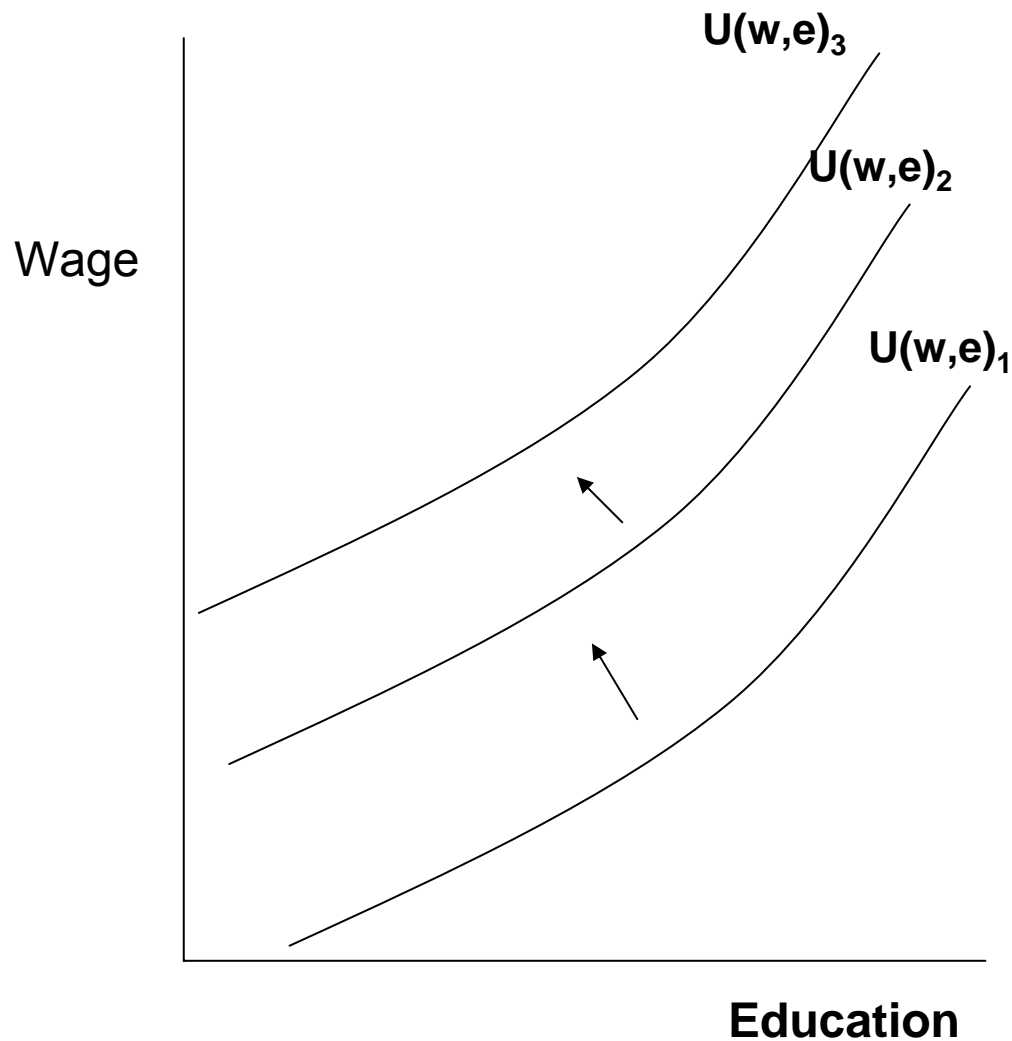


Adverse selection

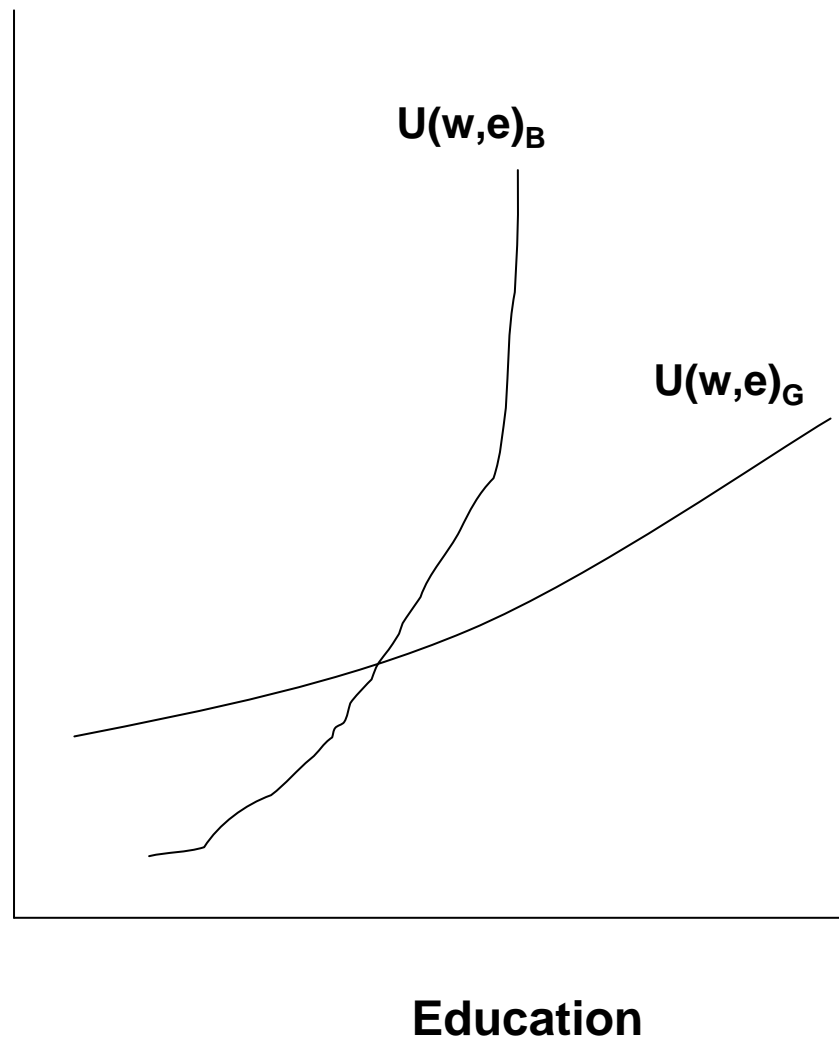
Education as Productivity Signalling Game

- Players consisting of {workers, firms and nature}.
- There are two types of workers [$t = \{1, 2\}$].
- Type 1 has less productive and type 2 more productive.
- Employer does not know which one is low or high quality worker but sees level of education
- Nature decides whether a worker is high or low productivity type.
- Level of education signals the quality of worker

Preference over wage and education



Single Crossing property



More Productive Worker Can Get Education Easily than Less Productive Worker

Preferences over Wage and Education

Workers choose level of education according to their beliefs about

its impact on wage offer: $w_t(e)$.

Utility from wage and education is given by

$$u_t(w, e)$$

Utility is rising in wage received

$$\partial u_t(w, e) / \partial w > 0$$

Utility falls in work efforts

$$\partial u_t(w, e) / \partial e < 0$$

It is costly to get education.

The utility function satisfies the single-crossing property

Specification of Utility Function

$$U_t(w, e) = f(w) - k_t g(e) \quad \text{for } k_t > 0 \quad \text{for } t = 1, 2.$$

k_t indicates the cost of education for the worker type t .

It is more expensive for less productive worker to produce education signal

$$k_1 > k_2.$$

More Specifically

$$U_t(w, e) = 42\sqrt{w} - k_t e^{1.5} \quad k_1 = 2 \quad k_2 = 1$$

Level of Education Chosen by Low Productive Worker

In perfect information equilibrium, firms pay according to the marginal productivity

$$w_1(e) = e$$

$$w_2(e) = 2e$$

The type 1 workers optimisation problem

$$U_t(w, e) = 42\sqrt{w} - k_1 e^{1.5} = 42\sqrt{e} - 2e^{1.5}$$

$$\frac{\partial U_t(w, e)}{\partial e} = 42 \times \frac{1}{2\sqrt{e}} - 3e^{0.5} = 0 \quad e^* = 7$$

Level of Education Chosen by More Productive Worker

The type 2 worker's optimisation problem

$$U_t(w, e) = 42\sqrt{w} - k_1 e^{1.5} = 42\sqrt{2e} - e^{1.5}$$

$$\frac{\partial U_t(w, e)}{\partial e} = 42 \times \frac{1}{2\sqrt{2e}} - 1.5e^{0.5} = 0$$

$$42 \times \frac{1}{\sqrt{2e}} = 1.5e^{0.5}; \quad \frac{42}{1.5\sqrt{2}} = e;$$

$$e^* = 19.8$$

Popular Principal Agent Games

| Principal | Agent | Action |
|-------------------|---------------|-------------------|
| Plantation owner | Share cropper | Labour input |
| Insurance company | Policy holder | Careful behaviour |
| Patient | Doctor | Intervention |
| Owner | Renter | Maintenance |
| Firms | Workers | Work effort |

Both principal and agents have their objective functions they like to maximize.

The principal is interested in maximising profit from the business,

Agents aim to maximise utility (payoff) choosing the best contract available from the principal with proper allowances for its efforts.

Rasmusen (2007) has interesting examples on this topic.

Incomplete Information and Adverse Selection

Principal wants to produce output employing workers with a scheme of wage contract that matches efforts put by a worker to produce.

Worker knows his type but the principal does not.

Principal knows the distribution of quality of workers $F(s)$, where s denotes either good or bad state such as probability of observing good is 0.5 and of bad 0.5.

Principal offers the agent a wage contract $W(q)$.

Worker accepts or rejects this contract based on self-selection and participation constraints.

Objectives of Principal and Agents

Basically worker evaluates the utility from the wage and disutility from work and decides the amount of work to put in.

Output from good workers is $q(e, good) = 3e$ and from bad state is $q(e, bad) = e$

If agent rejects the contract there is no work both worker and principal get zero payoff.

If worker accepts the contract

Agent's utility: $\pi_{agent} = U(e, w, s) = w - e^3$

Principal's profit: $\pi_{principal} = V(q - w) = q - w$.

Objectives and Optimal Efforts for Agents

Good worker maximises

$$\underset{e_g}{Max} \quad 3e_g - e_g^2$$

The first part is wage income and the second part of disutility of work.

The optimal level of efforts by good agent is:

$$3 - 2e_g = 0$$

$$e_g = 1.5$$

Bad worker's Objective and Optimal Efforts

$$\underset{e_b}{Max} \quad e_b - e_b^2$$

$$1 - 2e_b = 0 \quad ; \quad e_g = 0.5$$

The principal does not know what levels of efforts are appropriate for good and bad workers.

Principal's Objective and Contracts

Principal maximises expected profit

$$\underset{q_g, q_b, w_g, w_b}{Max} \left[0.5(q_g - w_g) + 0.5(q_b - w_b) \right]$$

by designing separate contracts for good and bad worker
 (q_g, w_g) and (q_b, w_b) .

Wage for good worker: $q(e, good) = 3e$ or $e = \frac{q_g}{3}$

Wage for bad worker: $q_b = e$

Incentive Compatibility Constraints for Agents

Self selection constraint for good worker

$$\pi_{agent}(q_g, w_g / good) = w_g - \left(\frac{q_g}{3}\right)^2 \geq \pi_{agent}(q_b, w_b / good) = w_b - \left(\frac{q_b}{3}\right)^2$$

Self selection constraint for bad worker

$$\pi_{agent}(q_b, w_b / bad) = w_b - (q_b)^2 \geq \pi_{agent}(q_g, w_g / bad) = w_g - (q_g)^2$$

Participation constraints for good worker

$$\pi_{agent}(q_g, w_g / good) = w_g - \left(\frac{q_g}{3}\right)^2 \geq 0$$

Participation constraint for bad worker

$$\pi_{agent}(q_b, w_b / bad) = w_b - (q_b)^2 \geq 0$$

Binding Constraints

Participation constraint of bad worker

$$w_b = (q_b)^2$$

Self selection constraint for good worker

$$w_g = \left(\frac{q_g}{3}\right)^2 + w_b - \left(\frac{q_b}{3}\right)^2 = \left(\frac{q_g}{3}\right)^2 + (q_b)^2 - \left(\frac{q_b}{3}\right)^2$$

Solving the Principal Agent Game

Principal's objective function

$$\underset{q_g, q_b, w_g, w_b}{Max} \left[0.5(q_g - w_g) + 0.5(q_b - w_b) \right]$$

Including binding constraints of agents:

$$\underset{q_g, q_b}{Max} \left[0.5 \left(q_g - \left(\frac{q_g}{3} \right)^2 - (q_b)^2 + \left(\frac{q_b}{3} \right)^2 \right) + 0.5(q_b - (q_b)^2) \right]$$

First order conditions with respect to q_g and q_b

$$0.5 \left(1 - \frac{2q_g}{9} \right) = 0 \quad \text{or} \quad q_g = 4.5$$

$$0.5 \left(-2q_b + \frac{2q_b}{9} \right) + 0.5(1 - 2q_b) = 0 \quad \text{or} \quad \left(-2q_b + \frac{2q_b}{9} \right) + (1 - 2q_b) = 0$$

$$\left(-4q_b + \frac{2q_b}{9} + 1 \right) = 0; \quad 34q_b = 9 \quad \text{or} \quad q_b = 0.265$$

Incentive Compatible First Best Choices of Good and Bad Worker

Now wages can be found from the constraints

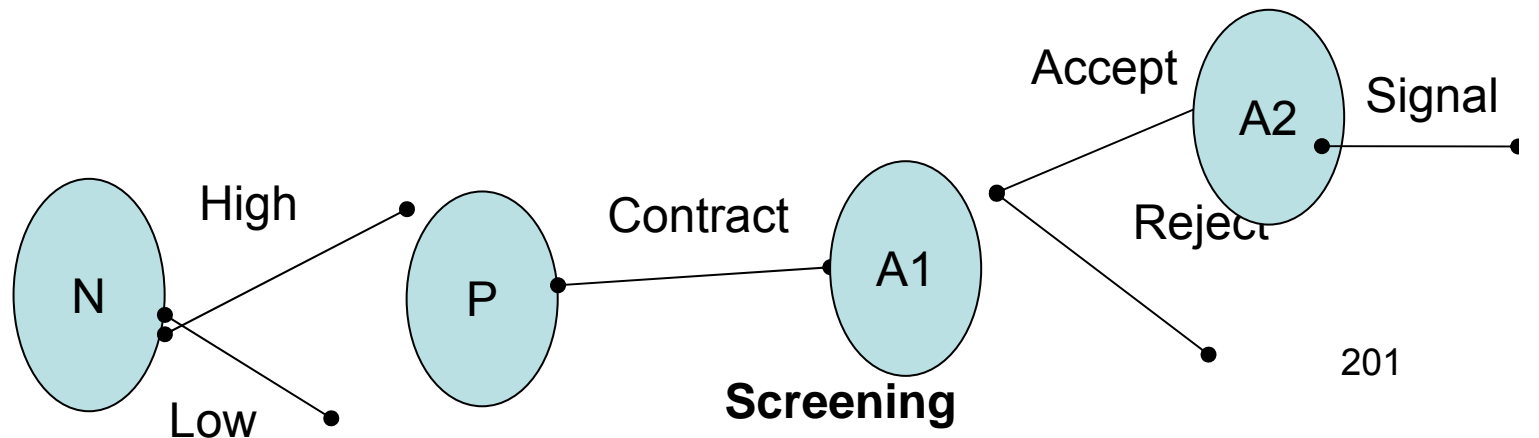
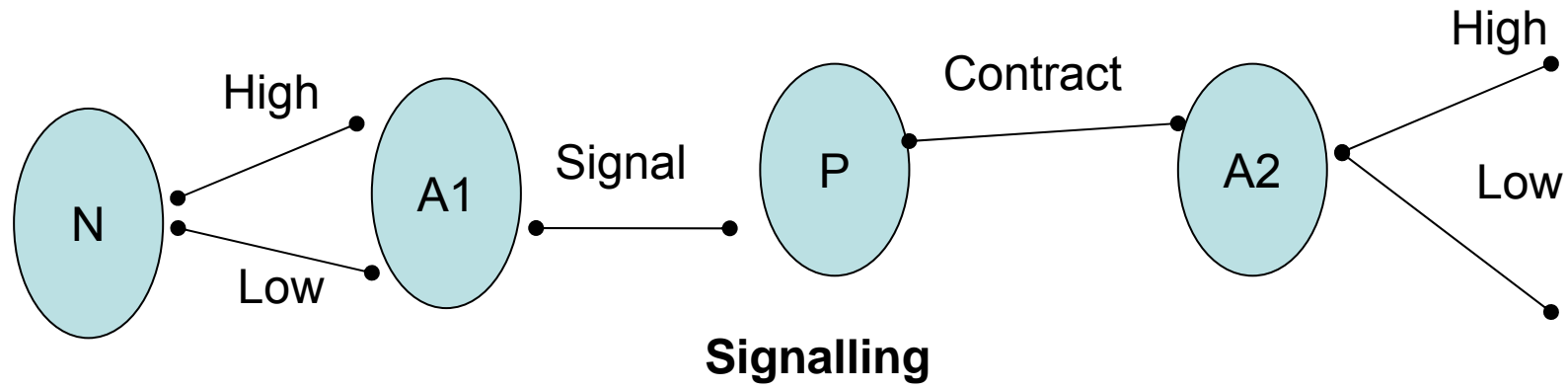
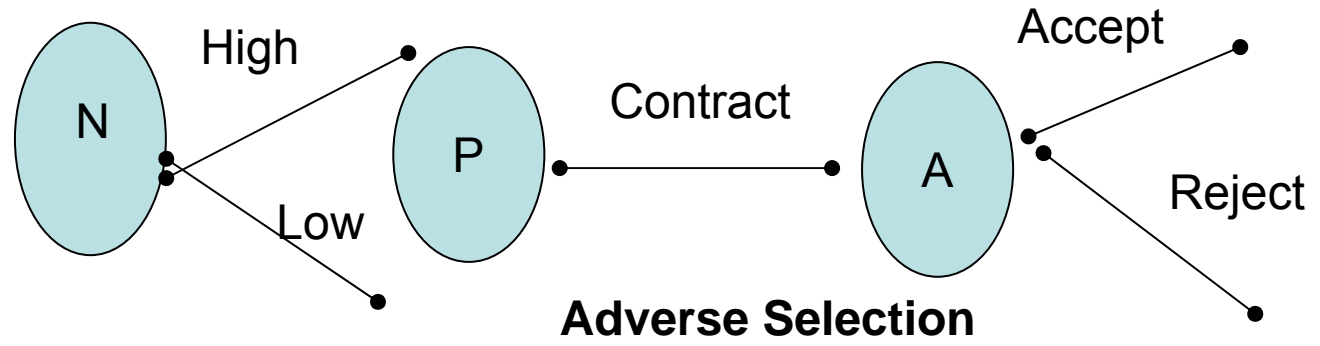
$$w_b = (q_b)^2 = (0.265)^2 = 0.07$$

$$w_g = \left(\frac{q_g}{3}\right)^2 + (q_b)^2 - \left(\frac{q_b}{3}\right)^2 = \left(\frac{4.5}{3}\right)^2 + (0.265)^2 - \left(\frac{0.265}{3}\right)^2 = 2.32$$

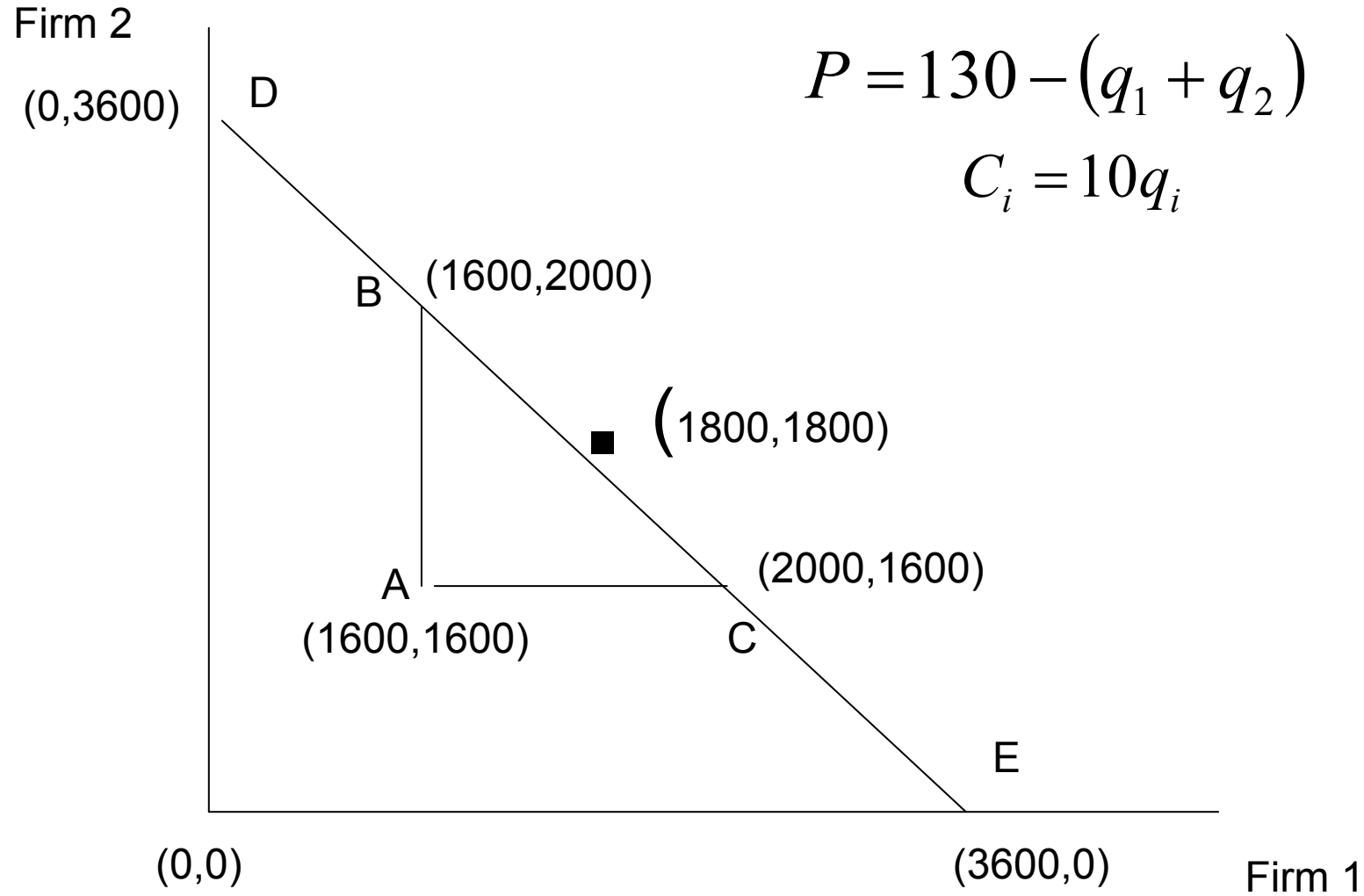
Thus in the presence of information asymmetry , the efforts by the good worker is at **the first best level** as the bad effort by him is not as attractive as the good effort.

It is not profitable for good worker to pretend as a bad worker. Good worker is not attracted by the contract for bad worker.

It is very costly for the bad worker to accept the contract of good worker. Bad worker's **first best to put low effort**.



Infinitely Repeated Game in a Duopoly Profits for firm 1 and 2



Cooperative Solution in Infinitely Played Repeated Game

Market demand for a product is $P = 130 - (q_1 + q_2)$

Cost of production for each of two firms is $C_i = 10q_i$.

If played infinite number of time two firms form a cartel and monopolise the market.

Each will supply only 30, set market price to monopoly level at £70 and divide total profit £3600 equally; each getting £1800.

This is shown by (1800,1800) point in the diagram.

It pays to cooperate in the long run; it is sub-game perfect equilibrium.

$$\Pi = (130 - Q)Q - 10Q ; \frac{\partial \Pi}{\partial Q} = 130 - 2Q - 10 = 0 ; Q = \frac{120}{2} = 60 ;$$

$$P = 130 - Q = 130 - 60 = 70 ; C = 10Q = 600 ;$$

$$\Pi = PQ - C = 70 \times 60 - 10 \times 60 = 4200 - 600 = 3600$$

Non-Cooperative Nash Equilibrium

If any one firm cheats and tries to supply more in order to get more profit; it will be found out by another firm.

It will react to this.

Game will be non-cooperative with resulting in a Cournot Nash equilibrium.

with each firm producing 40 units, market price of 50 and each getting £1600 profits.

$$\Pi_1 = (130 - (q_1 + q_2))q_1 - 10q_1 \quad \text{and} \quad \Pi_2 = (130 - (q_1 + q_2))q_2 - 10q_2$$

$$\text{with reaction functions } 2q_1 + q_2 = 120 \text{ and } q_1 + 2q_2 = 120$$

Total supply is 80, each supplying 40 and making profit is 1600 and market price 50.

Trigger Strategy and Perpetual Punishment

If firm 1 plays Cournot game but firm 2 still plays cartel and supply just 30.

Then from the firm 1' reaction function $2q_1 + q_2 = 120$

$$q_1 = 60 - \frac{1}{2}q_2 = 60 - \frac{1}{2}(30) = 45.$$

If firm 1 supplies 45, market price will be $P = 130 - (q_1 + q_2) = 130 - 45 - 30 = 55$.

This makes profit margin of firm 1 to be 45 and its profit $\Pi_1 = 45 \times 45 = 2025$.

Firm 2 will find out that firm 1 has cheated.

It will also produce according to its reaction curve.

Thus the Nash equilibrium will result with each firm producing 40 and earning 1600 profit for the rest of the periods and the market price will be 50.

For whom is it profitable to Cheat?

Does firm 1 gain or lose by deviation from the agreement. For this evaluate the infinite series of profits in deviation and in compliance with agreement.

Present value of profit in case of cheating

$\Pi_1 = (1 - \delta)[2025 + 1600\delta + 1600\delta^2 + \dots + \dots]$ by adding and subtracting 1600 and applying the formula for infinite series

$$\Pi_1 = (1 - \delta)[2025 - 1600 + 1600 + 1600\delta + 1600\delta^2 + \dots + \dots] = (1 - \delta)\left[425 + \frac{1600}{1 - \delta}\right]$$

$$\Pi_1 = 425(1 - \delta) + 1600 = 425 - 425\delta + 1600 = 2025 - 425\delta$$

By comparing profits with and without cheating

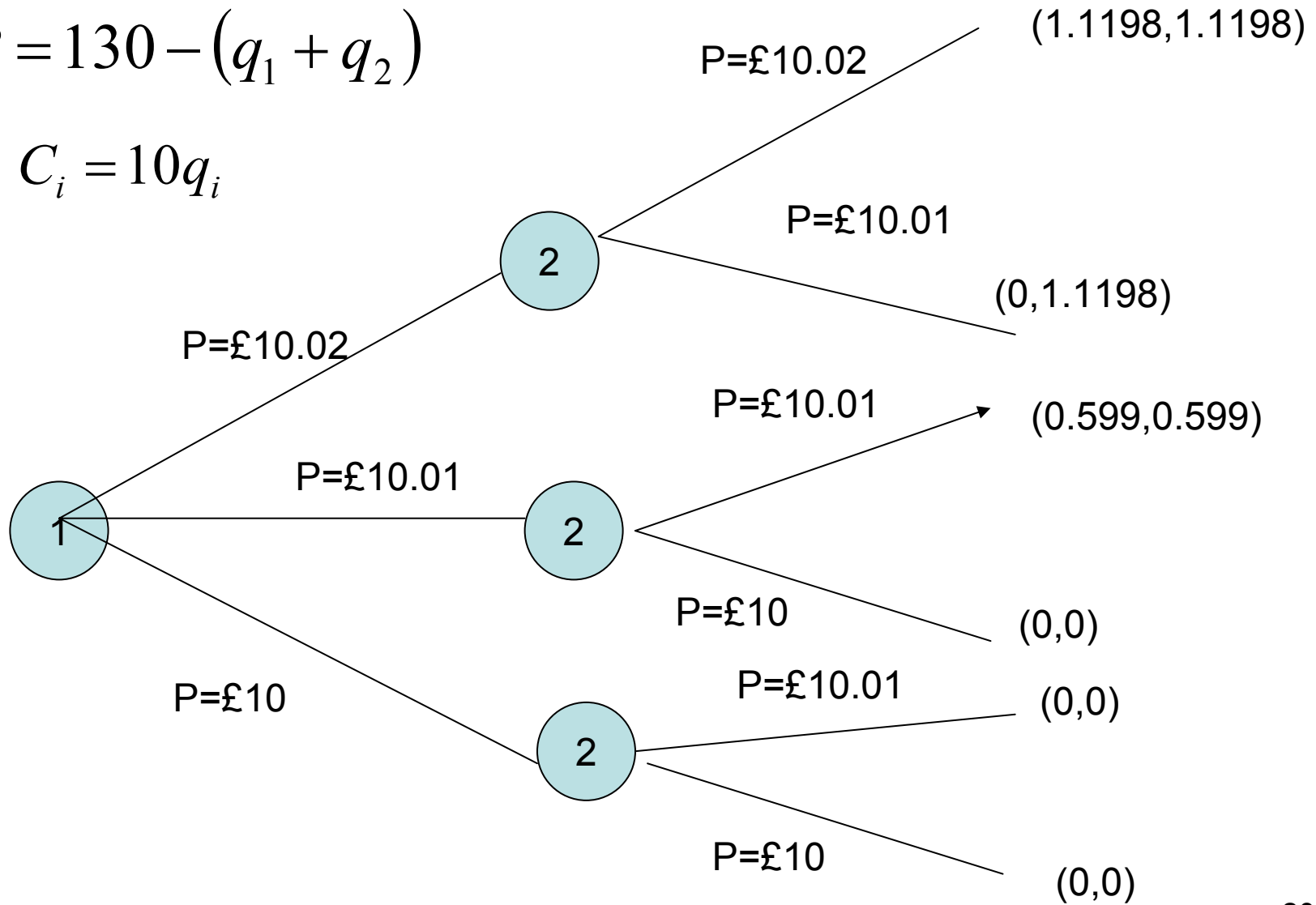
$$2025 - 425\delta < 1800 \text{ or } 425\delta > 2025 - 1800; \delta > \frac{225}{425}; \delta > \frac{9}{17}$$

Whether the firm 1 will stick to agreement or not depends on whether its discount factor is greater than $\delta > \frac{9}{17}$. For discount factor $\delta < \frac{9}{17}$ it benefits from sticking to the agreement.

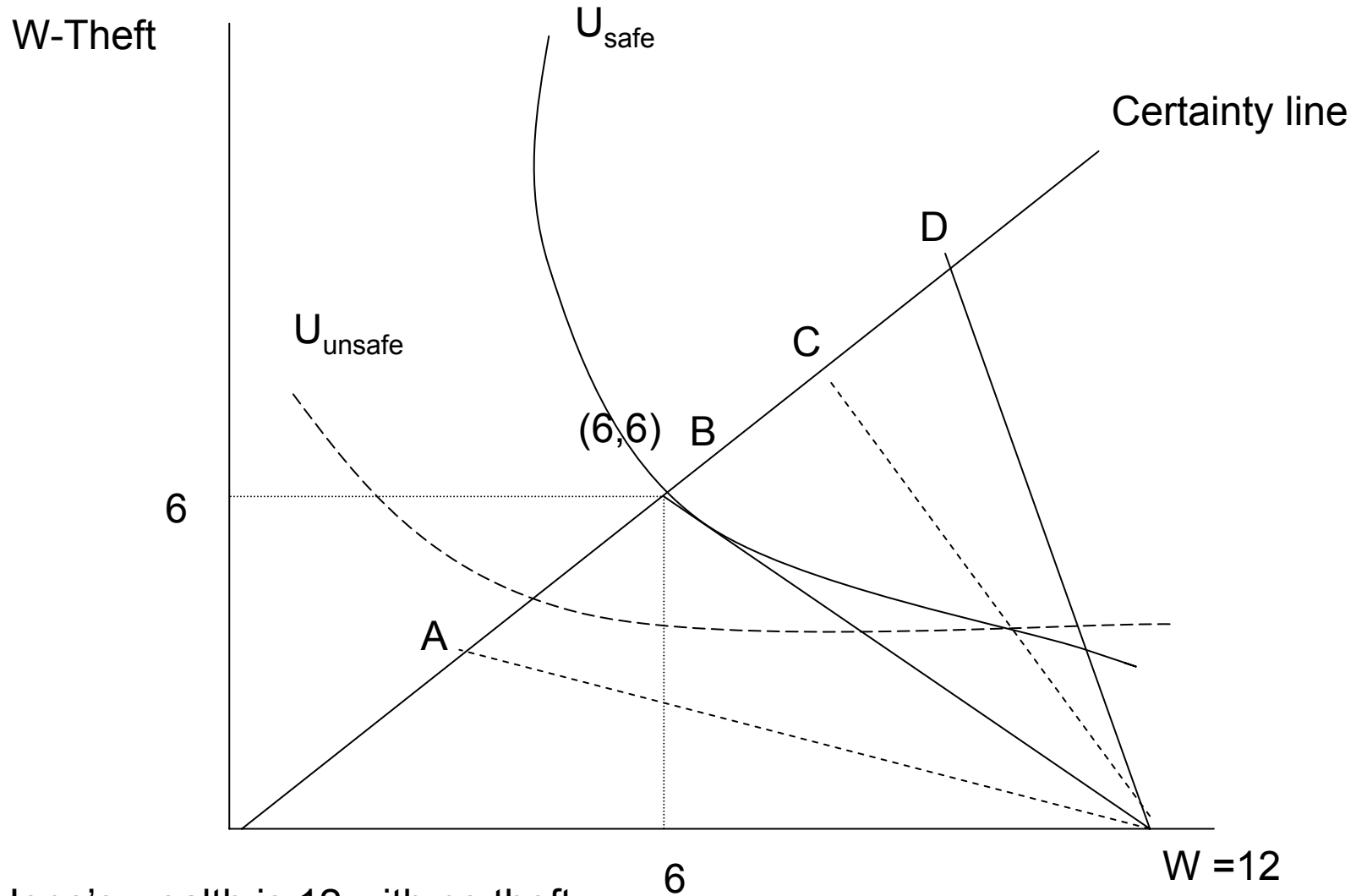
Bertrand-Stackelberg Competition

$$P = 130 - (q_1 + q_2)$$

$$C_i = 10q_i$$



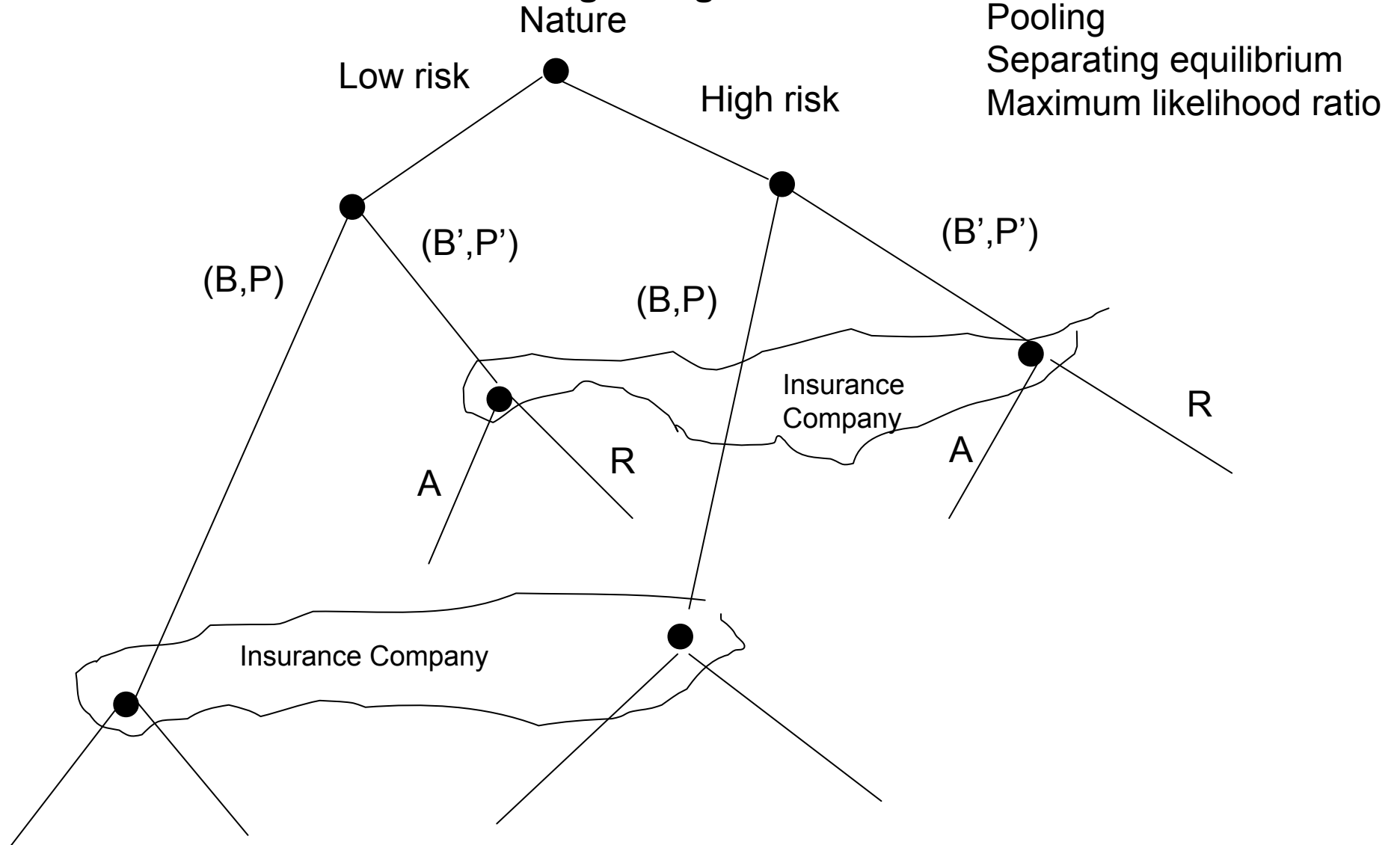
Theft Insurance



Jone's wealth is 12 with no theft
 0 with theft; premium will live him at
 (6,6) in both states.

W = 12
 Wealth no theft

Insurance Signalling Game



Nature decides whether a customer is high or low risk type but the insurance company does not know which is of which type. It offers insurance schemes suitable for low and high risk individual. Choice of the customer signals the insurance company of its type. Then the insurance company can make an incentive mechanism so that it is beneficial to buy high risk type insurance for high risks and low risk type for low risks.

Insurance Game with Moral Hazard

Let p be insurance premium.

$\pi(e)$ probability of accident with effort e , this diminishes with greater care (higher e).

Level of benefit offered in case of accident is B_L specific to losses $L = 1, 2, \dots, L$.

The Moral hazard problem is for insurance company to set the premium according to efforts

$$\text{Max}_{e, p, B_0, \dots, B_L} p - \sum_{l=0}^L \pi_l(e) B_l \quad (1)$$

subject to participation constraint:

$$\sum \pi_l(e) u(W - p - l + B_l) - d(e) \geq \bar{u} \quad (2)$$

Lagrangian function

$$L = p - \sum_{l=0}^L \pi_l(e) B_l + \lambda \left[\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) - \bar{u} \right] \quad (3)$$

Solving Moral Hazard for Insurance

$$\frac{\partial L}{\partial p} = 1 - \lambda \left[\sum \pi_l(e) u'(W - p - l + B_l) \right] = 0 \quad (4)$$

$$\frac{\partial L}{\partial B_l} = -\pi_l(e) + \lambda \pi_l(e) u'(W - p - l + B_l) = 0 \quad (5)$$

$$\frac{\partial L}{\partial \lambda} = \sum_l \pi_l(e) u(W - p - l + B_l) - d(e) - \bar{u} = 0 \quad (6)$$

From (5) $u(W - p - l + B_l) = d(e) + \bar{u}$

Under full insurance $B_l = l$

this implicitly defines the insurance premium for effort level

$$u(W - p(e)) = d(e) + \bar{u} .$$

Since low effort is less costly than more effort for the customer
 $d(0) \leq d(1)$;

the premium under lower effort must be set higher than for the higher effort: $p(0) \geq p(1)$ for profit maximisation

$$p - \sum_{l=0}^L \pi_l(e) \cdot l$$

This is the prediction of moral hazard with complete information but uncertainty with consumer's hidden action.

Insurance company cannot observe the consumer's choice of accident prevention efforts.

But the insurance company continues to seek maximize he expected profit.

It now need to add incentive compatibility constraint.

$$\underset{e, p, B_0, \dots, B_L}{Max} \quad p - \sum_{l=0}^L \pi_l(e) B_l \quad (1)$$

subject to participation constraint:

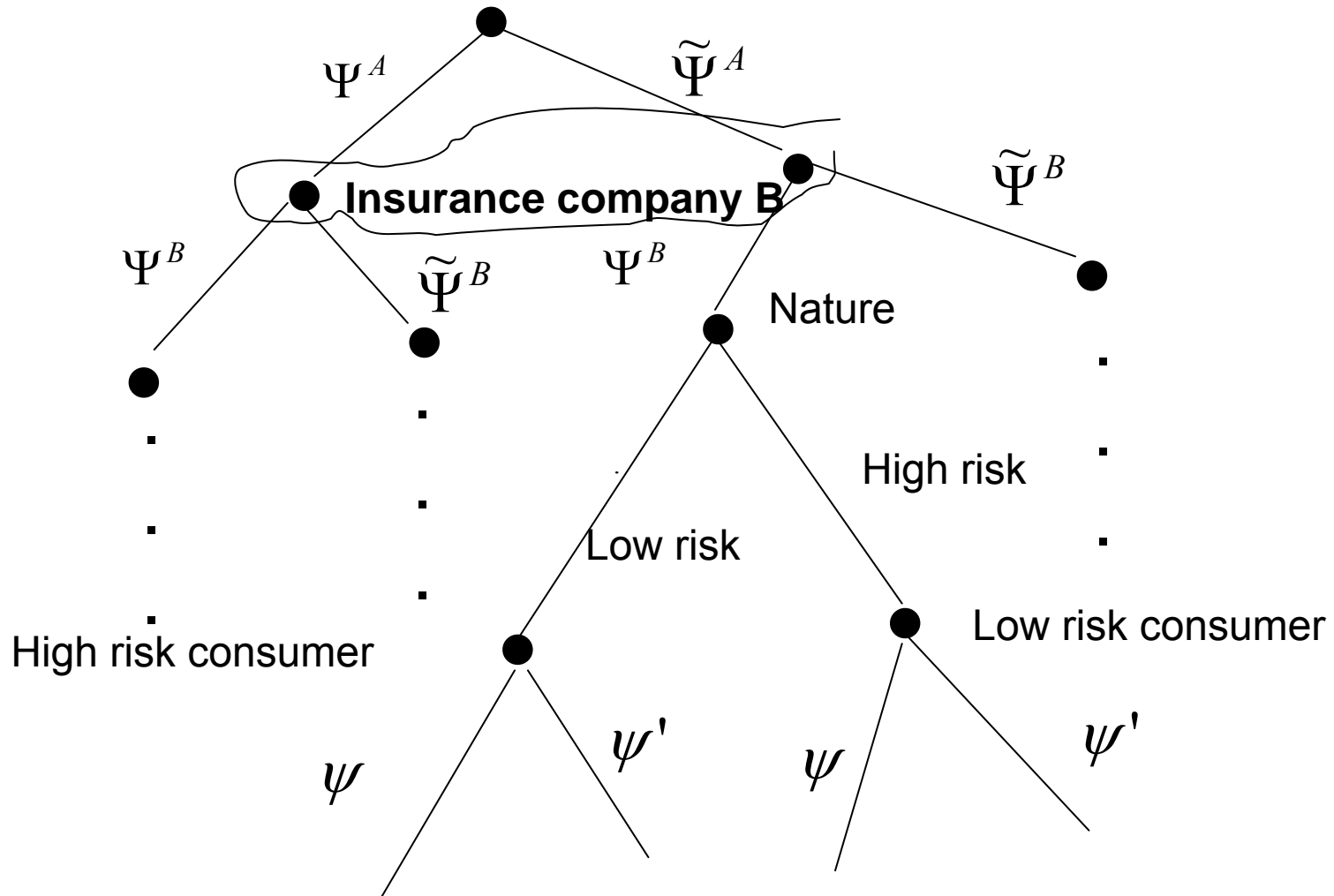
$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \sum_l \pi_l(e') u(W - p - l + B_l) - d(e')$$

$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \bar{u} \quad (2)$$

Incentive compatibility constraint

$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \sum_l \pi_l(e') u(W - p - l + B_l) - d(e')$$

Solving Screening Game for an Insurance Company



Insurance company A and B move simultaneously. They do not know who is risky and who is not risky consumer. Consumers reveal their choice by choosing the insurance policy. Then companies guess who is risky and who is safe.²¹⁴

Optimisation Conditions

$$L = p - \sum_{l=0}^L \pi_l(e)B_l + \lambda \left[\sum_l \pi_l(e)u(W - p - l + B_l) - d(e) - \bar{u} \right] + \beta \left[\left\{ \sum_l \pi_l(e)u(W - p - l + B_l) - d(e) \right\} - \left\{ \sum_l \pi_l(e')u(W - p - l + B_l) - d(e') \right\} \right] \quad (3)$$

$$\frac{\partial L}{\partial p} = 1 - \left[\sum (\lambda \pi_l(1) + \beta(\pi_l(1) - \pi_l(0))u'(W - p - l + B_l)) \right] = 0 \quad (4)$$

$$\frac{\partial L}{\partial B_l} = -\pi_l(1) + [\lambda \pi_l(1) + \beta(\pi_l(1) - \pi_l(0))]u'(W - p - l + B_l) = 0 \quad (5)$$

$$\frac{\partial L}{\partial \lambda} = \sum_l \pi_l(e)u(W - p - l + B_l) - d(e) - \bar{u} \geq 0$$

$$\frac{\partial L}{\partial \beta} = \left\{ \sum_l (\pi_l(1) - \pi_l(0))u(W - p - l + B_l) + d(0) - d(1) \right\} \geq 0$$

$$\frac{1}{u'(W - p - l + B_l)} = \lambda + \beta \left[1 - \frac{\pi_l(0)}{\pi_l(1)} \right]$$

$$\frac{1}{u'(W - p - l + B_l)} = \lambda + \beta \left[1 - \frac{\pi_l(0)}{\pi_l(1)} \right]$$

$$\sum_l \pi_l(e) u(W - p - l + B_l) - d(e) \geq \sum_l \pi_l(e') u(W - p - l + B_l) - d(e')$$

since $\beta > 0$ the RHS is strictly decreasing, this implies that $u'(W - p - l + B_l)$ must be strictly increasing for this to happen $l - B_l$ must increase with effort levels and losses

$l = 0, 1, 2, \dots, L$. Optimal high policy does not provide full insurance but the deductible payment increases size of loss (Jehle and Reny (2001, Chapter 8).

Optimal policy is crafted so that the utility of benefit from high efforts equals higher costs.

Main Feature of Simultaneous Equation System

- So far we only discussed regression model in which a dependent variable Y is explained by one or a set of explanatory variables with assumption that there one way causation from independent variables to the dependent variables.
- However, many variables in economics are interdependent.
- Consider a market model with demand and supply. How much price can a firm charge for a particular product from its customers depends on the quantity sold in the market and how much quantity is demanded by customers depend on the market price. Price determines quantity and quantity determines price. Same is true in national income determination model.

Main Feature of Simultaneous Equation System-2

- Consumption is a component of income that determines income, but income is the major determinant of consumption. Both quantities and prices and income and consumption are determined simultaneously.
- We need to estimate a system of equations, not a single equation, in order to be able to capture this interdependency among variables.
- The main features of a simultaneous equation model are:
 1. two or more dependent (endogenous) variables
 2. A set of equations
 3. Computationally cumbersome, highly non-linearity in parameters and errors in one equation transmitted through the whole system

Simultaneity-an Example

- Consider a relation between quantity and price

$$\bullet Q_t = \alpha_0 + \alpha_1 P_t + u_t$$

- A priori it is impossible to say whether this a demand or supply model, both of them have same variables.
- If we estimate a regression model like this how can we be sure whether the parameters belong to a demand or supply model?

We need extra information. Economic theory suggests that demand is related with income of individual and supply may be respond

Example of an Identified Model

Demand

$$Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1,t} \quad (1)$$

Supply

$$Q_t^s = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2,t} \quad (2)$$

where Q_t^d is quantity demanded and Q_t^s is quantity supplied, P_t is the price of commodity, P_{t-1} is price lagged by one period, I_t is income of an individual,

$u_{1,t}$ and $u_{2,t}$ are independently and identically distributed

(iid) error terms with a zero mean and a constant variance.

Q_t and P_t are endogenous variables and P_{t-1} and I_t are exogenous variables, $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$ and β_2 are six parameters defining the system.

Example of an Identified Model

- In equilibrium $Q_t^d = Q_t^s$, however note that quantities bought and sold (Q) depends upon market prices (P) and the equilibrium prices is determined by quantity supplied and demanded.
- The random terms of quantity and price equations above, $u_{1,t}$ and $u_{2,t}$, are not independent of each other. There is simultaneity problem.

A Simple Keynesian Model of Income Determination

A simple version of Keynesian income determination model of the following form:

$$C_t = \beta_0 + \beta_1 Y_t + u_t \quad (3)$$

$$Y_t = C_t + I_t \quad (4)$$

where Y_t is income, c_t is consumption I_t is investment, and u_t is the random error term. The subscript t refers to time period. β_0 and β_1 are structural parameters, and $0 < \beta_1 < 1$. Y_t and c_t are endogenous variables and I_t is an exogenous variable.

Reduced form of the Market Model

- a. Reduced form is obtained by expressing endogenous variable in terms of exogenous variables. In the demand and supply equation in example 1 the reduced form takes the following form:

$$Q_t = \pi_{10} + \pi_{11} P_{t-1} + \pi_{12} I_t + v_{1,t} \quad (5)$$

$$P_t = \pi_{20} + \pi_{21} P_{t-1} + \pi_{22} I_t + v_{2,t} \quad (6)$$

Where the reduced form coefficients are defined as:

$$\pi_{10} = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}; \quad \pi_{11} = \frac{-\alpha_2}{\alpha_1 - \beta_1}; \quad \pi_{12} = \frac{\beta_2}{\alpha_1 - \beta_1};$$

$$v_{1,t} = \frac{u_{2,t} - u_{1,t}}{\alpha_1 - \beta_1};$$

$$\pi_{20} = \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1}; \quad \pi_{21} = \frac{\alpha_1 \beta_1}{\alpha_1 - \beta_1}; \quad \pi_{22} = \frac{\alpha_2 \beta_1}{\alpha_1 - \beta_1};$$

$$v_{2,t} = \frac{\alpha_1 u_{2,t} - \beta_1 u_{1,t}}{\alpha_1 - \beta_1};$$

These coefficients are obtained by using equation (2) in (1) for quantity (Q) and using that information in (2) for price level.

Reduced form of the National Income Model

In the income determination model (example 2) the reduced form is obtained by expressing C and Y endogenous variables in terms of I which is the only exogenous variable in the model.

$$C_t = \pi_{11} + \pi_{12}I_t + v_{1,t} \quad (7)$$

$$Y_t = \pi_{21} + \pi_{22}I_t + v_{2,t} \quad (8)$$

Where $\pi_{11} = \frac{\beta_0}{\beta_1}$; $\pi_{12} = \frac{\beta_1}{1-\beta_1}$; $\pi_{21} = \frac{\beta_0}{1-\beta_1}$ and

$$\pi_{22} = \frac{1}{1-\beta_1}.$$

Simultaneity bias

$$b_1 = \frac{\sum_t c_t y_t}{\sum_t y_t^2} = \frac{\sum_t (c_t - \bar{c}) y_t}{\sum_t y_t^2} = \frac{\sum_t c_t y_t - \bar{c} \sum_t y_t}{\sum_t y_t^2} = \frac{\sum_t c_t y_t}{\sum_t y_t^2}$$

$$b_1 = \frac{\sum_t c_t y_t}{\sum_t y_t^2} \Rightarrow b_1 = \frac{\sum_t (\beta_0 + \beta_1 Y_t + e_t) y_t}{\sum_t y_t^2}$$

$$\text{cov}(Y_t, e_t) = E[Y_t - E(Y_t)][e_t - E(e_t)] = E\left[\frac{e_t}{1 - \beta_1}\right] e_t = \frac{\sigma_e^2}{1 - \beta_1}$$

$$p \lim(b_1) = \beta_1 + \frac{\left(\frac{\sum_t e_t y_t}{n}\right)}{\left(\frac{\sum_t y_t^2}{n}\right)} = \beta_1 + \frac{\sigma_e^2 / (1 - \beta_1)}{\sigma_y^2}$$

A simple Example of the Simultaneous Equation System

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} e_{1i} \\ e_{2i} \end{bmatrix}$$

$$y_{1i} = \frac{(a_{22}b_{11} - a_{12}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(a_{22}b_{12} - a_{12}b_{22})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{1i}$$

$$y_{2i} = \frac{(-a_{21}b_{11} + a_{11}b_{21})}{(a_{11}a_{22} - a_{12}a_{21})} x_{1i} + \frac{(-a_{21}b_{12} + a_{11}b_{22})}{(a_{11}a_{22} - a_{12}a_{21})} x_{2i} + \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} e_{2i}$$

General Form for the Simultaneous Equation System

$$a_{11}y_{1i} + a_{12}y_{2i} + \dots + a_{1m}y_{mi} + b_{11}x_{1i} + b_{12}x_{2i} + \dots + b_{1k}x_{ki} = e_{1i}$$

$$a_{21}y_{1i} + a_{22}y_{2i} + \dots + a_{2m}y_{mi} + b_{21}x_{1i} + b_{22}x_{2i} + \dots + b_{2k}x_{ki} = e_{2i}$$

.....

$$a_{m1}y_{1i} + a_{m2}y_{2i} + \dots + a_{mm}y_{mi} + b_{m1}x_{1i} + b_{m2}x_{2i} + \dots + b_{mk}x_{ki} = e_{mi}$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \cdot \\ \cdot \\ y_{mi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mk} \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \\ \cdot \\ \cdot \\ x_{mi} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}^{-1} \begin{bmatrix} e_{1i} \\ e_{2i} \\ \cdot \\ \cdot \\ e_{mi} \end{bmatrix}$$

Impact and Shock Analysis in a Simultaneous Equation System

$$AY_i + BX_i = U_i$$

$$Y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \cdot \\ \cdot \\ y_{mi} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \quad X_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \cdot \\ \cdot \\ x_{mi} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mk} \end{bmatrix} \quad U_i = \begin{bmatrix} e_{1i} \\ e_{2i} \\ \cdot \\ \cdot \\ e_{mi} \end{bmatrix}$$

$$Y_i = -A^{-1}BX_i + A^{-1}U_i \quad \hat{Y}_i = -A^{-1}BX_i$$

$$\text{Var}(Y_i - \hat{Y}_i) = A^{-2}\sigma_u^2$$

Macroeconometric (IS-LM) Model

Consumption function: $C_t = \beta_0 + \beta_1(Y_t - T_t) + \varepsilon_{1t}$ (1)

Taxation function: $T_t = c_0 + c_1 Y_t + \varepsilon_{2t}$ (2)

Import function: $M_t = m_0 + m_1 Y_t + \varepsilon_{3t}$ (3)

Investment function: $I_t = \mu_0 + \mu_1 R_t + \phi \Delta Y_{t-1} + \varepsilon_{4t}$ (4)

National income identity: $Y_t = C_t + I_t + G_t + X_t - M_t$ (5)

Money market (LM curve): $\left(\frac{MM_t}{P_t}\right) = b_0 + b_1 Y_t - b_2 R_t + \varepsilon_{6t}$ (6)

Money market (LM curve): $R_t = \frac{b_0}{b_2} - \frac{1}{b_2} \left(\frac{MM_t}{P_t}\right) + \frac{b_1}{b_2} Y_t + \varepsilon_{6t}$ (6)

Y_t, C_t, M_t, I_t, R_t and T_t are endogenous variables and G_t, X_t, MM_t and ΔY_{t-1} are predetermined or exogenous variables representing government expenditure, exports, money stock and change in Y_t in the previous period respectively. Set the price level P_t equal to unity for simplicity.

Reduced form of the Macro Model

$$C_t = \Pi_{10} + \Pi_{11}G_t + \Pi_{12}X_t + \Pi_{13}MM_t + \Pi_{14}\Delta Y_{t-1}$$

$$M_t = \Pi_{20} + \Pi_{21}G_t + \Pi_{22}X_t + \Pi_{23}MM_t + \Pi_{24}\Delta Y_{t-1}$$

$$I_t = \Pi_{30} + \Pi_{31}G_t + \Pi_{32}X_t + \Pi_{33}MM_t + \Pi_{34}\Delta Y_{t-1}$$

$$R_t = \Pi_{40} + \Pi_{41}G_t + \Pi_{42}X_t + \Pi_{43}MM_t + \Pi_{44}\Delta Y_{t-1}$$

$$T_t = \Pi_{50} + \Pi_{51}G_t + \Pi_{52}X_t + \Pi_{53}MM_t + \Pi_{54}\Delta Y_{t-1}$$

$$Y_t = \Pi_{60} + \Pi_{61}G_t + \Pi_{62}X_t + \Pi_{63}MM_t + \Pi_{64}\Delta Y_{t-1}$$

Rank and Order Conditions of Identification

Order condition: $K - k \geq m - 1$

Rank condition: $\rho(A) \geq (M - 1)(M - 1) \Rightarrow$ order of the matrix

M = number of endogenous variables in the model

K = number of exogenous variables in the model

including the intercept

m = number of endogenous variable in an equation

k = number of exogenous variables in a given equation

Rank condition is defined by the rank of the matrix, which should have a dimension $(M-1)$, where M is the number of endogenous variables in the model.

How to Determine the Rank of the Matrix?

Rank of matrix is the order of non-singular matrix

Rank matrix is formed from the coefficients of the variables (both endogenous and exogenous) excluded from that particular equation but included in the other equations in the model.

The rank condition tells us whether the equation under consideration is identified or not.

The order condition tells us if it is exactly identified or overidentified.

Order and rank conditions of identification

1. If $K - k > m - 1$ and the rank of the $\rho(A)$ is $M-1$ then the concerned equation is overidentified.
2. If $K - k = m - 1$ and the rank of the $\rho(A)$ is $M-1$ then the equation is exactly identified.
3. If $K - k \geq m - 1$ and the rank of the $\rho(A)$ is less than $M-1$ then the equation is underidentified.

If $K - k < m - 1$ the structural equation is unidentified. The rank of the $\rho(A)$ is less than $M-1$ in this case.

Steps for rank condition

1. Write down the system in the tabular form
2. Strike out the coefficients corresponding to the equation to be identified
3. Strike out the columns corresponding to those coefficients in 2 which are nonzero.
4. The entries left in the table will give only the coefficients of the variables included in the system but not in the equation under consideration. From these coefficients form all possible A matrices of order $M-1$ and obtain a corresponding determinant. If at least one of these determinants is non-zero then that equation is identified

Rank condition: $\rho(A) \geq (M - 1)(M - 1) \Rightarrow$ order of the matrix.

Table of Coefficients in a Macro Econometric Model

| | Constant | Y_t | C_t | M_t | I_t | R_t | T_t | G_t | X_t | MM_t | ΔY_{t-1} |
|-------|--------------------|--------------------|-------|--------|-------|----------|-----------|-------|-------|-----------------|------------------|
| C_t | $-\beta_0$ | $-\beta_1$ | 1 | 0 | 0 | 0 | β_1 | 0 | 0 | 0 | 0 |
| T_t | $-t_0$ | $-t_1$ | 0 | $-t_2$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| M_t | $-m_0$ | $-m_1$ | 1 | 1 | 0 | 0 | $-t_2$ | 0 | 0 | 0 | 0 |
| I_t | $-\mu_0$ | 0 | 0 | 0 | 1 | $-\mu_1$ | 0 | 0 | 0 | 0 | 0 |
| R_t | $-\frac{b_0}{b_2}$ | $-\frac{b_1}{b_2}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | $\frac{1}{b_2}$ | 0 |
| Y_t | 0 | 1 | -1 | 1 | -1 | 0 | 0 | -1 | -1 | 0 | 0 |

Checking Rank Identification Condition for the above System

Consumption function:

$$A_1 = \begin{bmatrix} t_2 & 0 & 0 & 0 \\ 1 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad |A_1| = -\frac{1}{b_2} \phi m_2 t_2 \quad \rho(A_1) = 4$$

It is obvious that there exists at least one non-singular matrix of order $M-1$.

Tax function:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & \mu_1 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad \rightarrow \quad |A_1| = -\frac{1}{b_2} \phi m_2 \quad \rho(A_1) = 4$$

Checking Rank Identification Condition for the above System

Import

$$A_1 = \begin{bmatrix} 1 & \beta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \phi \\ 0 & 0 & \frac{1}{b_2} & 0 \end{bmatrix} \quad |A_1| = -\frac{1}{b_2} \phi \quad \rho(A_1) = 4$$

Interest Rate

$$A_1 = \begin{bmatrix} \beta_1 & 0 & -\beta_1 & 0 \\ t_1 & t_2 & 1 & 0 \\ m_1 & 1 & m_3 & 0 \\ \frac{b_1}{b_2} & 0 & 0 & \frac{1}{b_2} \end{bmatrix} \quad |A_1| = \beta_1 t_2 m_3 \frac{1}{b_2} + \beta_1 t_2 m_1 \frac{1}{b_2} \quad \rho(A_1) = 4$$

Investment

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 1 & m_2 & 0 \\ 0 & 0 & -\mu_1 & \phi \end{bmatrix} \quad |A_1| = t_2 m_2 \phi \quad \rho(A_1) = 4 \quad 238$$

Some Techniques for Estimation of the Simultaneous Equations System

Single Equations Methods

Ordinary Least Squares

Indirect Least Squares

Two Stage Least Squares Method

System Method

Three Stage LS: Generalised Least Square

Seemingly Unrelated Regression Equations

Full Information Maximum Likelihood

Estimation by the Generalised Least Square Method

$$\text{cov}[v v'] = \begin{bmatrix} \text{var}(v_1) & \text{cov}(v_1 v_2) & \text{cov}(v_1 v_3) & \cdot & \text{cov}(v_1 v_5) \\ \text{cov}(v_2 v_1) & \text{var}(v_2) & \text{cov}(v_2 v_3) & \cdot & \text{cov}(v_2 v_5) \\ \text{cov}(v_3 v_1) & \text{cov}(v_3 v_2) & \text{var}(v_3) & \cdot & \text{cov}(v_3 v_5) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \text{cov}(v_5 v_1) & \text{cov}(v_5 v_2) & \text{cov}(v_5 v_3) & \cdot & \text{var}(v_v) \end{bmatrix} = \Omega$$

$$P'P = \Omega^{-1}$$

$$PY = PX\beta + Pe \qquad Y^* = X^*\beta + e^*$$

$$\beta_{GLS} = (X^{*'} X^*)^{-1} X^{*'} Y^* = (X' P' P X) X' P' P Y$$

$$\beta_{GLS} = (X^{*'} \Omega^{-1} X^*)^{-1} X^{*'} \Omega^{-1} Y^*$$

Estimation of Simultaneous System using PcGive

$$C = + 0.2128*M4 + 1.407*G + 0.1767*X + 8.059e+004$$

(SE) (0.0266) (0.212) (0.154) (1.63e+004)

$$Y = + 0.2617*M4 + 2.86*G + 1.348*X + 8.862e+004$$

(SE) (0.0454) (0.361) (0.262) (2.78e+004)

$$T = + 0.3204*M4 + 0.9521*G - 0.08909*X - 7.533e+004$$

(SE) (0.02) (0.159) (0.116) (1.23e+004)

$$M = + 0.06508*M4 - 0.4738*G + 1.003*X + 4.34e+004$$

(SE) (0.0198) (0.157) (0.114) (1.21e+004)

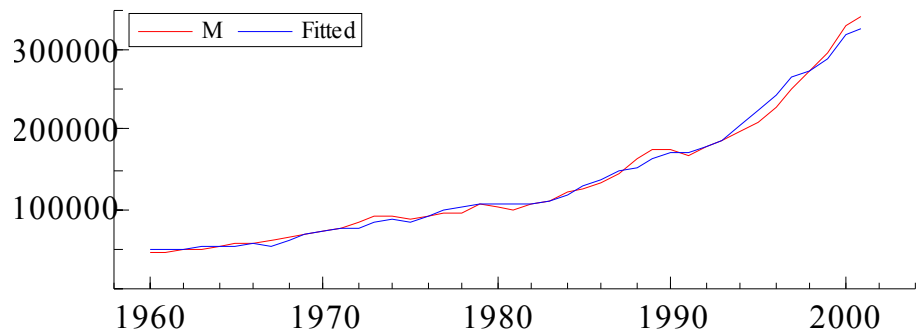
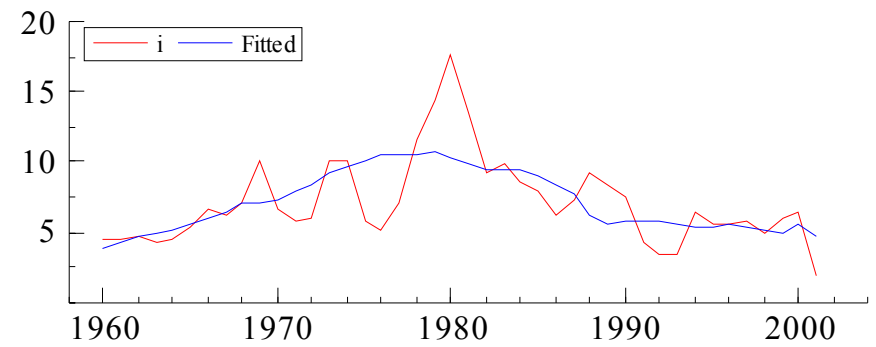
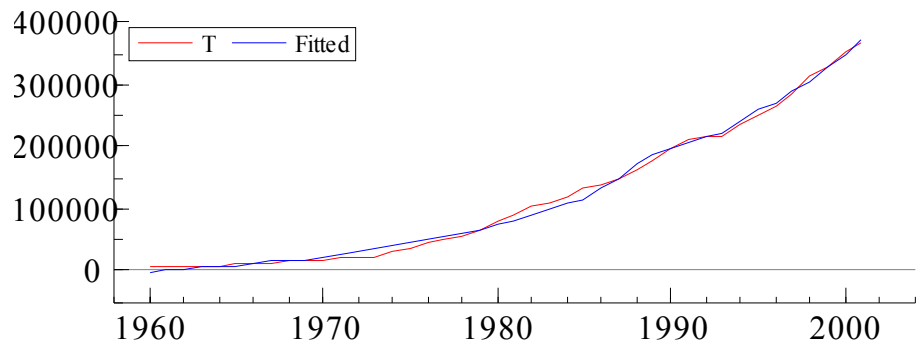
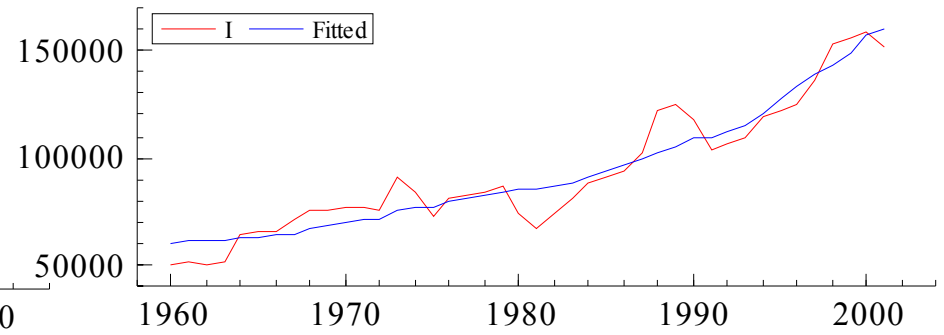
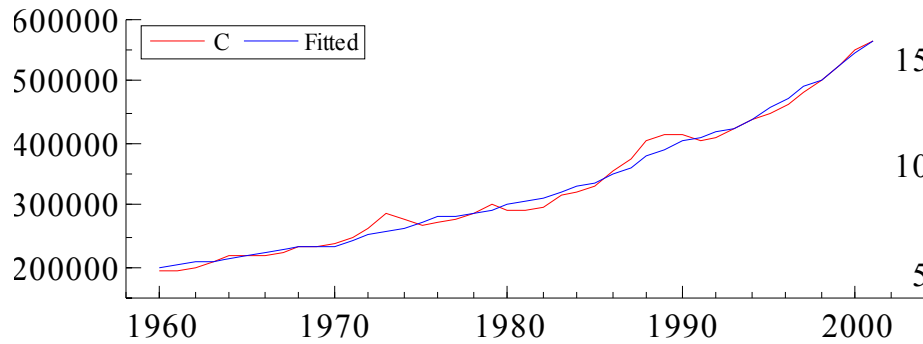
$$r = - 2.384e-005*M4 + 0.0001148*G + 6.273e-005*X - 7.408$$

(SE) (6.2e-006) (4.93e-005) (3.58e-005) (3.79)

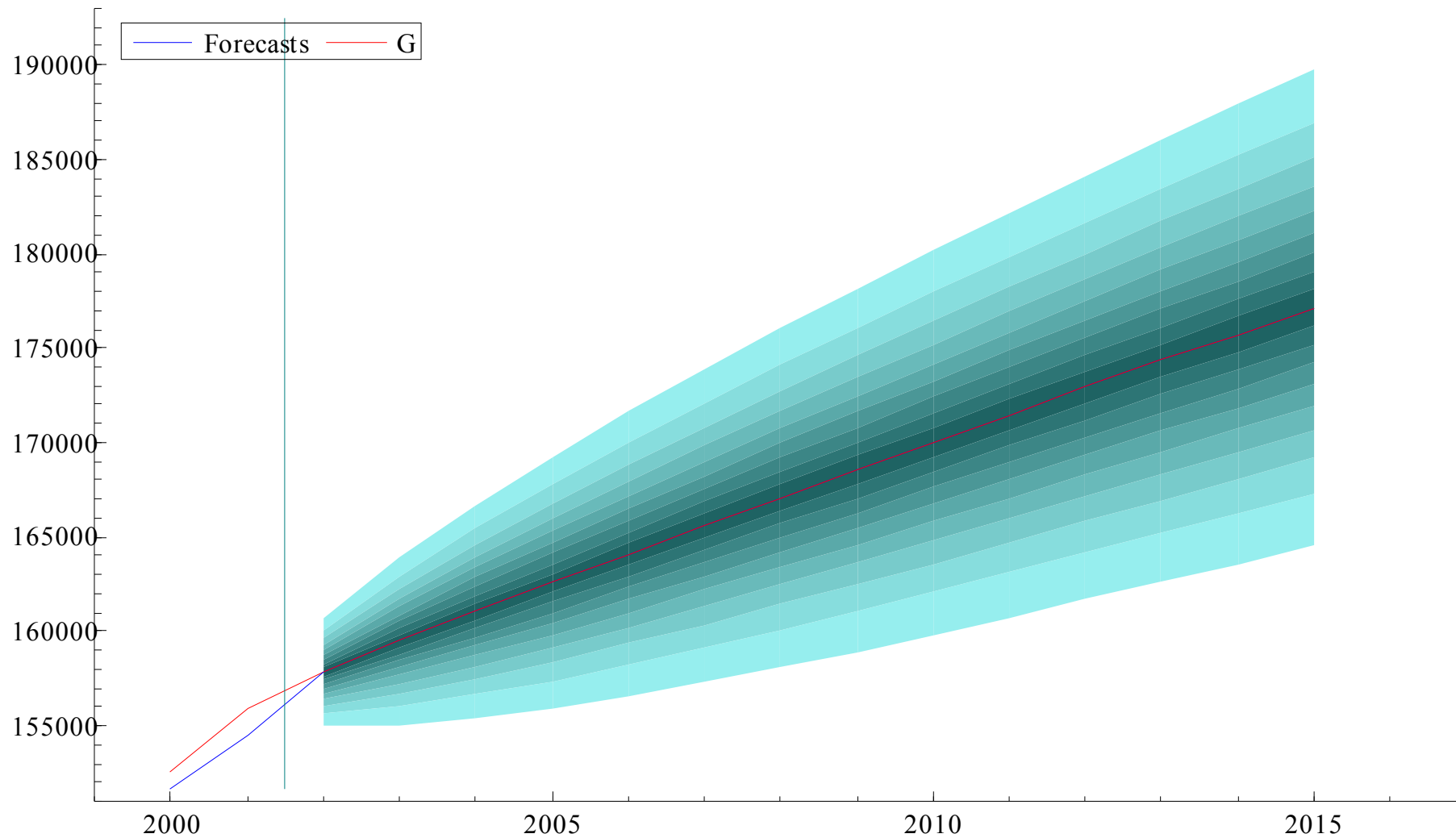
$$I = + 0.02907*M4 + 0.0684*G + 0.2681*X + 4.292e+004$$

(SE) (0.0229) (0.182) (0.132) (1.4e+004)

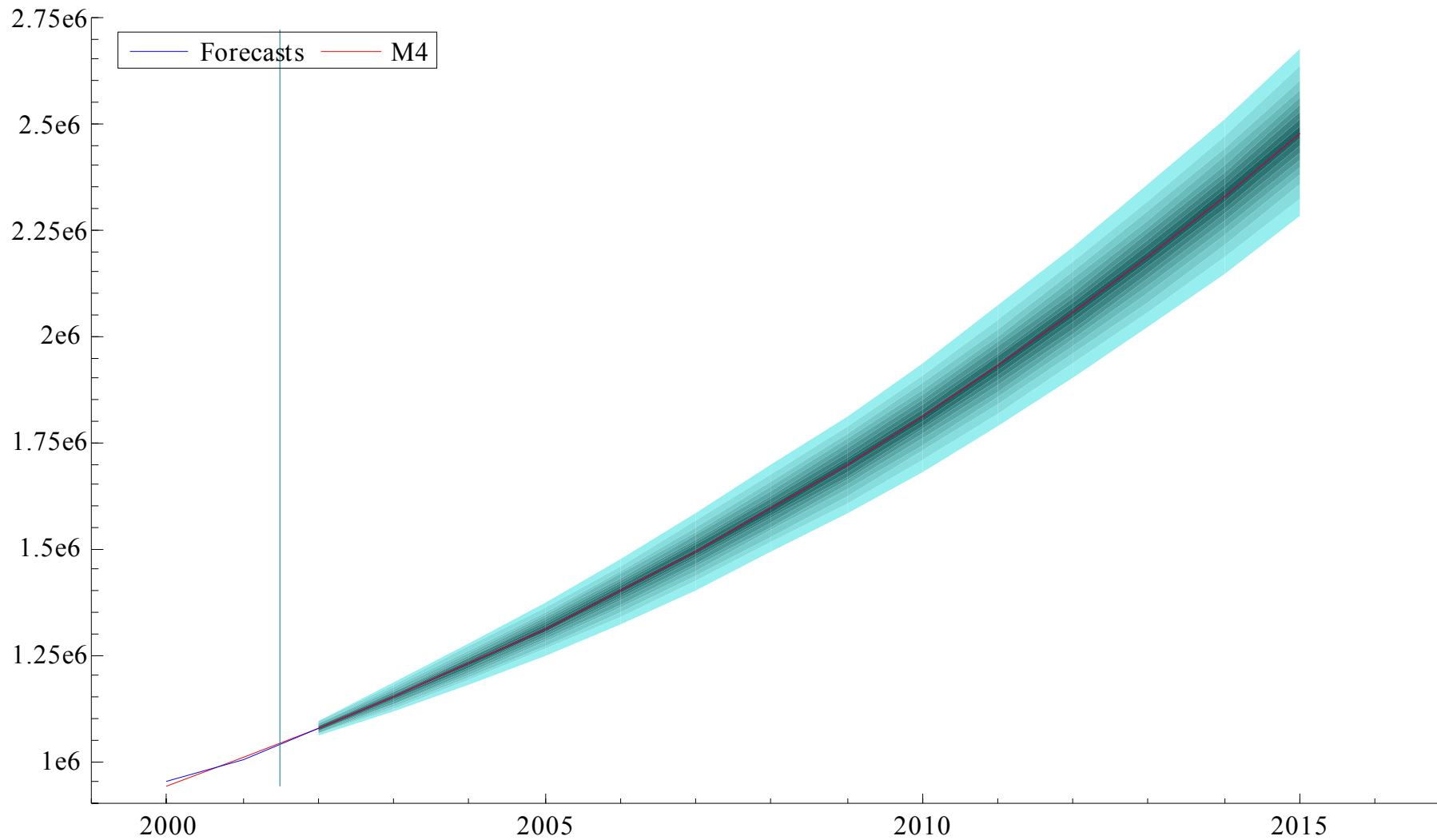
Actual and Predicted Values in above Macroeconomic Model Historical Simulations



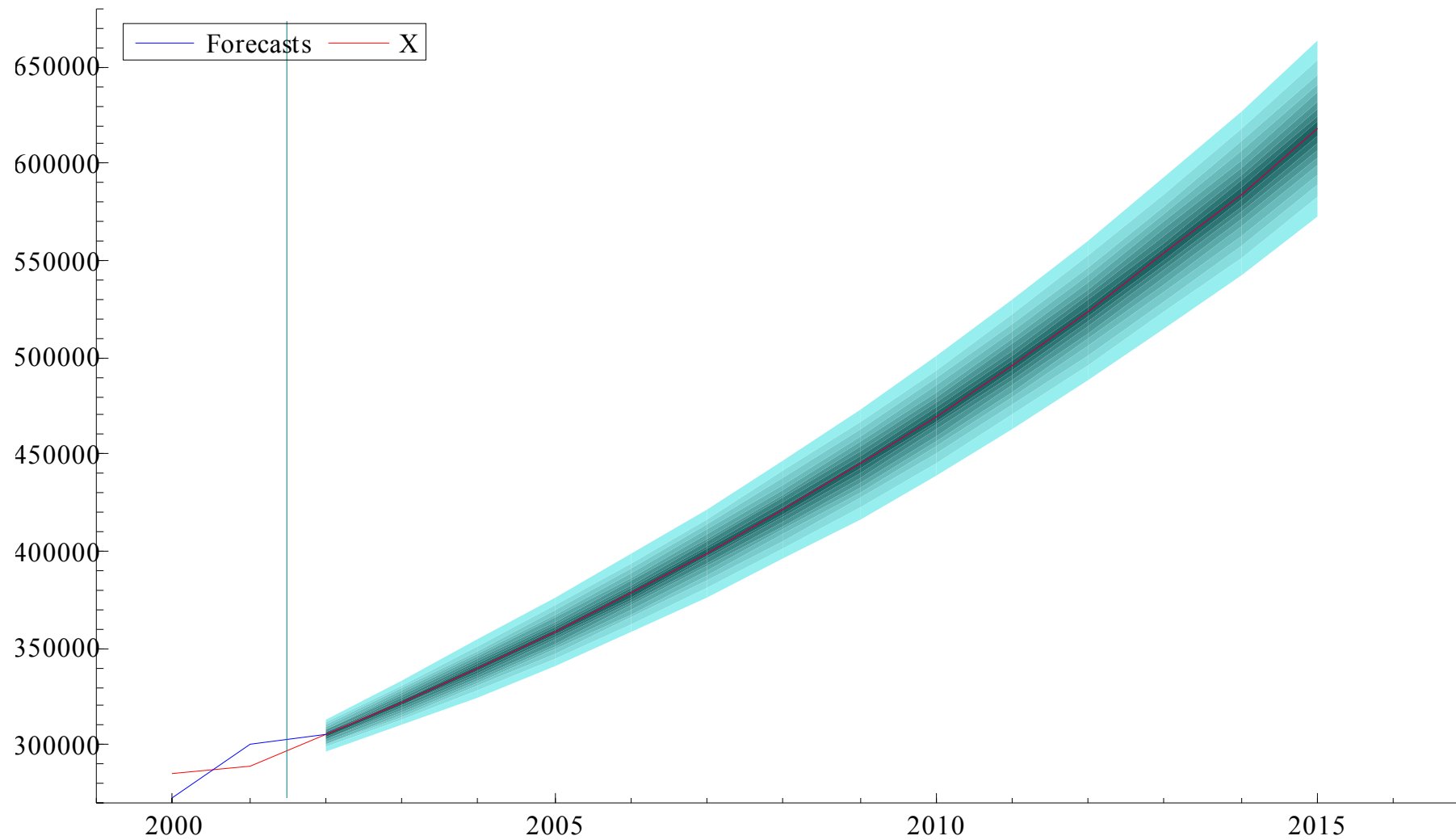
Forecast of Exogenous Variable AR(2) : Government Spending



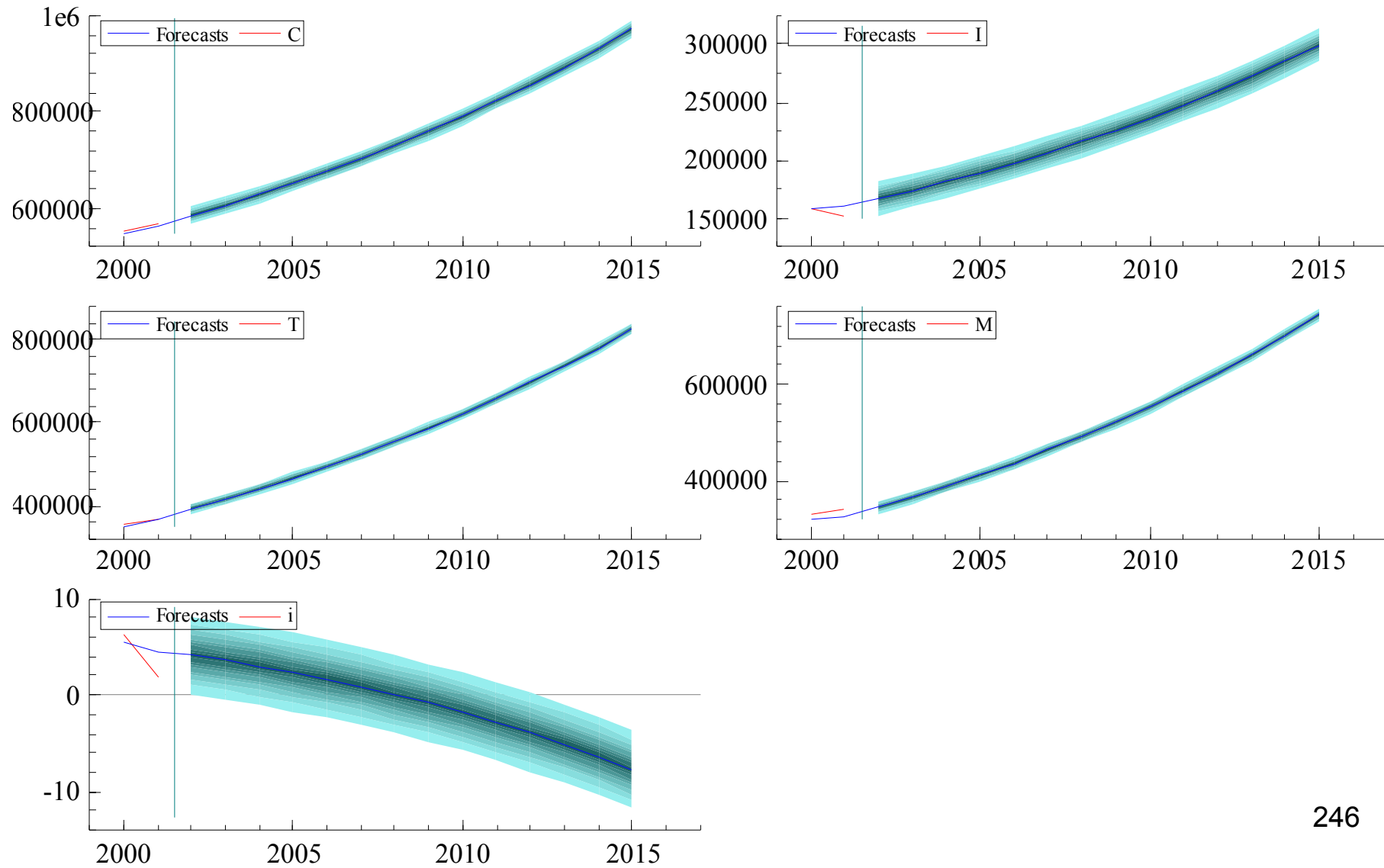
Forecast of Exogenous Variable AR(2) : Money Supply



Forecast of Exogenous Variable AR(2) : Exports



Ex-ante Forecast Endogenous Variables from the Simultaneous Equation Model Consumption, Tax, Investment and the Interest Rate



Classical Econometrics

Classical econometric methods assume that there is true parameter underlying the data generating process such as θ and its true value is unknown.

The objective of sample statistics $\hat{\theta}$ is to represent this unknown parameter as best as possible.

The estimated parameter $\hat{\theta}$ is a random variable and has its own distribution where as the true parameter θ is a fixed number but unknown.

The estimator should be unbiased, $[E(\hat{\theta} - \theta) = 0]$, and

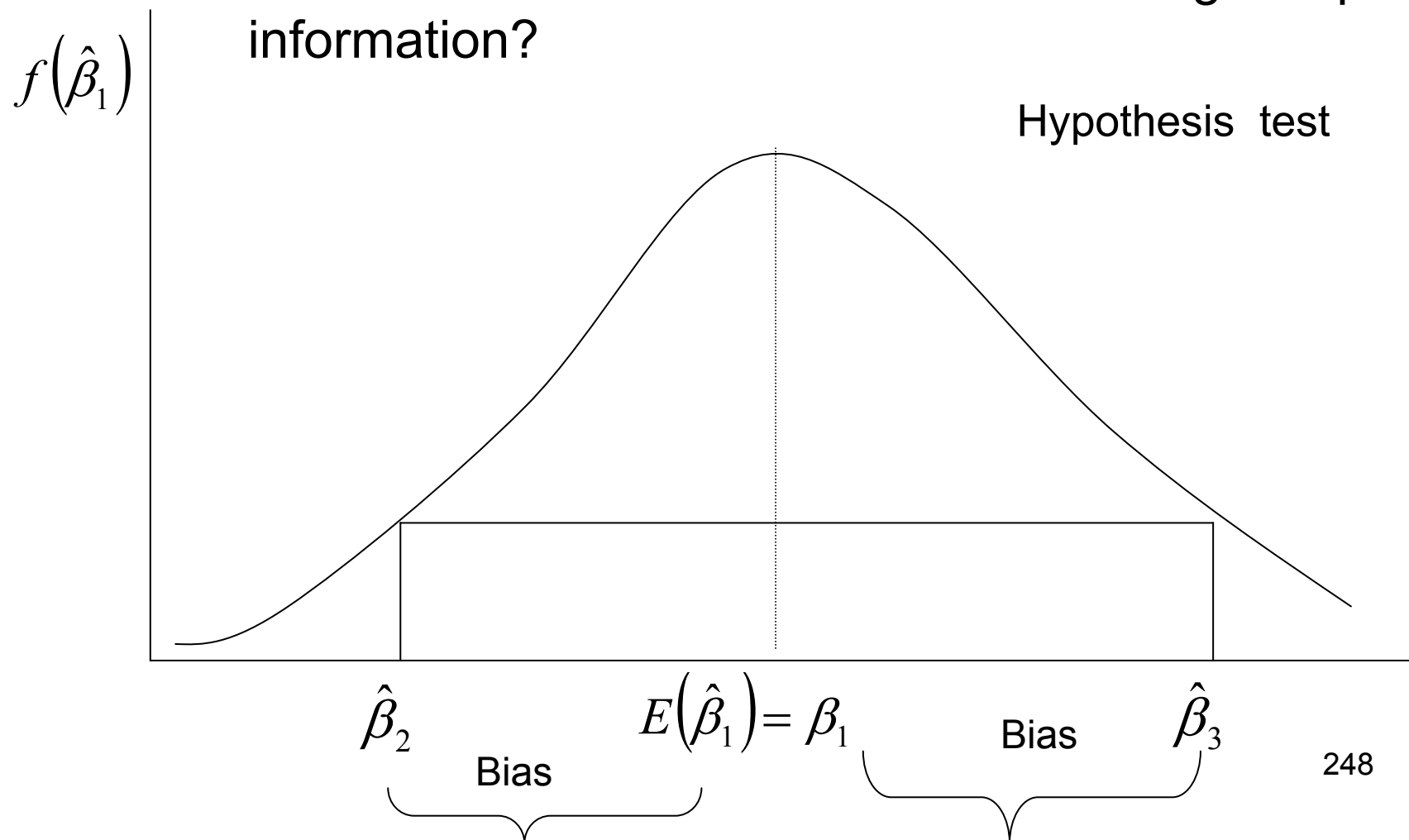
Efficiency of an estimator is judged by the minimum square error $[E(\hat{\theta} - \theta)(\hat{\theta} - \theta)]$

and the data generating process is given by

$$f(y; \theta) = \prod_{i=1}^T \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y - \mu)^2}{2\sigma^2}\right]$$

What is the best way to use sample information to infer about the characteristic of population?

How to minimise the bias or error in using sample information?



Bayesian Econometrics

In Bayesian analysis the value of true parameter θ unknown like in the classical approach but it is not fixed.

Instead θ has a probability distribution and it is updated continuously based on sample information – priors.

The prior density is given by $f(\theta)$ and this may represent all available information up to that point.

$$f(\theta_1) = \int_{-\infty}^{\infty} f(\theta) d\theta_2 d\theta_3 \dots d\theta_n$$

The sample density of variable y is treated as conditional on the random variable θ given by $f(y/\theta)$.

The joint density of y and θ is result of the product of prior density and the sample likelihood function.

$$f(y, \theta) = f(y/\theta)f(\theta)$$

Updating the Priors

Probability statements regarding θ after the sample on y has been observed given by a posterior density

$$f(\theta/y) = \frac{f(y, \theta)}{f(y)} = \frac{f(y/\theta)f(\theta)}{\int_{-\infty}^{\infty} f(y/\theta)f(\theta)d\theta} \text{ or simply}$$

$$f(y, \theta) = f(\theta/y)f(y)$$

This is the Bayes's law named after the eighteenth century mathematician Thomas Bayes.

Prior density function of β is $f(\beta)$

Sample information $f(y_t/\beta) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(y_t - \beta)^2\right\}$

$$f(y/\beta) = f(y_1, y_2, y_3, \dots, y_T, /\beta) = f(y_1/\beta).f(y_2/\beta)\dots f(y_T/\beta)$$

Joint density $= (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=1}^T (y_t - \beta)^2\right\}$

Post sample density that represents revised prior

$$f(\beta/y) = \frac{f(y/\beta)f(\beta)}{f(y)} = \mathbf{k} \times \text{sample information} \times \text{prior information.}$$

Bayesian Updating Procedure

$f(\beta) = 1$ for $-\infty < \beta < \infty$ assume the variance of distribution σ^2 is known.

$$f(\beta/y) = kf(y/\beta)f(\beta) = k(2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=1}^T (y_t - \beta)^2\right\} \times 1$$

Convert this to a density function of β

$$\begin{aligned} \sum_{t=1}^T (y_t - \beta)^2 &= \sum_{t=1}^T [(y_t - \bar{y}) - (\beta - \bar{y})]^2 = \sum_{t=1}^T (y_t - \bar{y})^2 + \sum_{t=1}^T (\beta - \bar{y})^2 - 2(\beta - \bar{y}) \sum_{t=1}^T (y_t - \bar{y}) \\ &= \sum_{t=1}^T (y_t - \bar{y})^2 + T(\beta - \bar{y})^2 \end{aligned}$$

substituting this into the posterior density function

$$f(\beta/y) = kf(y/\beta)f(\beta) = k(2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2} \left[\sum_{t=1}^T (y_t - \bar{y})^2 + T(\beta - \bar{y})^2 \right]\right\} \times 1$$

$$f(\beta/y) = k(2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=1}^T (y_t - \bar{y})^2\right\} \exp\left\{-\frac{T}{2\sigma^2} T(\beta - \bar{y})^2\right\}$$

$$f(\beta/y) = c_1 \exp\left\{-\frac{T}{2\sigma^2} (\beta - \bar{y})^2\right\} \text{ where } c_1 = k(2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=1}^T (y_t - \bar{y})^2\right\}$$

Distribution of Bayesian Parameter in a Single Variable Regression

$$z_1 = \frac{\beta_2 - \hat{\beta}_2}{\sqrt{\text{var}(\hat{\beta}_2)}} \text{ and } \beta_2 = \hat{\beta}_2 - z_1 \sqrt{\text{var}(\hat{\beta}_2)}$$

β_2 is now a random variable

$$E[\beta_2] = \hat{\beta}_2 - E(z_1) \sqrt{\text{var}(\hat{\beta}_2)} = \hat{\beta}_2$$

$$\text{var}[\beta_2] = \text{var}(z_1) \text{var}\left(\sqrt{\text{var}(\hat{\beta}_2)}\right)^2 = \text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum_i x_{1,i}^2} = \sigma_{\beta_2}^2$$

$$\beta_2 \sim N(\hat{\beta}_2, \sigma_{\beta_2}^2)$$

$$\text{Similarly for } \beta_1 \sim N(\hat{\beta}_1, \sigma_{\beta_1}^2); \sigma_{\beta_1}^2 = \frac{\sigma^2 \sum_i X_{1,i}^2}{T \sum_i x_{1,i}^2}$$

Classical Estimation

Take a simple regression Model $Y_i = \beta_1 + \beta_2 X_{1,i} + e_i$, with following data.

| Observations | Y | X_1 | X_2 | YX_1 | YX_2 | X_1X_2 | X_1^2 | X_2^2 | Y^2 |
|--------------|-----------|-----------|-----------|-----------|------------|------------|------------|------------|------------|
| Total | 28 | 35 | 28 | 97 | 126 | 117 | 245 | 126 | 140 |

The easiest way to estimate regression with one explanatory variable is by using the

equation in deviation form. $y_i = \beta_2 x_{1,i} + e_i$. This implies $\hat{\beta}_2 = \frac{\sum_i x_{1i} y_i}{\sum_i x_{1i}^2}$

where using the deviation formula and the information given in the table

$$\sum_i x_{1i} y_i = \sum_i (X_{1i} - \bar{X}_{1i})(Y_i - \bar{Y}_i) = \sum_i X_{1i} Y_i - N \bar{X}_{1i} \bar{Y}_i = 97 - 7(4)(5) = 97 - 140 = -43$$

$$\text{and } \sum_i x_{1i}^2 = \sum_i (X_{1i} - \bar{X}_1)^2 = \sum_i X_{1i}^2 - N \bar{X}_1^2 = 245 - 7(5)^2 = 245 - 175 = 70;$$

$$\hat{\beta}_2 = \frac{-43}{70} = -0.6143$$

$$\beta_1 = \bar{Y} - \hat{\beta}_2 \bar{X}_1 = 4 - (-0.6143)5 = 4 + 3.0715 = 7.07$$

Classical Hypothesis Test

$$y_i = \beta_2 x_{1,i} + e_i \quad y_i = \hat{y}_i + e_i \Rightarrow \sum y_i^2 = \sum \hat{y}_i^2 + \sum e_i^2$$

$$\sum y_i^2 = \hat{\beta}_2^2 \sum x_{1,i}^2 + \sum \hat{e}_i^2 \Rightarrow 28 = (-0.6143)^2 * 70 + \sum \hat{e}_i^2$$

$$\sum \hat{y}_i^2 = (-0.6143)^2 * 70 = 26.41 \quad \text{and} \quad \sum \hat{e}_i^2 = 28 - 26.41 = 1.584$$

$$R^2 = \frac{\sum \hat{y}_i^2}{\sum y_i^2} = \frac{26.41}{28} = 94.32\%;$$

$$R^2 = 1 - \frac{\sum \hat{e}_i^2 / (N - K)}{\sum y_i^2 / (N - 1)} = 1 - \frac{1.584 / (7 - 2)}{28 / (7 - 1)} = 1 - \frac{0.3168}{4.667} = 93.21\%$$

$$\text{var}(e_i) = \hat{\sigma}^2 = \frac{\sum \hat{e}_i^2}{N - K} = \frac{1.584}{5} = 0.3168;$$

The variance of the slope parameter is given by:

$$\text{var}(\hat{\beta}_2) = \frac{\hat{\sigma}^2}{\sum x_{1,i}^2} = \frac{0.3168}{70} = 0.00453$$

Calculate t value for the given estimates to test $H_0 : \beta_2 = 0$ against $H_1 : \beta_2 \neq 0$.

$$t_{calc} = \frac{\beta_2 - \hat{\beta}_2}{\sqrt{\text{var}(\hat{\beta}_2)}} = \frac{-0.6143}{\sqrt{0.0043}} = -9.367; \quad \text{The theoretical t value for } t_{5,0.05} = 2.571$$

Distribution of Bayesian Parameter in a Single Variable Regression

$$z_1 = \frac{\beta_2 - \hat{\beta}_2}{\sqrt{\text{var}(\hat{\beta}_2)}} \text{ and } \beta_2 = \hat{\beta}_2 - z_1 \sqrt{\text{var}(\hat{\beta}_2)}$$

β_2 is now a random variable

$$E[\beta_2] = \hat{\beta}_2 - E(z_1) \sqrt{\text{var}(\hat{\beta}_2)} = \hat{\beta}_2$$

$$\text{var}[\beta_2] = \text{var}(z_1) \text{var}\left(\sqrt{\text{var}(\hat{\beta}_2)}\right)^2 \quad \text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum_i x_{1,i}^2} = \sigma_{\beta_2}^2 = (\sqrt{0.0043})^2 = 0.0043$$

$$E[\beta_2] = -0.6143 - 9.367 \sqrt{0.004} = 1.207$$

$$\beta_2 \sim N(\hat{\beta}_2, \sigma_{\beta_2}^2)$$

$$\text{Similarly for } \beta_1 \sim N(\hat{\beta}_1, \sigma_{\beta_1}^2); \quad \sigma_{\beta_1}^2 = \frac{\sigma^2 \sum_i X_{1,i}^2}{T \sum_i x_{1,i}^2}$$

Asymptotic Theory

$$\lim P\left(|\hat{\theta} - \theta| \leq \varepsilon\right) = 1 \quad t \rightarrow \infty; \quad P\lim(\hat{\theta}) = \theta$$

Chebycheff inequality $P\{|z - E(z)| > \delta\} < \frac{V\{z\}}{\delta^2}$ for all $\delta > 0$.

For the OLS estimator this implies that

$$P\{|b_k - \beta_k| > \delta\} < \frac{V\{\beta_k\}}{\delta^2} = \frac{\sigma^2 (X'X)^{-1}}{\delta^2} \quad \text{as the sample size increases}$$

$(X'X)^{-1} = \left(\sum_i^n x_i x_i'\right)^{-1}$ decreases as $\left(\sum_i^n x_i x_i'\right)$ becomes larger. When N become infinitely larger then

$$\lim_{N \rightarrow \infty} P\{|b_k - \beta_k| > \delta\} < \frac{V\{\beta_k\}}{\delta^2} = \frac{\sigma^2 (X'X)^{-1}}{\delta^2} = 0 \quad \text{for all } \delta > 0.$$

This is the same thing as saying that the b converges in probability in β .

$P\lim(b) = \beta$ when the estimator of β converges to its true value it is a **consistent estimator** with asymptotic distribution as: $b \overset{a}{\sim} N\left(\beta, \left(\sum_i^n x_i x_i'\right)^{-1}\right)$

Sargan's General Instrumental Variable Estimator

If $E(X'u) \neq 0$ assume a set of instruments Z which has same dimension as X but uncorrelated with the error term, $E(Z'u) = 0$ and variance $Var(Z'u) = (Z'Z)\sigma^2$.

The instrumental variable estimator is obtained by setting $(Z'u) = 0$
or $Z'(Y - X\beta) = 0$ or $\hat{\beta}_{IV} = (Z'X)^{-1}Z'Y$

Sargan (1958) had proposed generalised instrumental variable estimator (GIVE)

$$\text{Min}(Y - X\beta)' Z'(Z'Z)^{-1} Z'(Y - X\beta)$$

$$E(u'u) = \Omega$$

Then GIVE minimizes $\text{Min}(Y - X\beta)' Z'(Z'\Omega Z)^{-1} Z'(Y - X\beta)$.

The GMM estimator proposed by Hansen (1982) is similar to GIVE estimator

It minimises $u'ZWZ'u$

$$\text{Min } (Y - X\beta)'Z'WZ'(Y - X\beta) \rightarrow (X'Z)W(Z'X)\hat{\beta} = (X'Z)WZ'Y$$

$$\hat{\beta}_{GMM} = (X'ZWZ'X)^{-1}(X'ZWZ'Y)$$

Hansen's weighting matrix $W = (Z'Z)^{-1} / \sigma^2 = \text{Var}(Z'u)^{-1}$ when $E(u'u) = \Omega$

Changing this weighting matrix

$$\hat{\beta}_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}(X'Z(Z'\Omega Z)^{-1}Z'Y)$$

$$\text{Cov}(\hat{\beta}_{GMM}) = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}$$

The popularity of GMM is due to its generality as it can be applied to linear and non-linear estimations.