Combining Belief Functions Issued from Dependent Sources

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Definition. A piece of information about a finite set of propositional variables is described by a basic belief assignment (bba)

$$m : 2^\Omega \to [0, 1] \text{ s.t. } m(\emptyset) = 0 \text{ and } \sum_A m(A) = 1,$$

where $\Omega$ is the set of valuations of the propositional language (i.e. $\Omega$ is the set of “possible worlds” ⇒ the “open-world assumption” does not make sense).

The respective belief and plausibility functions on $\Omega$ are defined by

$$\text{bel}(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad \text{pl}(A) = \sum_{B \cap A \neq \emptyset} m(B).$$

to pool the information issued from two sources

$\leadsto$ combine the respective bbas $m_1$ and $m_2$ in a new bba $m_{12}$

independence of the sources assumed

(this assumption can be justified only by analogies with other situations in which it proved to be sensible)

$\leadsto$ use Dempster’s rule of combination:

$$m_{12}(A) \propto \sum_{B \cap C = A} m_1(B) m_2(C) \text{ if } A \neq \emptyset$$
to allow the dependence of the sources
$\leadsto$ generalize Dempster’s rule

**Definition.** A joint belief assignment (jba) with marginal bbas $m_1$ and $m_2$ is a function

$$m : 2^\Omega \times 2^\Omega \to [0, 1] \text{ s.t. } \sum_B m(A, B) = m_1(A)$$

$$\text{and } \sum_A m(A, B) = m_2(B).$$

combination with respect to a jba $m$:

$$m_{12}(A) \propto \sum_{B \cap C = A} m(B, C) \text{ if } A \neq \emptyset$$

($\Rightarrow$ the independence assumption corresponds to the choice of the jba $m(A, B) = m_1(A) m_2(B)$)

nothing assumed about the sources
$\leadsto$ play safe and choose the “most conservative” combination
MINIMAL CONFLICT

Definition. A combination \( \text{bel}_{12} \) of two belief functions \( \text{bel}_1 \) and \( \text{bel}_2 \) is monotonic if

\[
\text{bel}_{12} \geq \text{bel}_1 \quad \text{and} \quad \text{bel}_{12} \geq \text{bel}_2.
\]

Definition. The conflict of the combination with respect to a jba \( m \) is

\[
\sum_{A \cap B = \emptyset} m(A, B).
\]

(no conflict \( \Rightarrow \) the combination is monotonic)

the conflict is a good index for the nonmonotonicity of a combination

\( \sim \) the “most conservative” combination has minimal conflict

Theorem. The minimal conflict of the combinations of \( \text{bel}_1 \) and \( \text{bel}_2 \) is

\[
\max_{A} \left( \text{bel}_1(A) - \text{pl}_2(A) \right).
\]

Corollary. The monotonicity of the combination of \( \text{bel}_1 \) and \( \text{bel}_2 \) is admissible (i.e. \( \exists \) \( \text{bel} \) s.t. \( \text{bel} \geq \text{bel}_1 \) and \( \text{bel} \geq \text{bel}_2 \)) if and only if they are compatible (i.e. \( \text{bel}_1 \leq \text{pl}_2 \)).

In this case, the combinations with minimal conflict are monotonic.
In the generalized Bayes’ theorem, combinations with minimal conflict lead to better results than combinations obtained from Dempster’s rule.

Consider $n$ hypotheses $h_1, \ldots, h_n$ implying the belief functions $bel_1, \ldots, bel_n$ on $\Omega$, respectively.
Let the belief function $bel_o$ on $\Omega$ represent an observation and let $c_1, \ldots, c_n$ be the conflicts of its combination with $bel_1, \ldots, bel_n$, respectively.
In the simplest case, the prior belief function on $\{h_1, \ldots, h_n\}$ is an epistemic probability $p_1, \ldots, p_n$. In this case, the posterior belief function is the epistemic probability $p'_1, \ldots, p'_n$, with

$$p'_i \propto (1 - c_i) p_i.$$ 

Thus the conflicts come out as the measure of the disagreement between the respective hypotheses and the observation.

If the $c_i$ are the minimal conflicts, then from $bel_o \leq pl_i$ (i.e. $h_i$ is compatible with the observation) follows $p'_i \geq p_i$.
This is not assured if we use Dempster’s rule: $p'_i < p_i$ is possible even if $bel_o = bel_i$ (i.e. $h_i$ is “perfect”).

As a measure of the disagreement between two belief functions, the minimal conflict (1) is much better than the conflict of Dempster’s rule.

**Example.** $\Omega = \{a, b\}$, $n = 4$, $bel_o = bel_1$, $bel_2$ is vacuous.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$m_i({a})$</th>
<th>$m_i({b})$</th>
<th>$m_i(\Omega)$</th>
<th>Dempster’s rule</th>
<th>minimal conflict</th>
</tr>
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</table>
MINIMAL SPECIFICITY

**Definition.** The *measure of nonspecificity* of a belief function with bba $m$ is

$$\sum_{A \neq \emptyset} m(A) \log_2 |A|.$$  

if the combination with minimal conflict is not unique

$\leadsto$ the "most conservative" combination has minimal specificity (i.e. it maximizes the measure of nonspecificity) among the ones with minimal conflict

**Definition.** $bel_2$ is a specialization of $bel_1$ if $m_2$ can be obtained through redistribution of $m_1(A)$ to the non-empty sets $B \subseteq A$, for all $A \subseteq \Omega$.

**Theorem.** $bel_1$ and $bel_2$ have a common specialization if and only if they are compatible (i.e. $bel_1 \leq pl_2$). In this case, the combinations with minimal specificity among the ones with minimal conflict are the least specific common specializations of $bel_1$ and $bel_2$.

To obtain a combination with minimal specificity among the ones with minimal conflict

$\leadsto$ maximize a linear functional on the convex polytope (in $\mathbb{R}^{2^{[\Omega]}}$) of the jbas

the solutions build a convex polytope

$\leadsto$ choose a point of the polytope in such a way that the obtained rule $(bel_1, bel_2) \mapsto bel_1 \odot bel_2$ satisfies some requirements of invariance (choose for instance the centre of the polytope)
Conservative Combination Rule

The obtained “most conservative” combination rule $\odot$ has the following properties.

- **commutativity:**
  
  $\text{bel}_1 \odot \text{bel}_2 = \text{bel}_2 \odot \text{bel}_1$

- **monotonicity** (if admissible, i.e. if $\exists \text{bel}$ s.t. $\text{bel} \geq \text{bel}_1$ and $\text{bel} \geq \text{bel}_2$):
  
  $\text{bel}_1 \odot \text{bel}_2 \geq \text{bel}_1$ and $\text{bel}_1 \odot \text{bel}_2 \geq \text{bel}_2$

- $\text{bel}_1 \odot \text{bel}_2$ is a least specific common specialization of $\text{bel}_1$ and $\text{bel}_2$
  
  (if a common specialization exists)

  $\Rightarrow$ **absorption:**

  $\text{bel}_s$ is a specialization of $\text{bel}$  $\Rightarrow$  $\text{bel}_s \odot \text{bel} = \text{bel}_s$

  $\Rightarrow$ **idempotency:**

  $\text{bel} \odot \text{bel} = \text{bel}$

But minimization of conflict and idempotency are both incompatible with associativity.

Thus the binary rule $\odot$ is not associative, but it can be easily extended to an $n$-ary rule for the simultaneous combination of any number of belief functions: simply consider the $n$-dimensional jbas instead of the 2-dimensional ones.